Unifying classical spin models using a quantum formalism

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Outline

- Motivation
- Completeness

  The 4D $\mathbb{Z}_2$ lattice gauge theory is complete

- Complexity

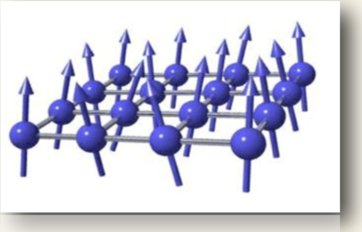
  Approximating the partition function of some models is BQP-complete

- Summary

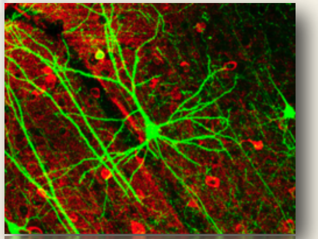
Motivation

Motivation

- Classical spin models:
  - Classical magnetism



  - Spin glasses



  - Neural networks
  - Econophysics

...

Motivation

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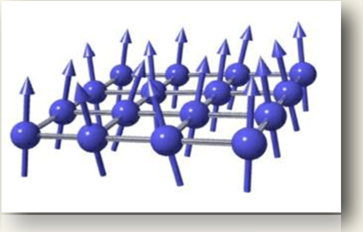
  - Classical magnetism



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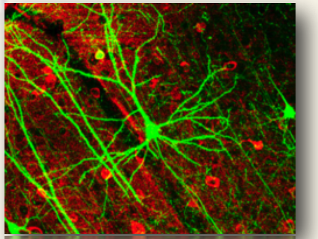


  - Neural networks

  - Econophysics

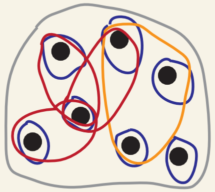
...

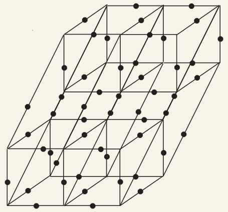
Toy models to tackle complex systems

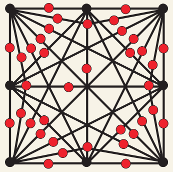
Make a simple microscopic model & study the macroscopic behavior

Motivation

- Many different kinds of classical spin models
  - Different dimensions, defined on complicated graphs...
  - Many-body interactions...

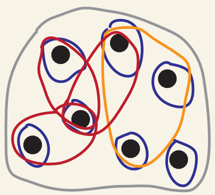
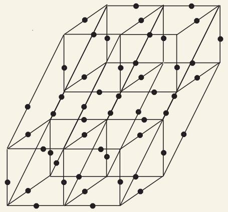
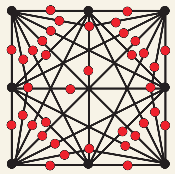






Motivation

- Many different kinds of classical spin models
  - Different dimensions, defined on complicated graphs...
  - Many-body interactions...
  - Symmetries:

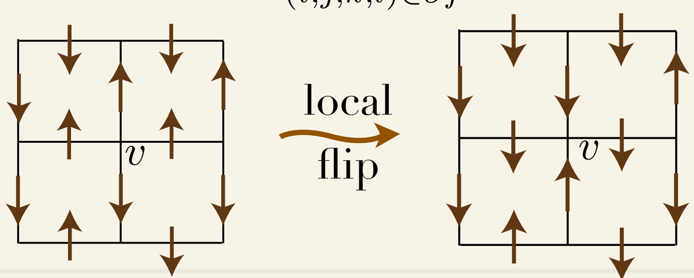
Global: Ising, Potts ...

$$H(\mathbf{s}) = -J \sum_{(i,j)\in E} s_i s_j$$

$H(\mathbf{s}) \quad = \quad H(\mathbf{s}')$

Local: lattice gauge theories

$$H(\mathbf{s}) = -J \sum_{(i,j,k,l)\in\partial f} s_i s_j s_k s_l$$



$H(\mathbf{s}) \quad = \quad H(\mathbf{s}')$

Motivation

Can one relate all these models?

By studying one model, can one learn something of *other* models?

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By studying one model, can one learn something of *other* models?

Yes!

Completeness results:

Models with different features can be mapped onto a single model

- Use Quantum Information tools to relate them
- In equilibrium the crucial quantity: partition function $Z = \sum_{\mathbf{s}} e^{-\beta H(\mathbf{s})}$

Completeness

Completeness

A model is 'complete'

⇕

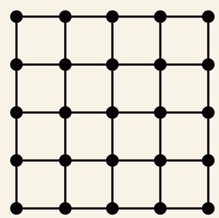
Its partition function can specialize
(by tuning its coupling strengths)
to the partition function of any other classical spin model

M. Van den Nest, W. Dür, H. J. Briegel, *Phys. Rev. Lett.* **100**, 110501(2008)

Completeness of the 2D Ising

- Result:

$$Z_{\text{2D Ising with } h}(J, J') \quad = \quad Z_{\text{any classical spin model}}(J)$$



J'

larger

↑

Ising, Potts, ...

- ✓ on an arbitrary graph
- ✓ q-level systems, any q
- ✓ any many-body int.

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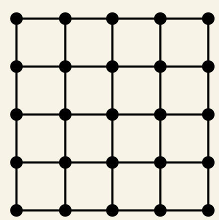

J' complex

larger

Ising, Potts, ...

✓ on an arbitrary graph

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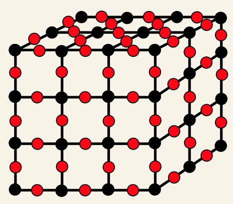
Completeness with real coupl.

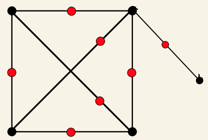
- Result:
  - Ising model:

$$Z_{\text{3D Ising}}(J, J') \quad = \quad Z_{\text{Ising, any } G}(J)$$



J' real ✓

larger



  - Analogous for q-level systems

GDlC, W. Dür, M. Van den Nest, H. J. Briegel, *JSTAT* P07001 (2009)

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⚠ same kind of interactions

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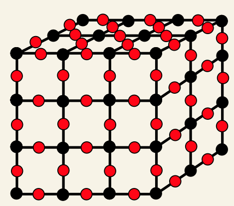
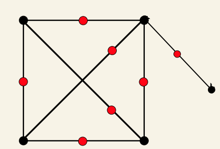


J' real ✓

larger



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Completeness with real coupl.

- Result:
  - Ising model:

tradeoff between 'completeness power' and real parameters?

⚠ same kind of interactions

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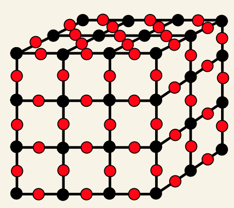

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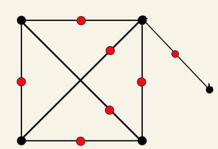
larger



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lattice gauge theory

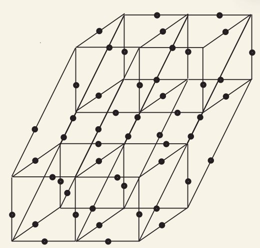
Completeness of the 4D $\mathbb{Z}_2$ LGT

- Main result:

constructive

$$Z_{4\mathrm{D}\mathbb{Z}_2\mathrm{LGT}}(J, J') = Z_{\text{any classical spin model}}(J)$$

real



larger

Abelian discrete

✓ any dimensions

✓ q-level systems, any q

✓ any many-body int.

global & local symmetries

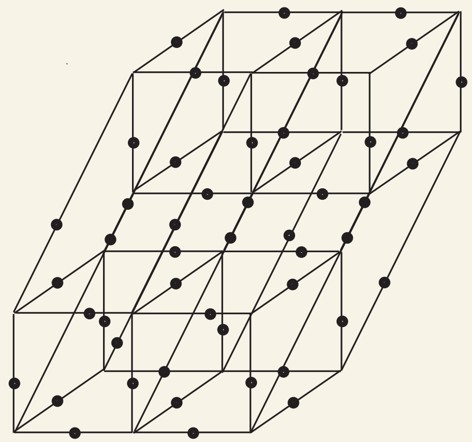
GDlC, W. Dür, H. J. Briegel, M. A. Martin-Delgado, *Phys. Rev. Lett.* **102**, 230502 (2009); *New J. Phys.* **12**, 043014 (2010)

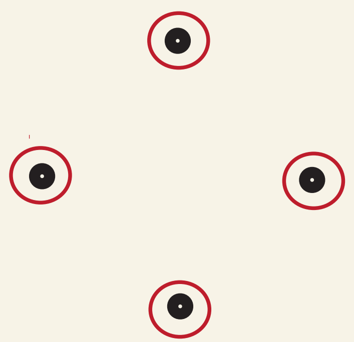
lattice gauge theory

Completeness of the 4D $\mathbb{Z}_2$ LGT

- Idea of the proof:

all k-cliques for $k=1,\ldots,n$ with Ising-type int.





4D $\mathbb{Z}_2$ LGT → Superclique

real couplings $J = 0, \infty$

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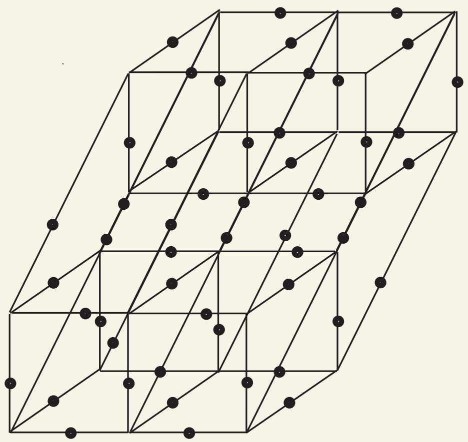

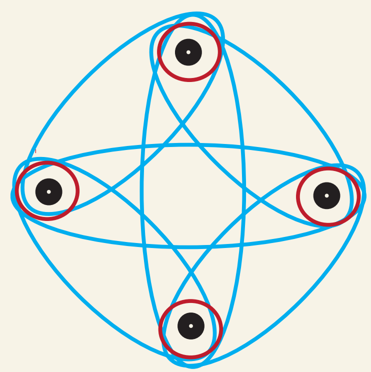

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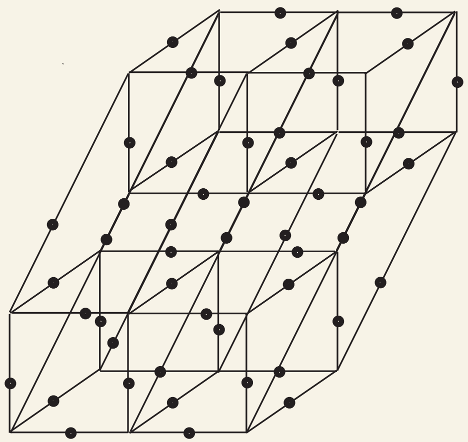

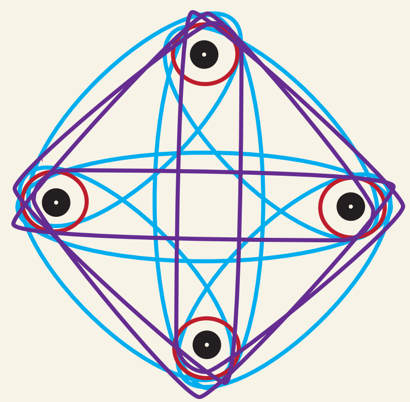

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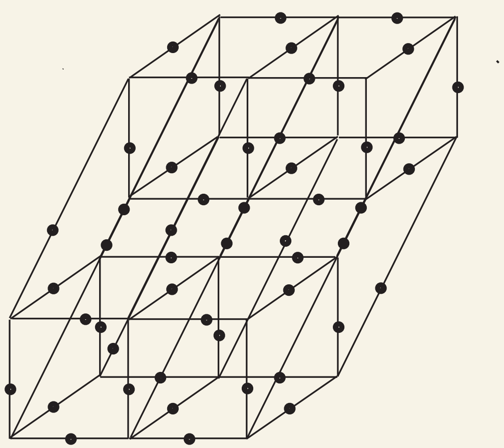

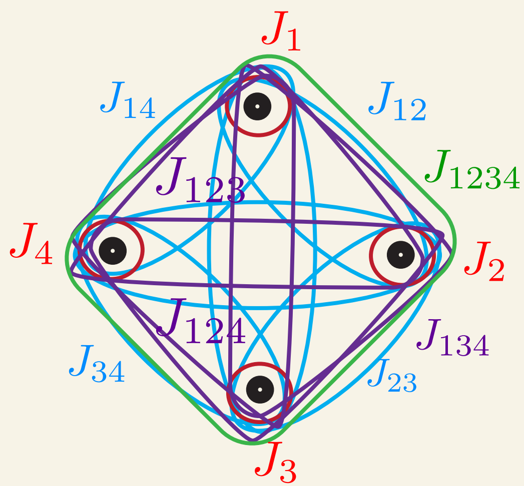

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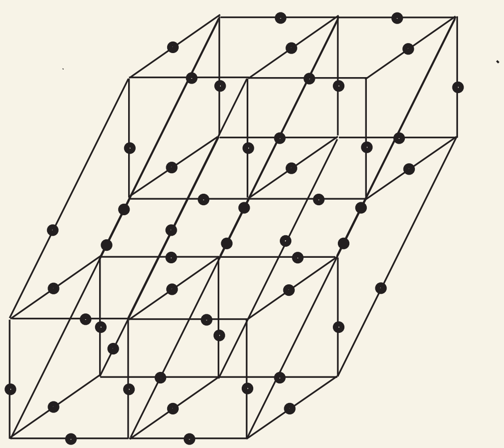

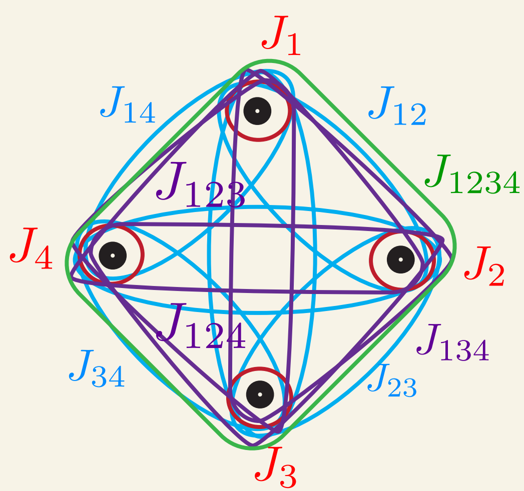

4D $\mathbb{Z}_2$ LGT → Superclique ⟷ Any Abelian discrete classical spin model

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Hamiltonian!

lattice gauge theory

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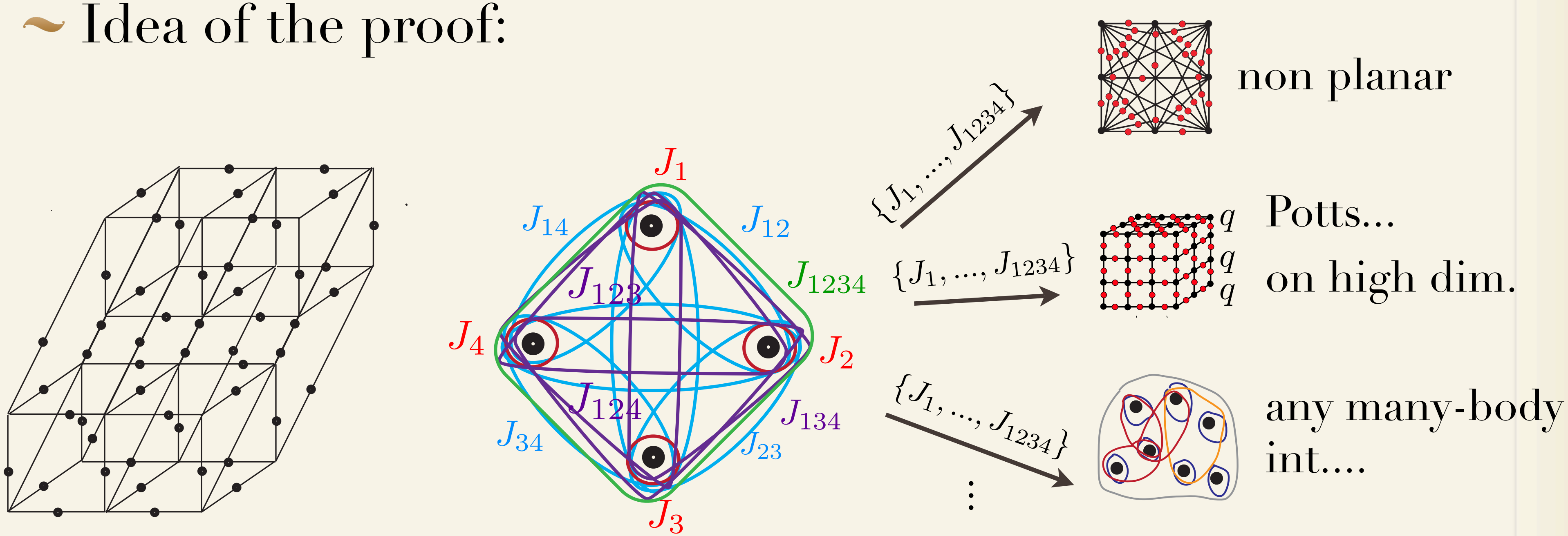

4D $\mathbb{Z}_2$ LGT ⇒ Superclique ⇔ Any Abelian discrete classical spin model

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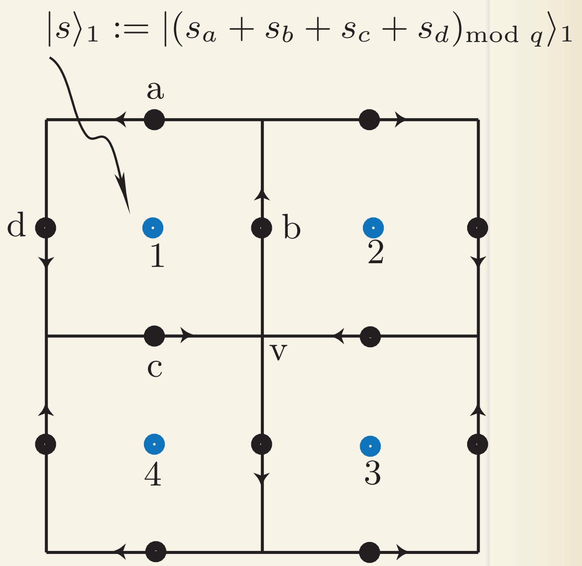
Completeness of the 4D $\mathbb{Z}_2$ LGT

- Quantum formulation of Abelian discrete LGTs
  - Hamiltonian $H(\mathbf{s}) = -\sum_{f\in F} J_f \cos\left[\frac{2\pi}{q}(s_1 + \ldots + s_k)_{\text{mod}q}\right]$

    Partition function: $Z_G(J) = \sum_{\mathbf{s}} e^{-\beta H(\mathbf{s})}$



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  - State defined on the faces:

    $$|\psi_G\rangle = \sum_{\mathbf{s}} \bigotimes_{f\in F} |(s_1+\cdots+s_k)_{\mathrm{mod}q}\rangle_f$$

    Product state with coefficients: $|\alpha(J)\rangle = \bigotimes_f |\alpha_f(J_f)\rangle$

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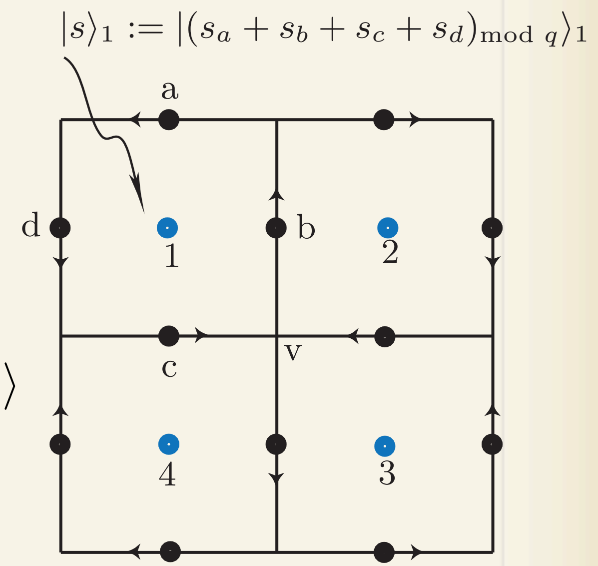

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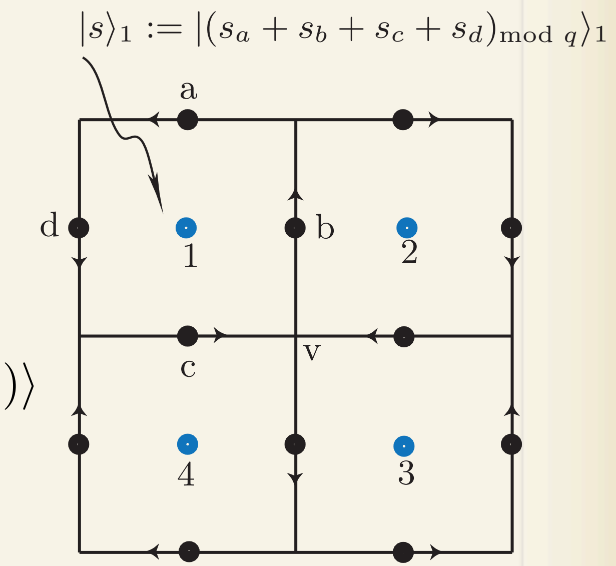
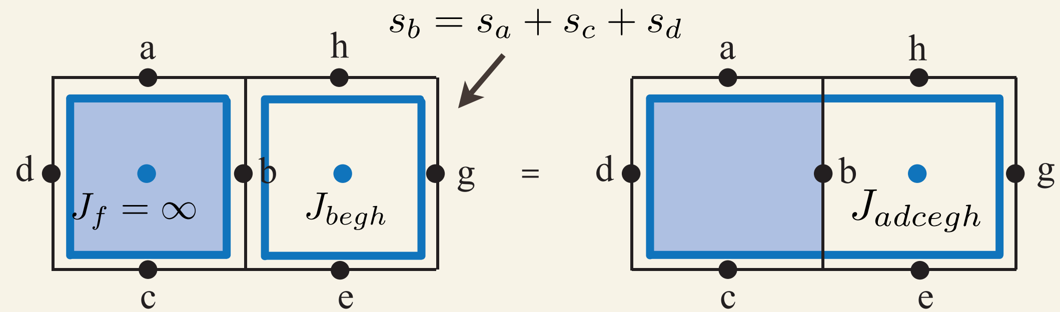


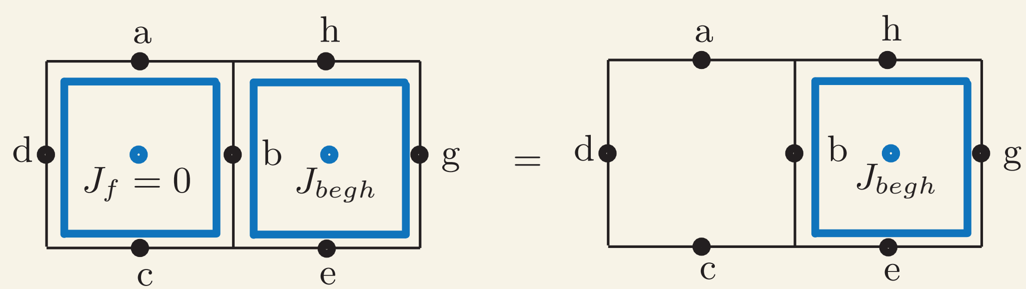
$Z_G(J) = \langle\alpha(J)|\psi_G\rangle$

Completeness of the 4D $\mathbb{Z}_2$ LGT

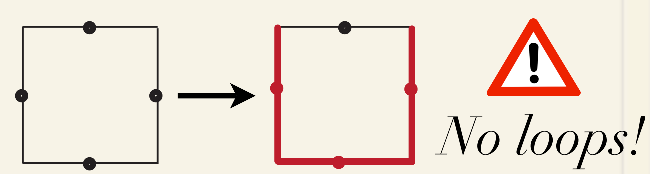
- Tools to 'transform' the model:
  - Merge rule:



  - Deletion rule:

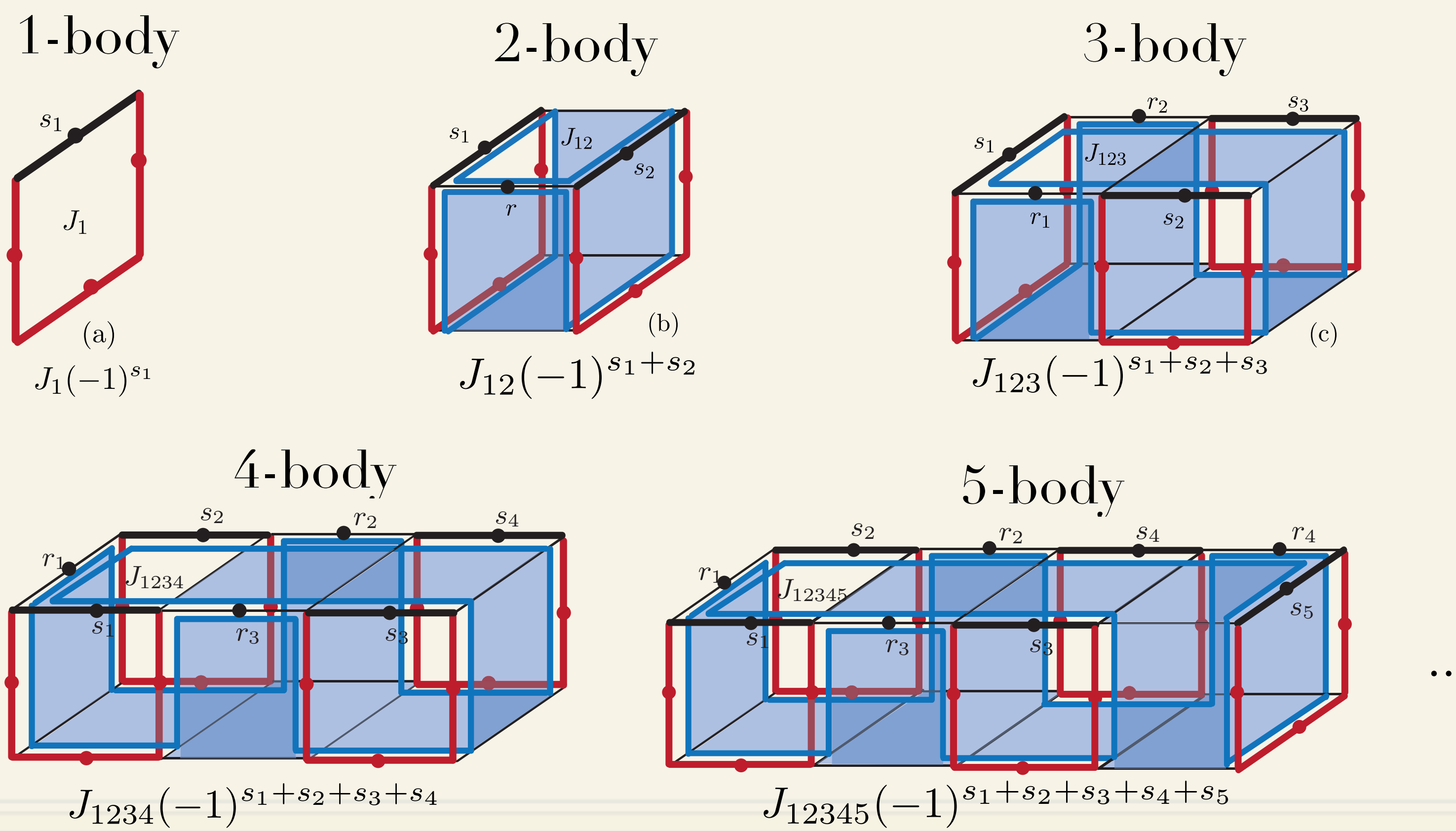


  - Fixing the spins using the gauge symmetry:



Completeness of the 4D $\mathbb{Z}_2$ LGT

- Construction of the superclique
  - Construction of many-body Ising-type int.:

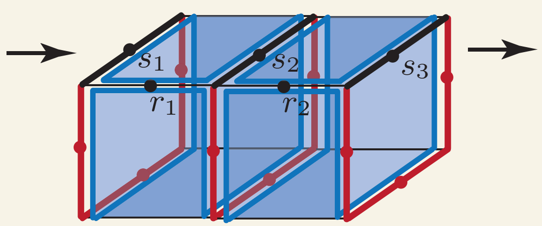


Completeness of the 4D $\mathbb{Z}_2$ LGT

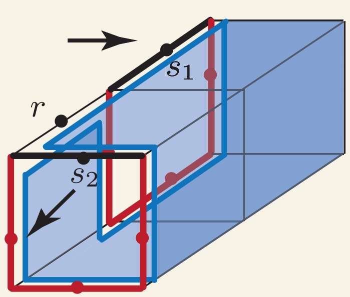
- Construction of the superclique
  - Transportation in the 4D lattice:

4th dimension required to avoid loops!

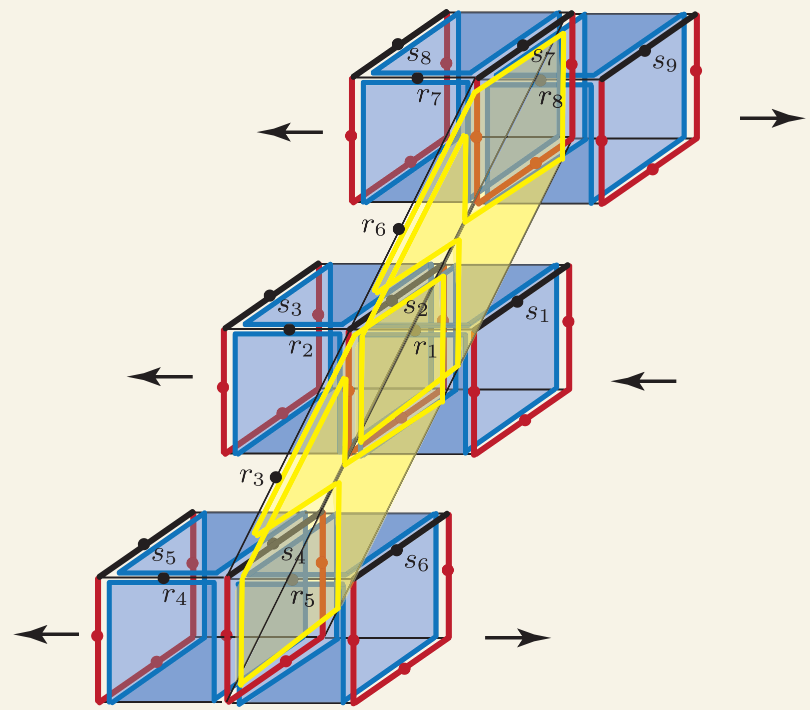
Propagation



Turns

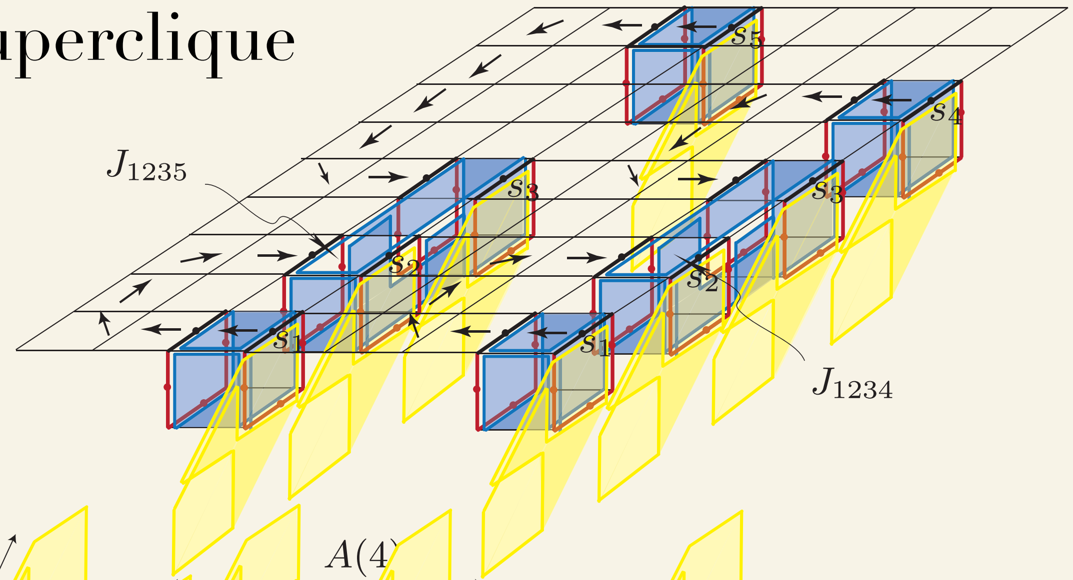


Replication

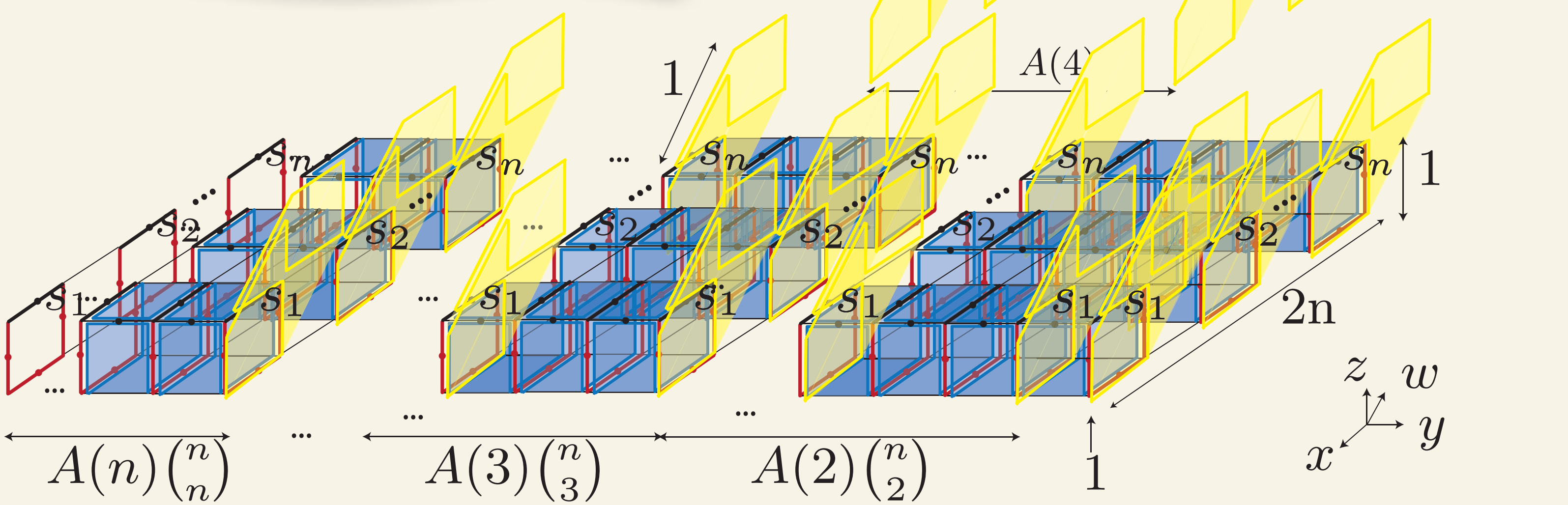


Completeness of the 4D $\mathbb{Z}_2$ LGT

- Construction of the superclique
  - Layout of the superclique:



Superclique:
complicated interaction pattern
with simple interactions



Completeness of the 4D $\mathbb{Z}_2$ LGT

Hamiltonian of superclique

$\Updownarrow$

Hamiltonian of any classical spin model

1. General Hamiltonian on n 2-level systems: different $E(\mathbf{s})$ for each $\mathbf{s}$
2. Show that one can invert the system of equations

$$\underbrace{\begin{pmatrix} 1 & (-1)^0 & \dots & (-1)^{0+0+\dots+0} \\ 1 & (-1)^0 & \dots & (-1)^{0+0+\dots+1} \\ & & \vdots & \\ 1 & (-1)^1 & \dots & (-1)^{1+1+\dots+1} \end{pmatrix}}_{C} \begin{pmatrix} J \\ J_1 \\ \vdots \\ J_{12\dots n} \end{pmatrix} = \begin{pmatrix} E(\mathbf{s} = (0,0,\dots,0)) \\ E(\mathbf{s} = (0,0,\dots,1)) \\ \vdots \\ E(\mathbf{s} = (1,1,\dots,1)) \end{pmatrix}$$

2^n different energies

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$ couplings

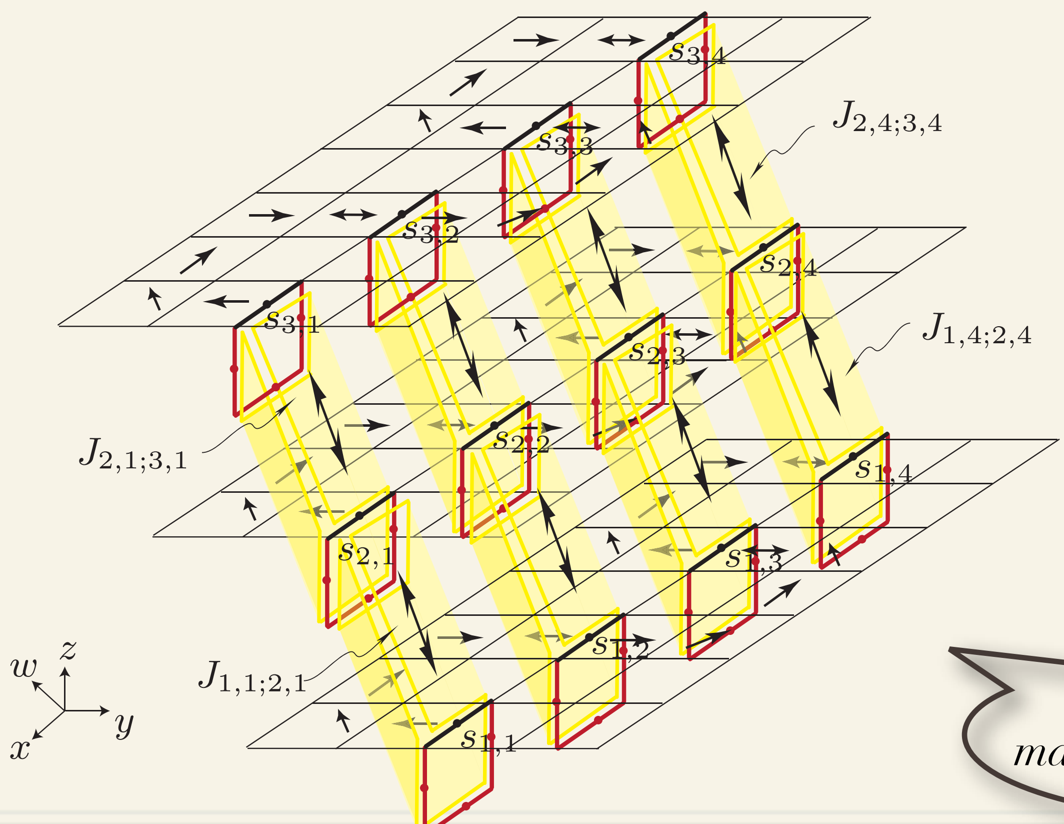
3. All rows are linearly independent, thus the determinant is non zero
4. q-level models: encode each q-level system into $\lceil \log_2 q \rceil$ 2-level sys.

q.e.d.

Completeness of the 4D $\mathbb{Z}_2$ LGT

- Note: efficient constructions for specific target models

  Example: 2D Ising model: linear overhead ✓



Note: 2D Ising can magnetize, but LGT cannot

Completeness of the 4D $\mathbb{Z}_2$ LGT

- We have proven that:

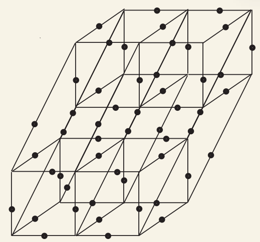
global & local symmetries

constructive

$$Z_{4\mathrm{D}\mathbb{Z}_2\mathrm{LGT}}(J, J') \doteq Z_{\text{any classical spin model}}(J)$$

real

Abelian discrete



✓ any dimensions

✓ q-level systems, any q

✓ any many-body int.

larger

Target hamiltonian with M terms and k-body int: scaling $poly(M, 2^k)$

- Result holds approx for continuous models: let $q \to \infty$

Applications of completeness

- Symmetries of the states ➡ symmetries of the partition function

$$Z_G(J) = \underbrace{\langle\alpha(J)|S}_{\langle\alpha(J')|}|\varphi_G\rangle = Z_G(J')$$

- Mapping models with poly overhead: infer comput. complexity

e.g. 2D Ising with fields $\xrightarrow[\textit{poly larger}]{}$ 4D $\mathbb{Z}_2$ LGT

#P-hard ➡ #P-hard

- Many different universality classes are mapped to a single model

They should be reproducible in the phase diagram of the complete model

this includes 'unexplored' models

Computational complexity

Ⓘ Mapping partition functions to quantum circuits

Classical spin model		Quantum circuit
Boltzmann weight of each int. $w^a = e^{-\beta h^a(s_1,s_2)}$	$\longrightarrow$	Quantum gate, e.g. $W^a_{(12)(12)} = \sum e^{-\beta h^a(s_1,s_2)} \vert s_1,s_2\rangle\langle s_1,s_2\vert$
Product of interactions $\prod_a w^a$	$\longrightarrow$	Contraction of quantum gates = Circuit $\mathcal{C}$
Left & Right bound. cond. $L=(s_1^L,\ldots,s_n^L)$, $R=(s_1^R,\ldots,s_n^R)$	$\longrightarrow$	Output & Input of circuit $\vert L\rangle = \vert s_1^L\rangle\ldots\vert s_n^L\rangle$, $\vert R\rangle = \vert s_1^R\rangle\ldots\vert s_n^R\rangle$

$$Z^{L,R} = \langle L|\mathcal{C}|R\rangle$$

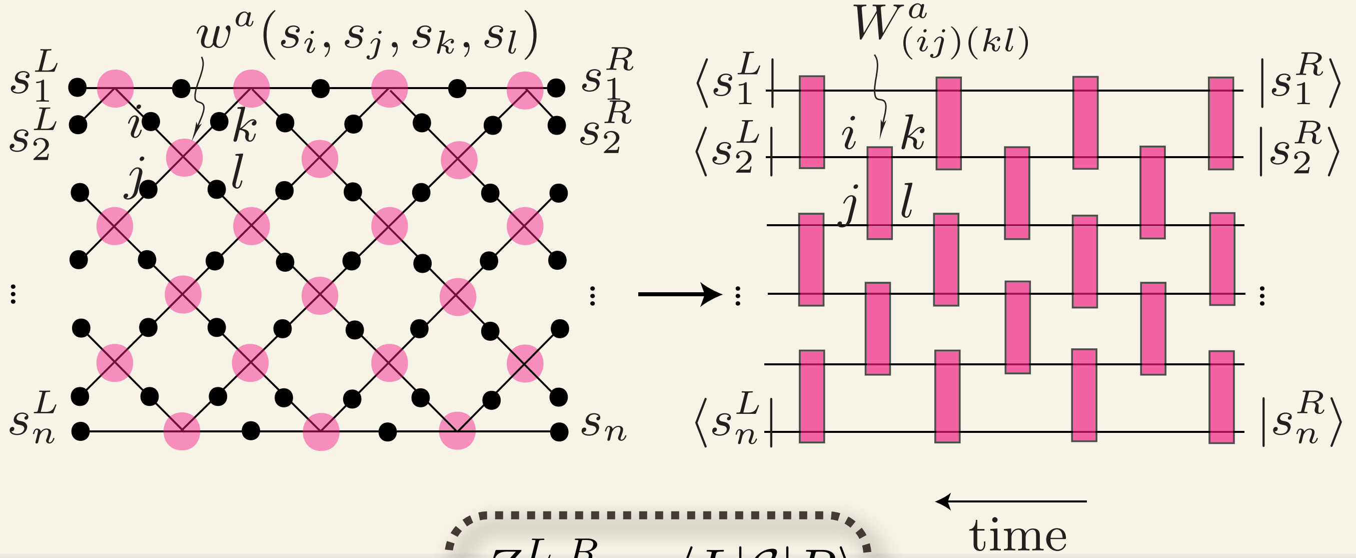
✓PBC $Z = \mathrm{Tr}\,\mathcal{C}$
✓other geometries

M. Van den Nest, W. Dür, R. Raussendorf, H. J. Briegel, *Phys. Rev. A* **80**, 052334 (2009)

Ⓘ Mapping partition functions to quantum circuits

- Mapping for vertex models
  - Particles at the edges
  - Interaction at vertex a → Two-qudit gate

$$w^a(\mathbf{s}) = \sum e^{-\beta h^a(s_i s_j s_k s_l)} \longrightarrow W^a_{(ij)(kl)} = \sum e^{-\beta h^a(s_i s_j s_k s_l)} |s_i, s_j\rangle\langle s_k s_l|$$



$$Z^{L,R}_{\mathrm{vm}} = \langle L|\mathcal{C}|R\rangle$$

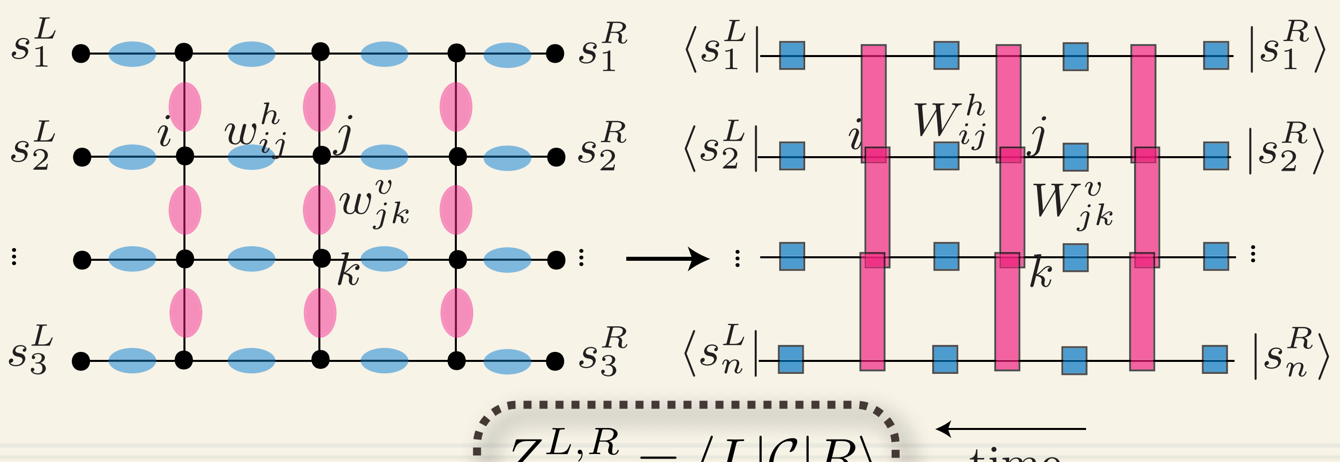
Ⅰ Mapping partition functions to quantum circuits

- Mapping for edge models
  - Particles at the vertices
  - Int. at edge in time dir. → Single qudit gate

$$w^{ij} = e^{-\beta h(s_i, s_j)} \longrightarrow w_{(i)(j)} = \sum e^{-\beta h(s_i, s_j)} |s_i\rangle\langle s_j|$$

  - Int. at edge in space dir. → Two qudit diagonal gate

$$w^{jk} = e^{-\beta h(s_j, s_k)} \longrightarrow w_{(jk)(jk)} = \sum e^{-\beta h(s_j, s_k)} |s_j s_k\rangle\langle s_j s_k|$$



$$Z_{\rm em}^{L,R} = \langle L|\mathcal{C}|R\rangle$$

Ⓘ Mapping partition functions to quantum circuits

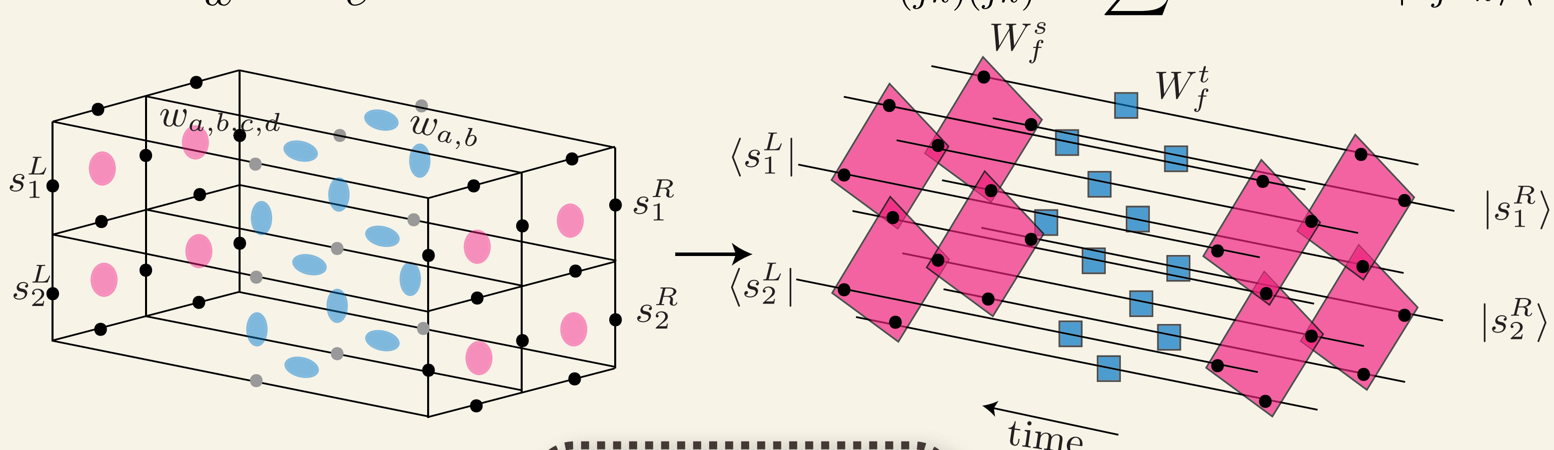
~ Mapping for lattice gauge theories

- Particles at the edges & Fixing the temporal gauge
- Int. at face in time dir. → Single qudit gate

$$w^{ij} = e^{-\beta h(s_i,s_j)} \quad \longrightarrow \quad w_{(i)(j)} = \sum e^{-\beta h(s_i,s_j)} |s_i\rangle\langle s_j|$$

- Int. at face in space dir. → Four qudit diagonal gate

$$w^{jk} = e^{-\beta h(s_j,s_k)} \quad \longrightarrow \quad w_{(jk)(jk)} = \sum e^{-\beta h(s_j,s_k)} |s_j s_k\rangle\langle s_j s_k|$$



$$Z_{\rm LGT}^{L,R} = \langle L|\mathcal{C}|R\rangle$$

Ⅱ BQP-completeness results

- Main idea:

Model ⟶ Gates corresp. to that model ⟶ Show that they form a universal gate set

⇕ $Z = \langle L|\mathcal{C}|R\rangle$

Approximating that partition function is as hard as simulating arbitrary quantum computation

II BQP-completeness results

- Main idea:

Model ⇝ Gates corresp. to that model ⇝ Show that they form a universal gate set

⇕ $Z = \langle L|\mathcal{C}|R\rangle$

Approximating that partition function is as hard as simulating arbitrary quantum computation

II BQP-completeness results

- Main idea:

Model ⟶ Gates corresp. to that model ⟶ Show that they form a universal gate set

⇕ $Z = \langle L | \mathcal{C} | R \rangle$

Approximating that partition function is as hard as simulating arbitrary quantum computation

- Prove BQP-completeness of computing Z
- Provide a quantum algorithm

Ⅱ BQP-completeness results

- Six vertex model
  - (Encoded) universal interaction $U = e^{itH_{\rm ex}}$ with $H_{\rm ex} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$
  - Encoding $|\mathbf{0}\rangle = \frac{1}{2}(|01\rangle - |10\rangle)^{\otimes 2}$
  - Preparation of $|\mathbf{0}\rangle|\mathbf{0}\rangle \ldots |\mathbf{0}\rangle$ from $|R\rangle = |0\rangle|1\rangle|0\rangle|1\rangle \ldots$ possible
  - The exchange int. is achieved with the six-vertex model-type gate:

$$W_{(ij)(jk)} = \begin{bmatrix} e^{i2t} & & & \\ & \cos(2t) & i\sin(2t) & \\ & i\sin(2t) & \cos(2t) & \\ & & & e^{i2t} \end{bmatrix}$$

II BQP-completeness results

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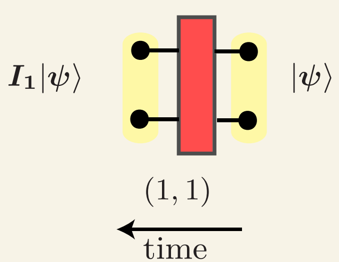
Approximating the partition function of the six vertex model on a certain complex parameter regime is BQP-complete

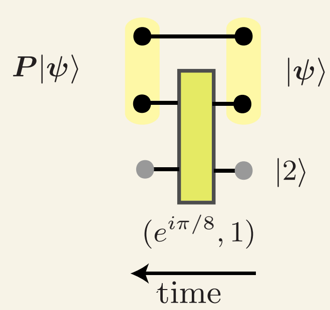
II BQP-completeness results

- Potts model
  - Encoding: $|\mathbf{0}\rangle = |0\rangle|1\rangle$
    $|\mathbf{1}\rangle = |1\rangle|2\rangle$
  - Trivial preparation of $|\mathbf{0}\rangle \dots |\mathbf{0}\rangle$ from $|R\rangle = |0\rangle|1\rangle \dots |0\rangle|1\rangle$
  - Each Potts gate is characterized by the pair $(e^{\beta J_{ii}}, e^{\beta J_{i \neq j}})$
  - Construct an (encoded) universal gate set:

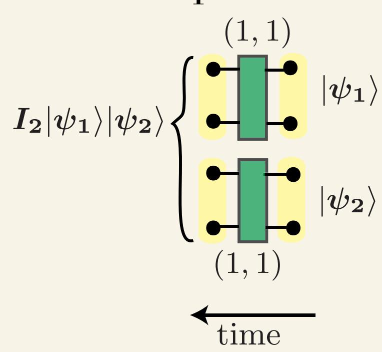
Single qubit identity



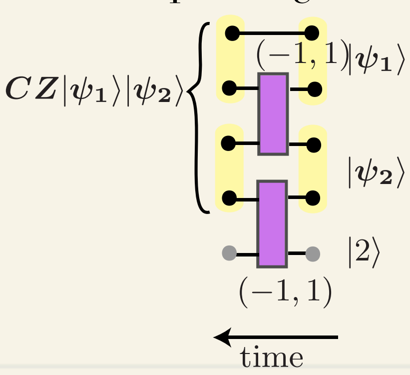
Phase gate



Two qubit identity



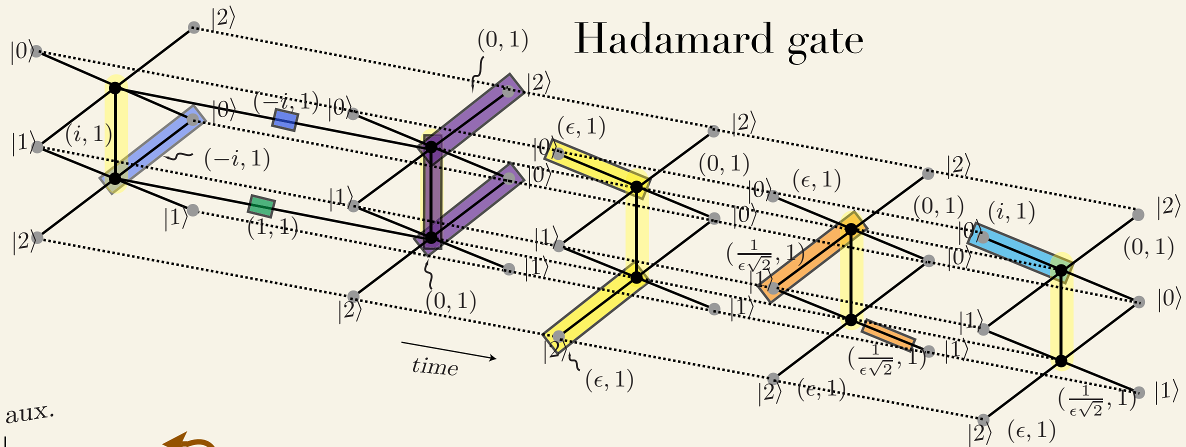
Controlled phase gate



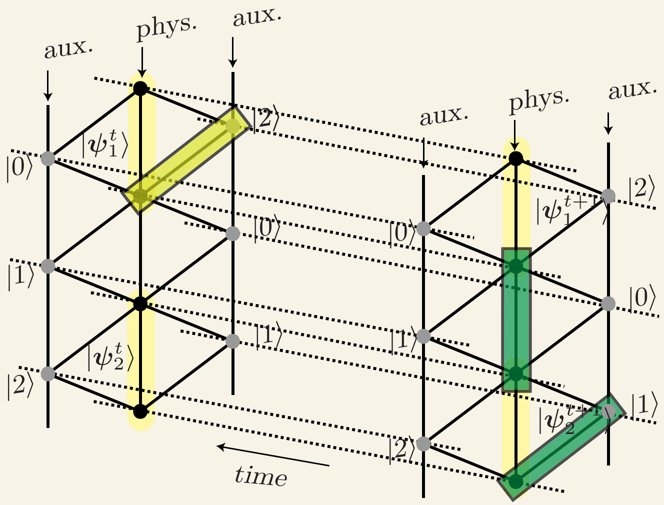
GDIC, M. Van den Nest, W. Dür, H. J. Briegel, M.A. Martin-Delgado (in preparation)

Ⅱ BQP-completeness results

Hadamard gate

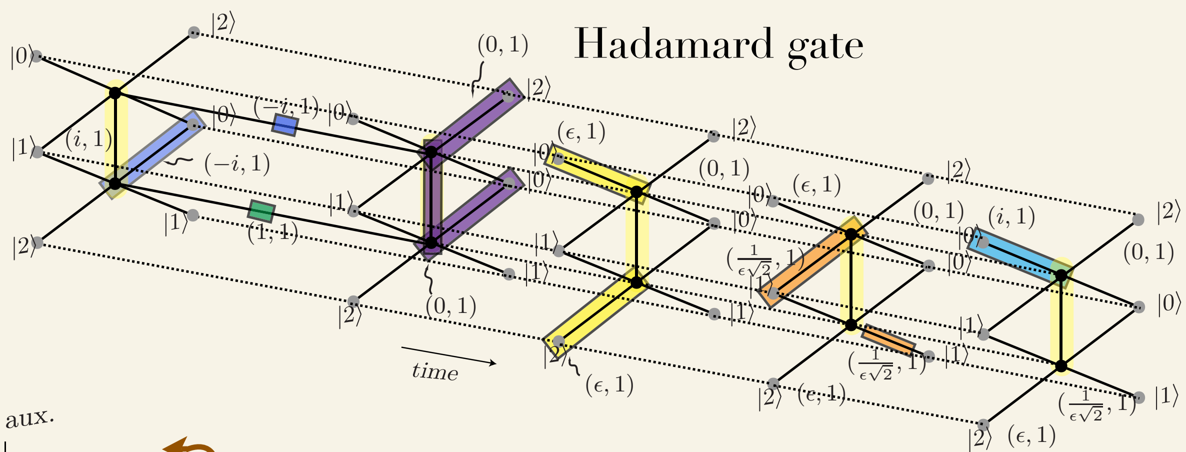


Example of part of a circuit:

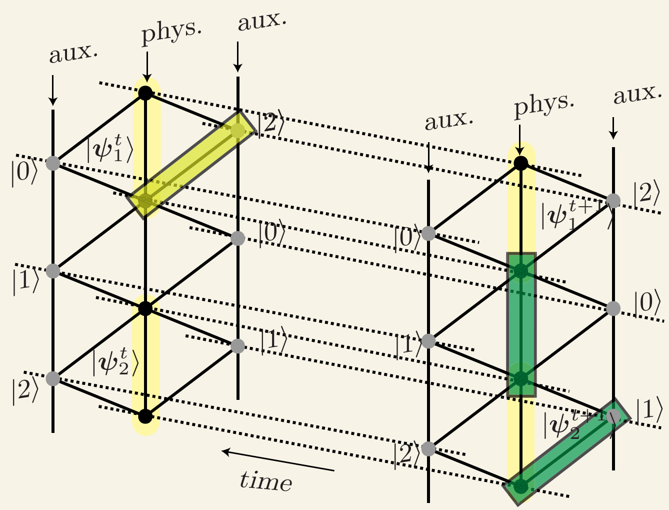


Note distribution of physical and auxiliary qubits

II BQP-completeness results



Example of part of a circuit:



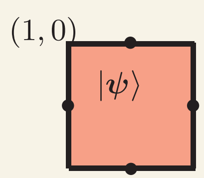
Note distribution of physical and auxiliary qubits

Approximating the partition function of a 2D 3-level Potts with auxiliary qubits on a certain complex parameter regime is BQP-complete

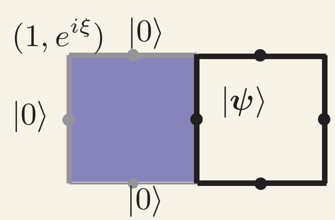
Ⅱ BQP-completeness results

- 3D $\mathbb{Z}_2$ LGT
  - Encoding: $|\mathbf{0}\rangle = |0\rangle|0\rangle|0\rangle|0\rangle$
    $|\mathbf{1}\rangle = |1\rangle|1\rangle|1\rangle|1\rangle$
  - Trivial preparation of $|\mathbf{0}\rangle \ldots |\mathbf{0}\rangle$ from $|R\rangle = |0\rangle \ldots |0\rangle$
  - Each $\mathbb{Z}_2$ LGT-type gate is characterized by the pair $(e^{\beta J_{ii}}, e^{\beta J_{i \neq j}})$
  - Construct an (encoded) universal gate set:

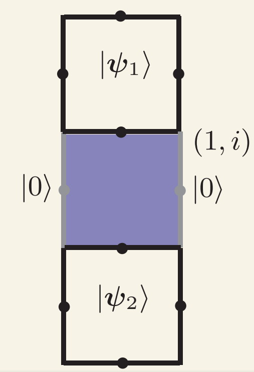
Single qubit identity



Z-Rotation



Controlled phase gate

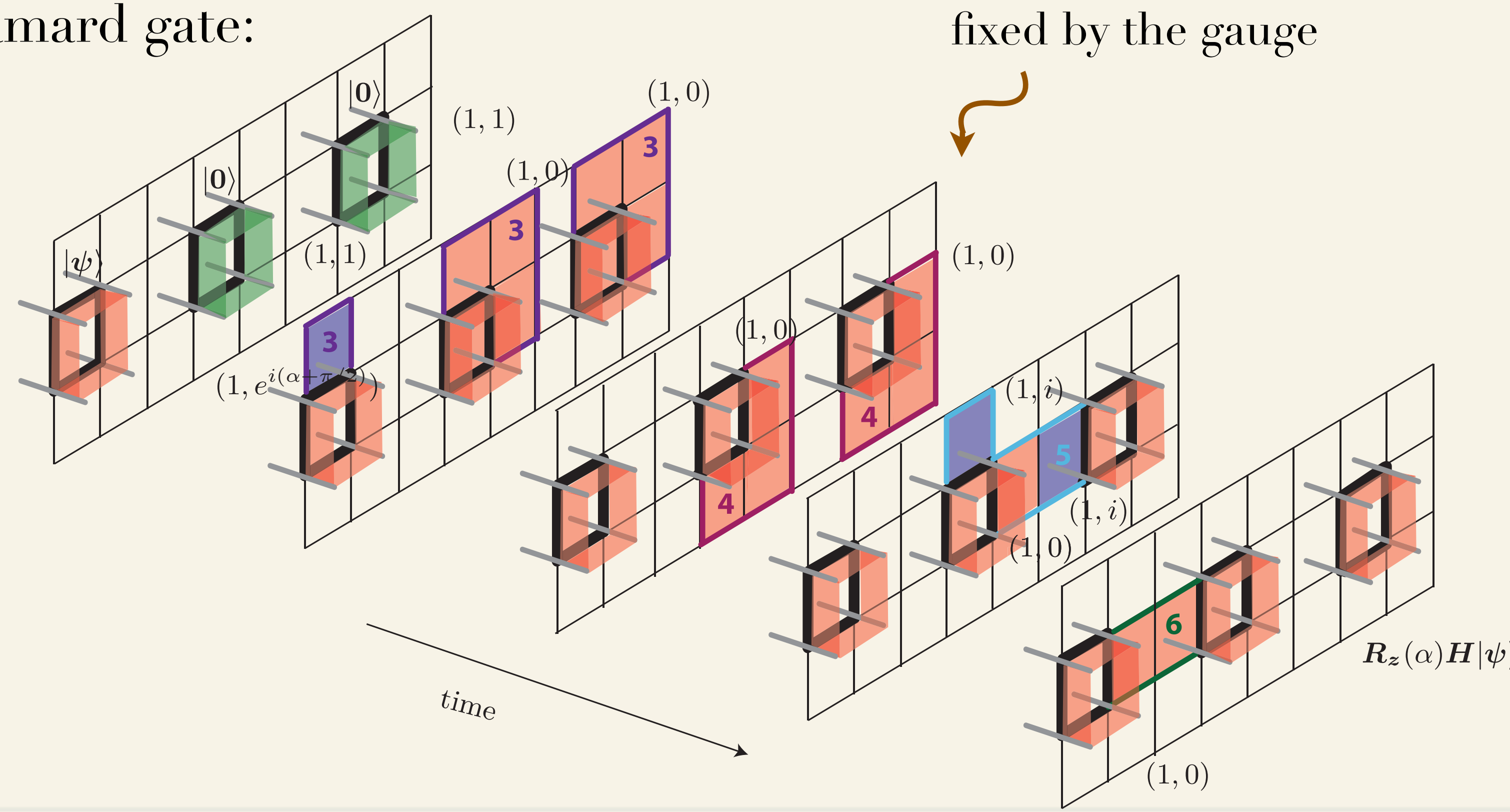


Ⅱ BQP-completeness results

- 3D $\mathbb{Z}_2$ LGT

Hadamard gate:

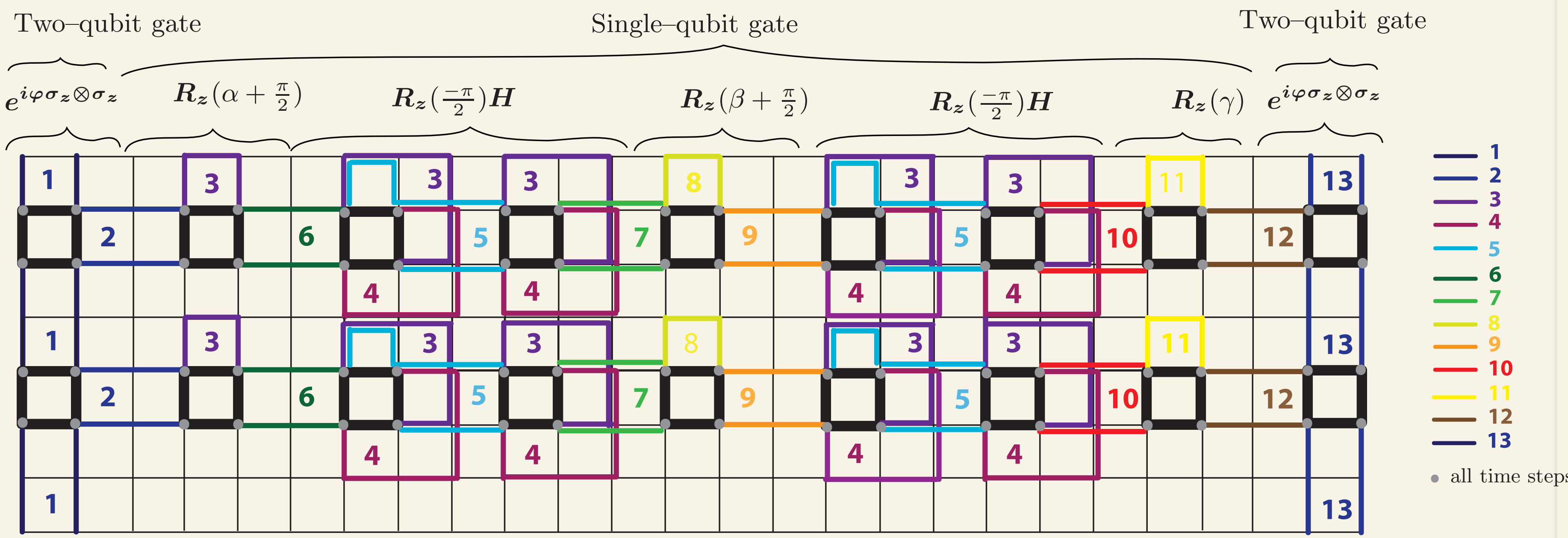
Note: many spins fixed by the gauge



Ⅱ BQP-completeness results

- 3D $\mathbb{Z}_2$ LGT

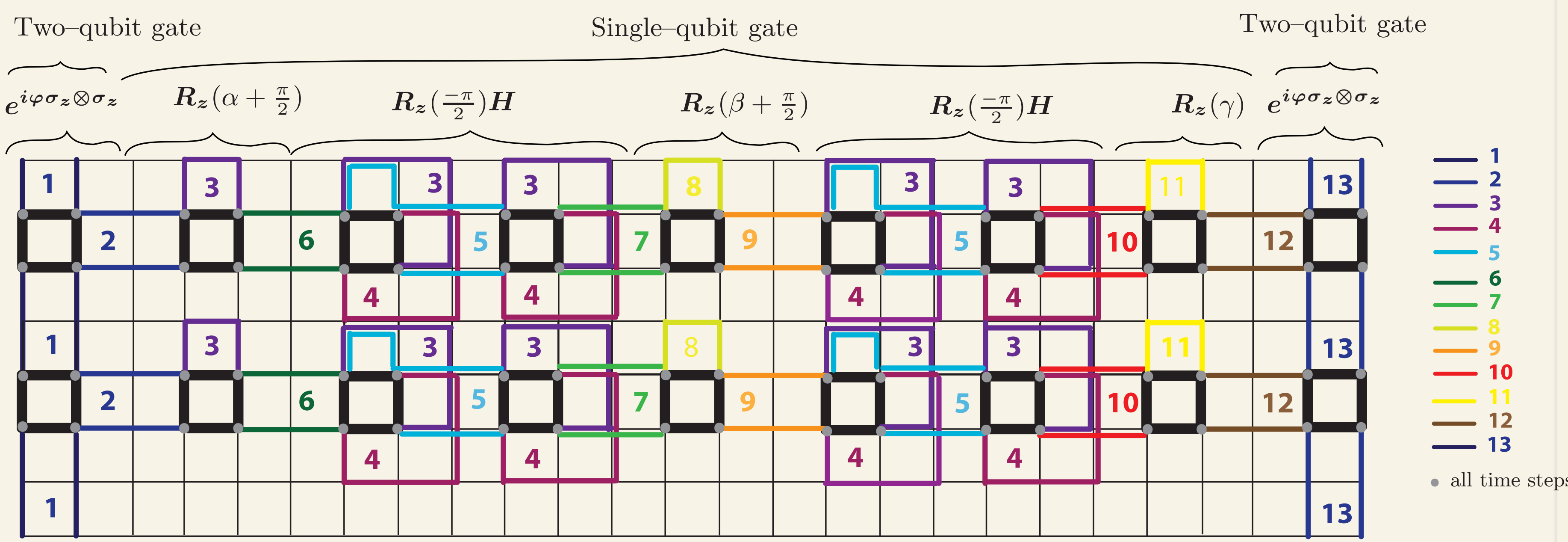
Verify that there are no loops of spins fixed by the gauge:



Ⅱ BQP-completeness results

3D $\mathbb{Z}_2$ LGT

Verify that there are no loops of spins fixed by the gauge:



Approximating the partition function of a 3D $\mathbb{Z}_2$ LGT on a certain complex parameter regime is BQP-complete

Summary

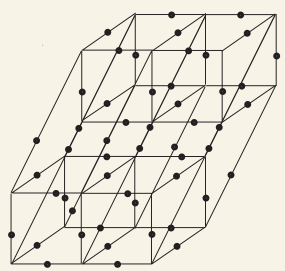
Summary

- Completeness

$Z = \langle \alpha | \psi \rangle$

$$Z_{4\mathrm{D}\mathbb{Z}_2\mathrm{LGT}}(J, J') \quad = \quad Z_{\text{any classical spin model}}(J)$$

real



Abelian discrete

✓ any dimensions

✓ q-level systems, any q

✓ any many-body int.

- Complexity

$Z = \langle L | \mathcal{C} | R \rangle$

Approximating Z of Six vertex model is BQP-complete

Potts model

3D $\mathbb{Z}_2$ LGT

in a certain complex parameter regime

Thank you for your attention!

Completeness

- M. Van den Nest, W. Dür, H. J. Briegel, *Phys. Rev. Lett.* **100**, 110501(2008)
- GDlC, W. Dür, M. Van den Nest, H. J. Briegel, *JSTAT* P07001 (2009)
- ** GDlC, W. Dür, H. J. Briegel, M. A. Martin-Delgado, *Phys. Rev. Lett.* **102**, 230502 (2009)
- ** GDlC, W. Dür, H. J. Briegel, M. A. Martin-Delgado, *New J. Phys.* **12**, 043014 (2010)
- Y. Xu, GDlC, W. Dür, H. J. Briegel, M.A. Martin-Delgado, *Completeness of a U(1) lattice gauge theory* (in preparation)

Complexity

- ** M. Van den Nest, W. Dür, R. Raussendorf, H. J. Briegel, *Phys. Rev. A* **80**, 052334 (2009)
- ** GDlC, M. Van den Nest, W. Dür, H. J. Briegel, M.A. Martin-Delgado, *Computational complexity of classical lattice models* (in preparation)