Fast Decoders for Topological Quantum Codes

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University of British Columbia, Vancouver, Ca

UNIVERSITÉ DE SHERBROOKE

MITACS

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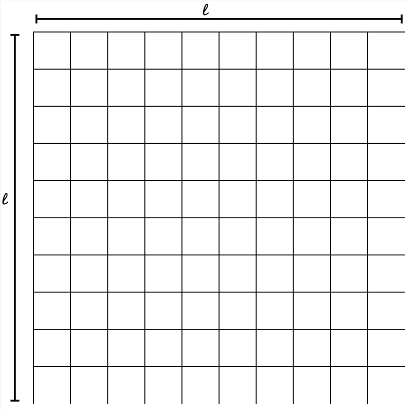
- Kitaev's toric code → useful toy model
- Quantum error-correction (QEC) → fast decoding algorithms

1. Kitaev's Toric Code
   - Stabilizer generators
   - Logical Operators
   - Topology
2. Concatenation
3. Topological Codes Decoding

Stabilizer Generators

Lattice



- 2D square lattice
- Periodic boundary conditions

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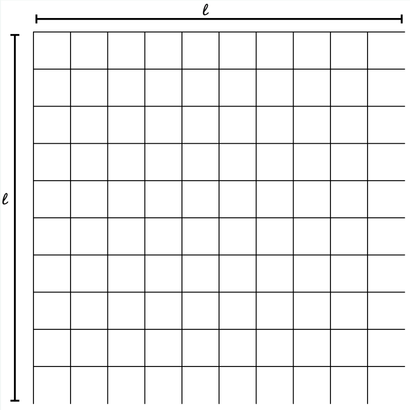

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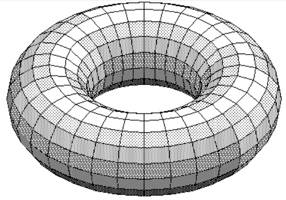
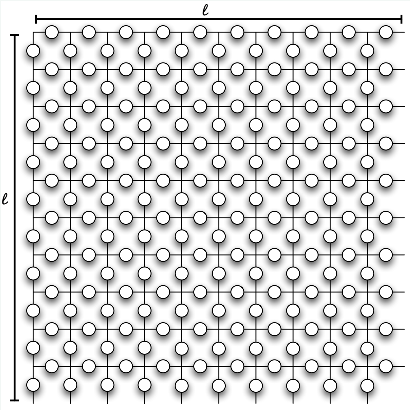


Lattice + Qubits



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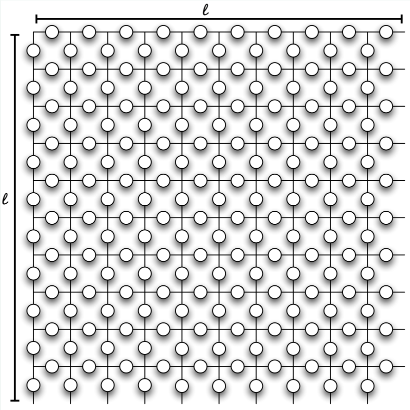

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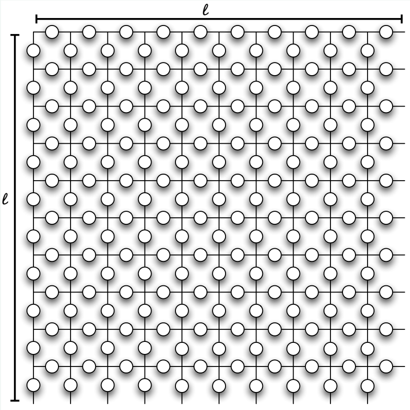

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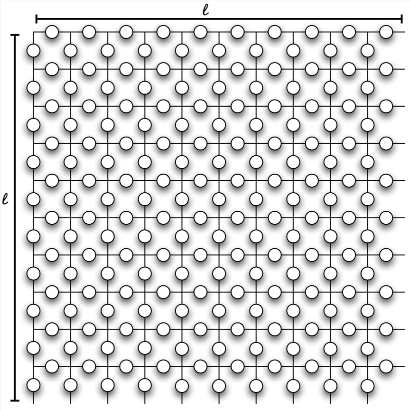
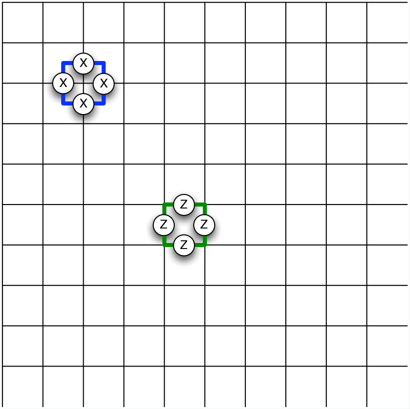


- 2D square lattice
- Periodic boundary conditions
- A qubit per *edge*
- $\Rightarrow 2\ell^2$ qubits

Stabilizer Generators



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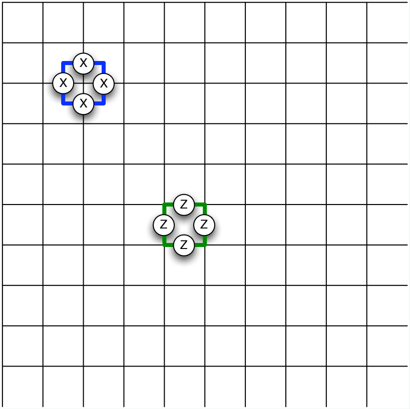

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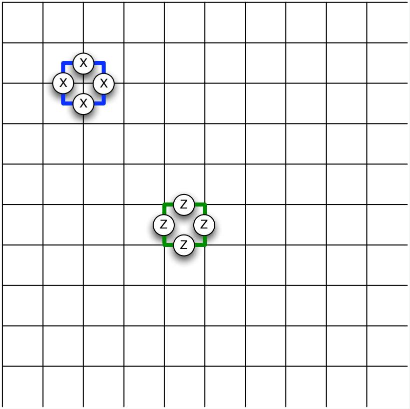

- Site (vertex) operator : $A_s = \prod_{i \in v(s)} X_i$
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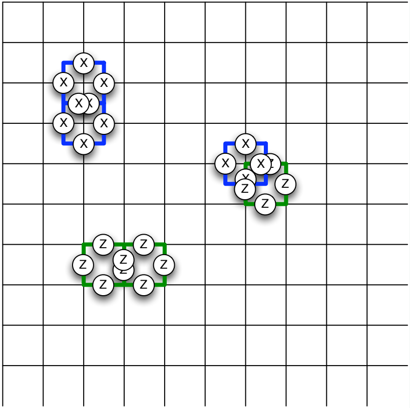
- ℓ^2 site and plaquette operators

Stabilizer Generators



- $[A_s, A_{s'}] = [B_p, B_{p'}] = 0$

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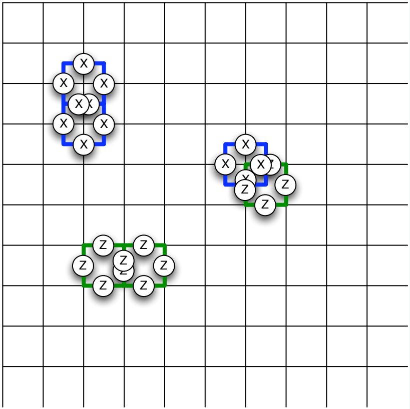

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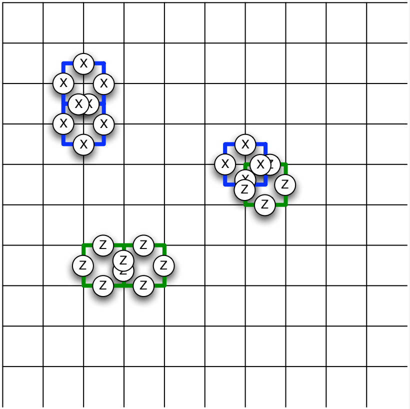
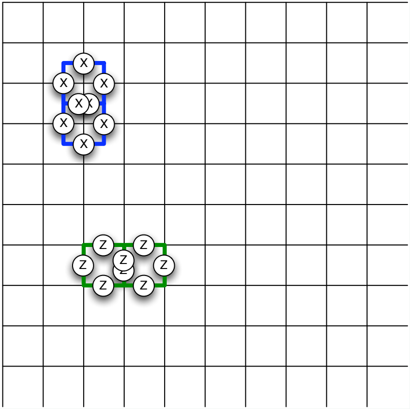


- $[A_s, A_{s'}] = [B_p, B_{p'}] = 0$
- $[A_s, B_p] = 0$
- The code is spanned by the simultaneous $+1$ eigenstates of all these

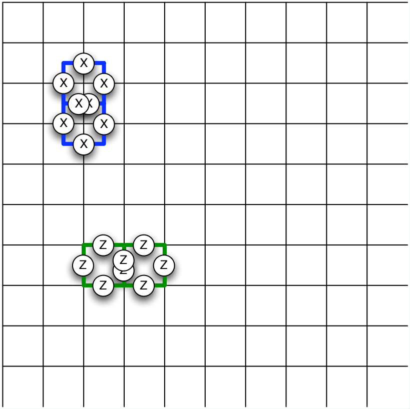
$$\mathcal{C} = \{|\psi\rangle : A_s |\psi\rangle = |\psi\rangle, B_p |\psi\rangle = |\psi\rangle \ (\forall s, p)\}$$

Stabilizer Generators



- $\prod_s A_s = I$

Stabilizer Generators



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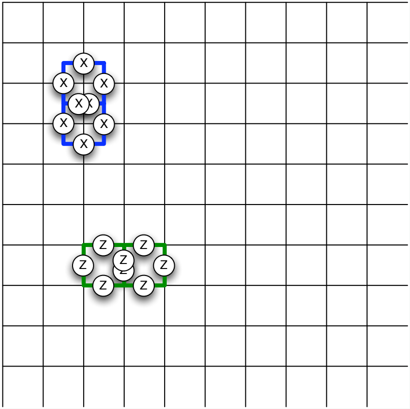

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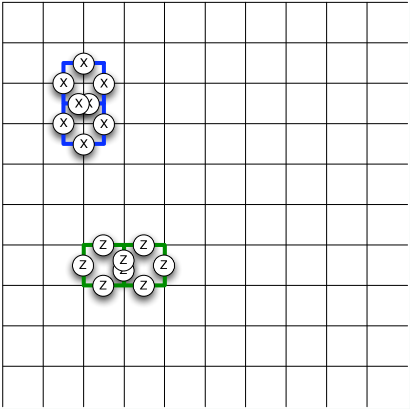

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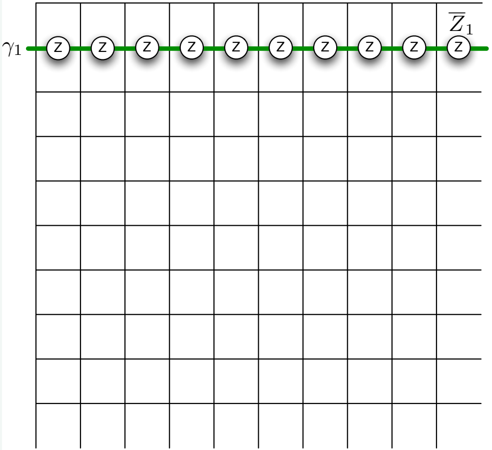
- $\Rightarrow 2\ell^2 - 2$ independent generators
- $\Rightarrow 2$ logical qubits

Logical Operators

First Logical Qubit

- $\overline{Z}_1 = \prod_{i \in \gamma_1} Z_i$



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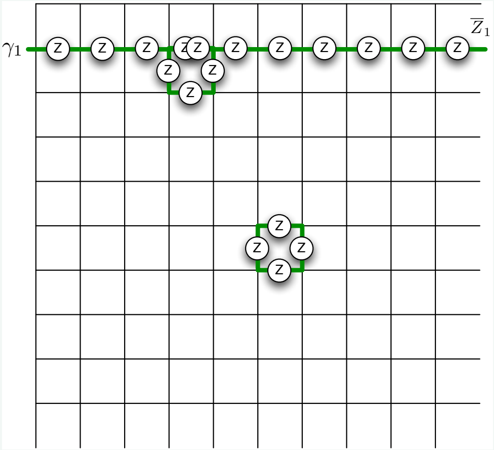

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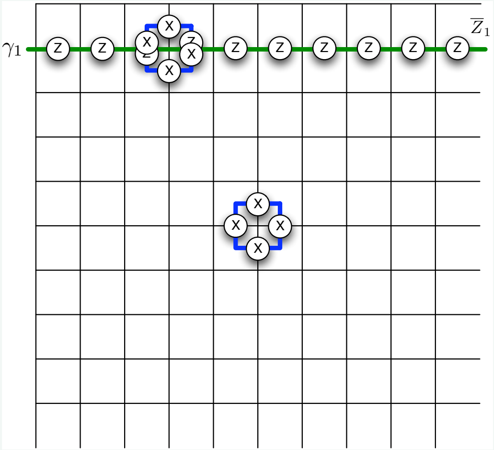

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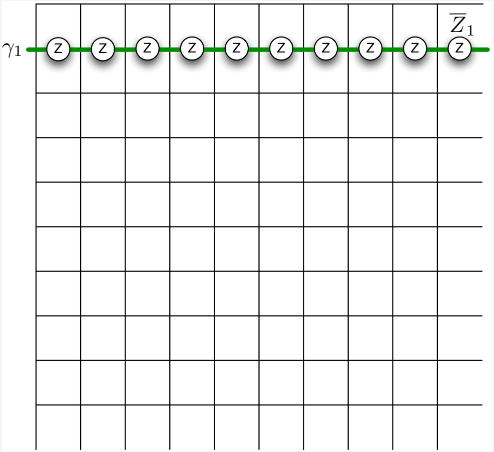

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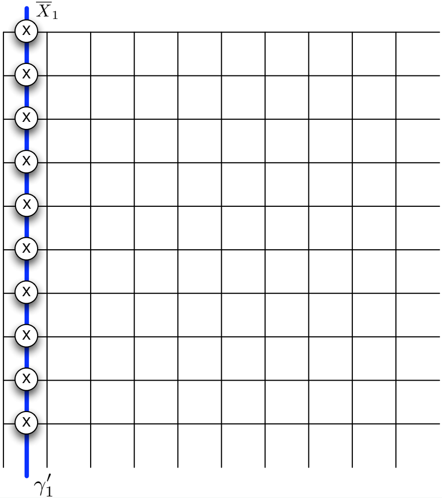

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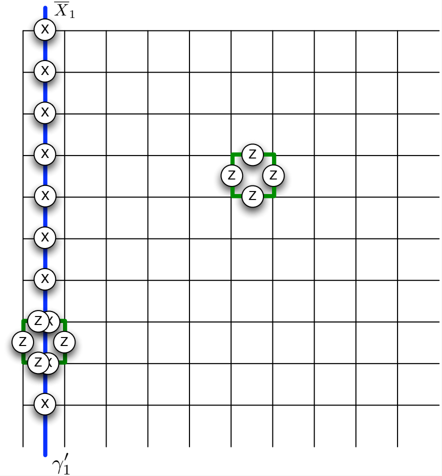

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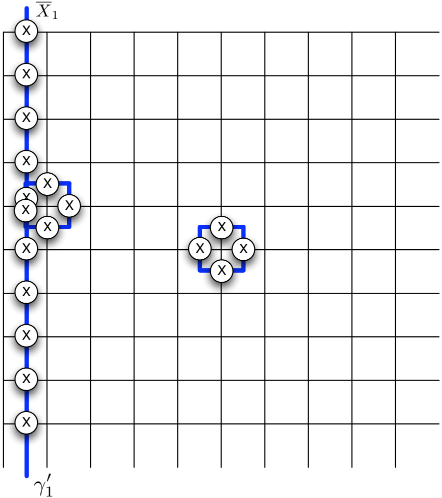

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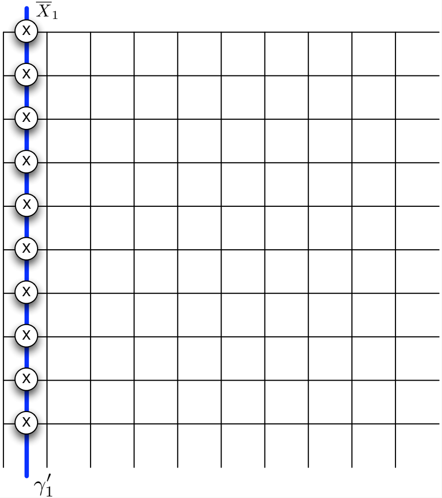

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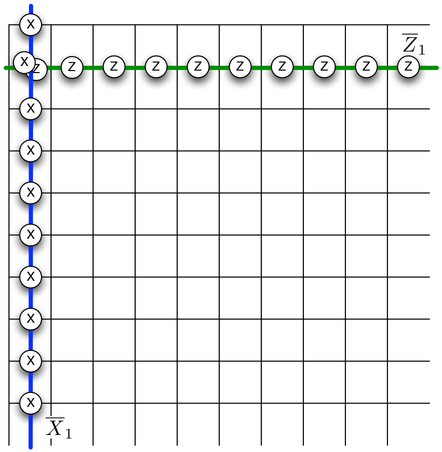
- $\{\overline{X}_1, \overline{Z}_1\} = 0$



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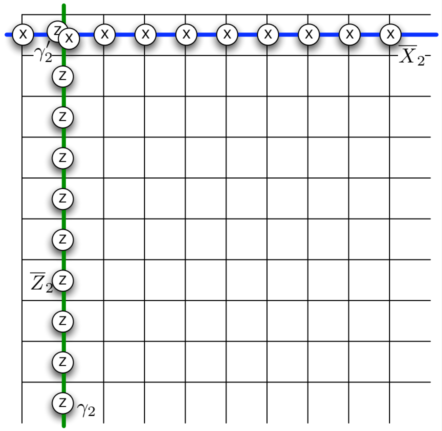

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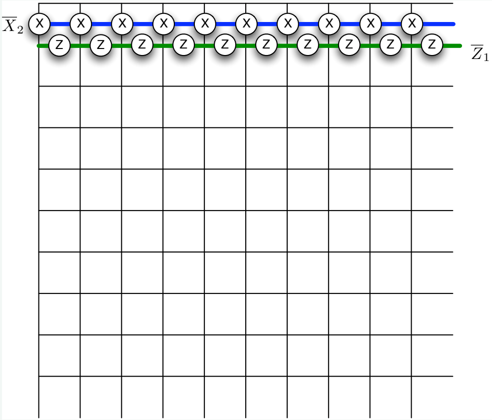

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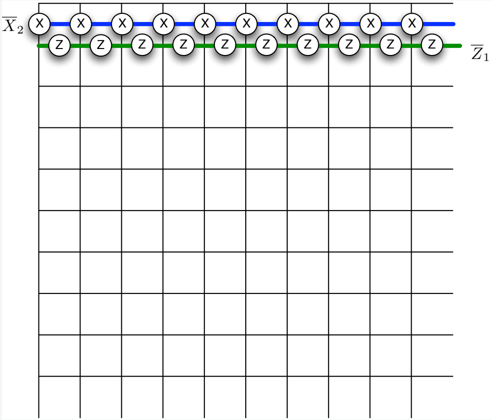

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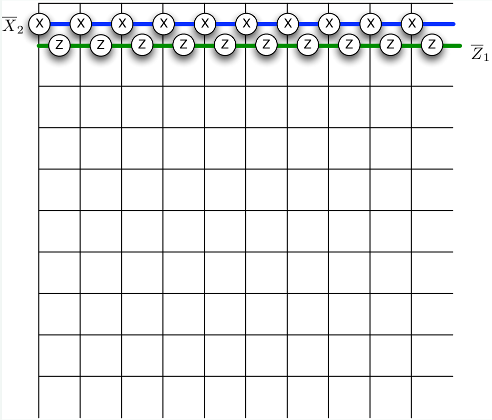
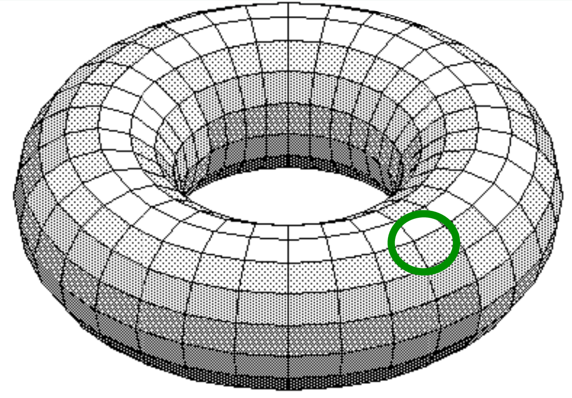


New Basis

$$
\begin{array}{cccccccc}
A_1, & \ldots, & A_{n/2-1}, & B_1, & \ldots, & B_{n/2-1}, & \bar{Z}_1, & \bar{Z}_2 \\
t_{A_1}, & \ldots, & t_{A_{n/2-1}}, & t_{B_1}, & \ldots, & t_{B_{n/2-1}}, & \bar{X}_1, & \bar{X}_2
\end{array}
$$

Topology ?

Trivial Cycles



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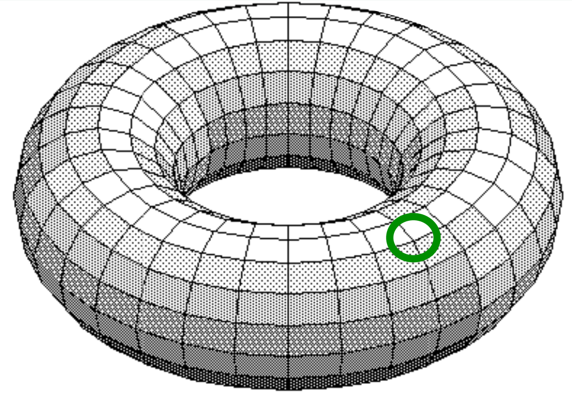

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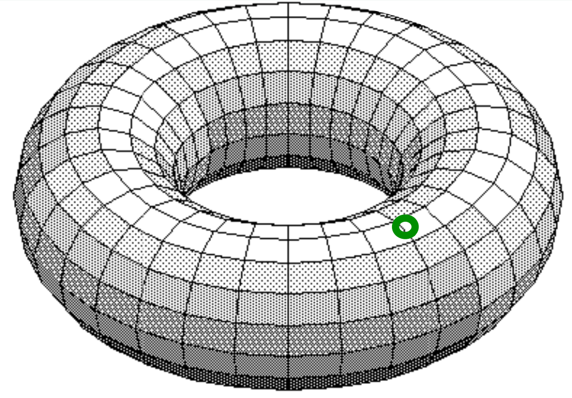

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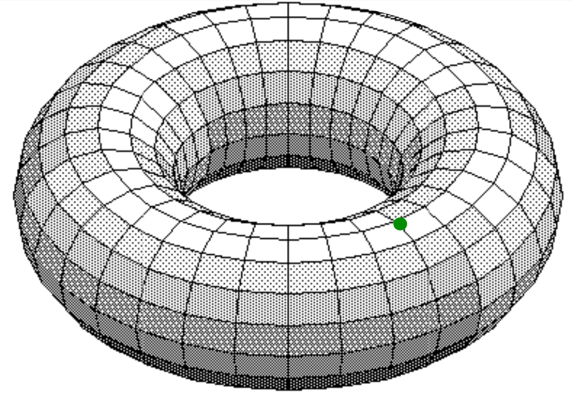
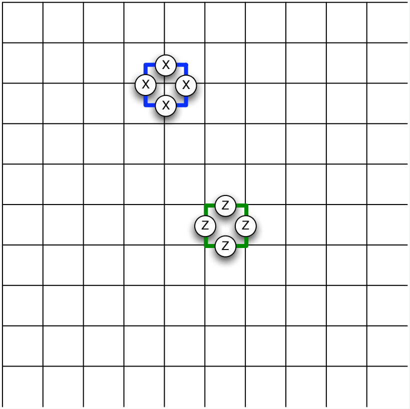


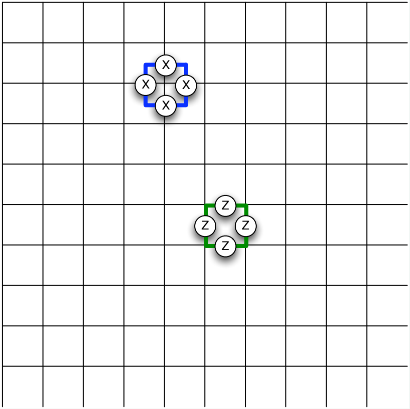
Trivial Cycles

- All A_s, B_p are trivial cycles



Trivial Cycles

- All A_s, B_p are trivial cycles
- They act as the identity on the code space : $A_s \left|\psi\right\rangle = B_p \left|\psi\right\rangle = +1 \left|\psi\right\rangle$



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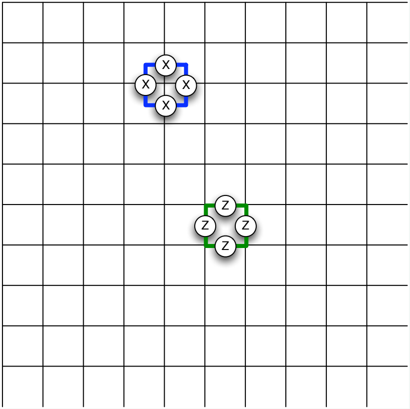
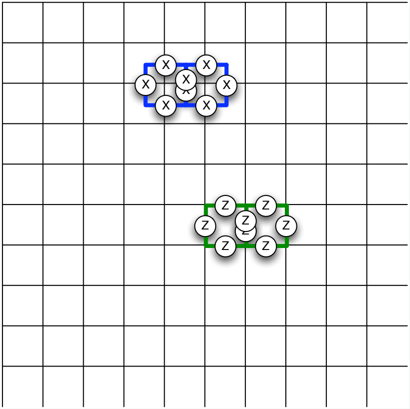
- All A_s, B_p are trivial cycles
- They act as the identity on the code space : $A_s\left|\psi\right\rangle = B_p\left|\psi\right\rangle = +1\left|\psi\right\rangle$
- Topologically and logically trivial



Trivial Cycles

- $\{A_s, B_p\}$ span the set of trivial cycles



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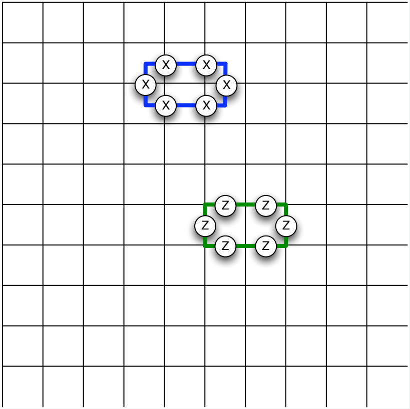
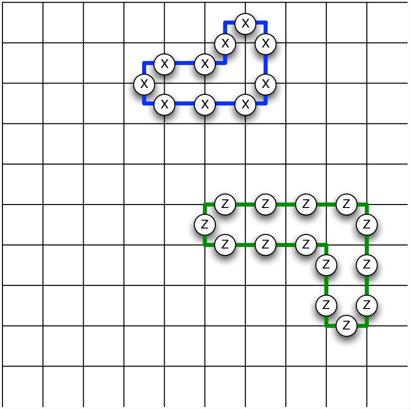
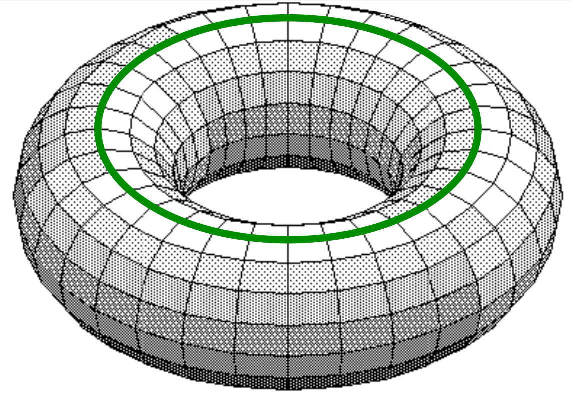


Trivial Cycles

- $\{A_s, B_p\}$ span the set of trivial cycles
- $\Rightarrow$ all trivial cycles are equivalent to the identity on the code space



Non-Trivial Cycles



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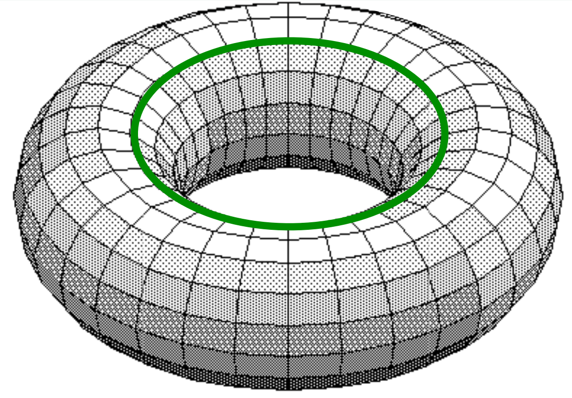

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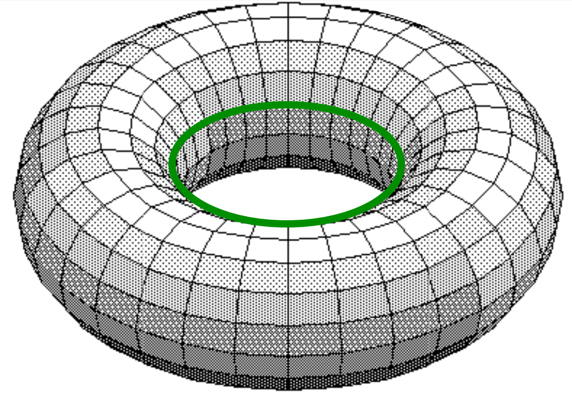

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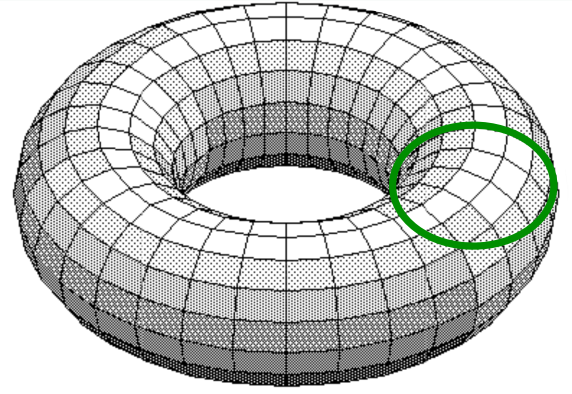
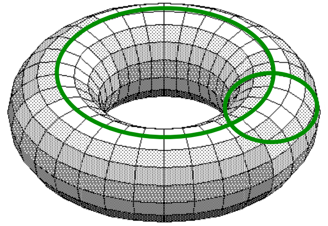
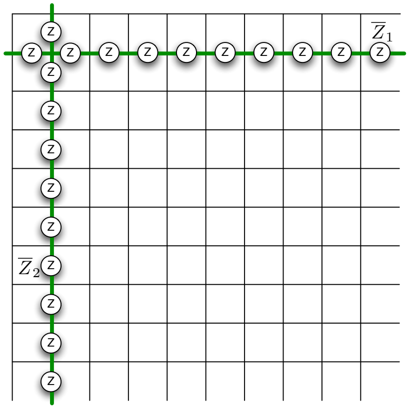


Non-Trivial Cycles

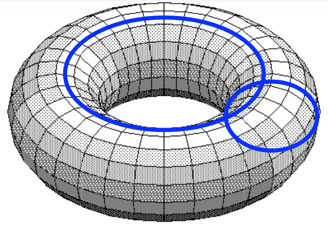
- $\overline{Z}_1$ and $\overline{Z}_2$ wind around the torus : non-trivial cycles
- They live on the lattice

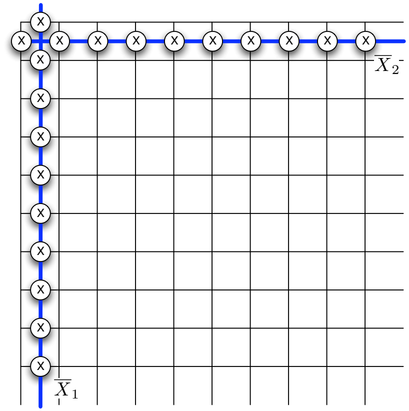




Non-Trivial Cycles

- $\overline{X}_1$ and $\overline{X}_2$ are conjugate to $\overline{Z}_1$ and $\overline{Z}_2$
- They live on the dual lattice





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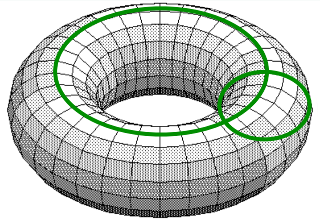
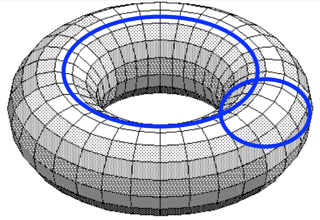
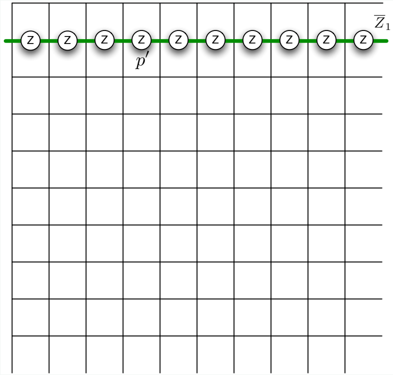




- Non-trivial cycles have non-trivial effects on the code space

Homological/Logical Classes

- $|\psi\rangle = B_{p'} |\psi\rangle$



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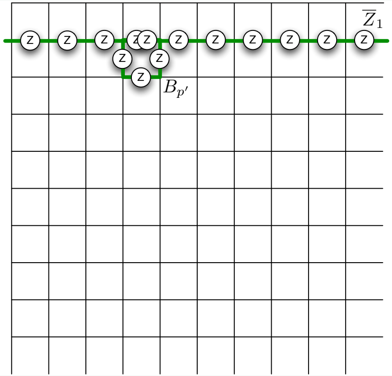
- $\overline{Z}_1 |\psi\rangle = \overline{Z}_1 B_{p'} |\psi\rangle$



Homological/Logical Classes

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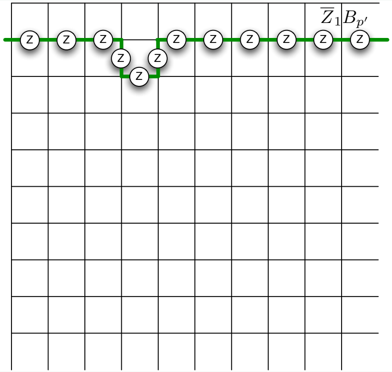
- $\overline{Z}_1 \equiv \overline{Z}_1 B_{p'}$



Homological/Logical Classes

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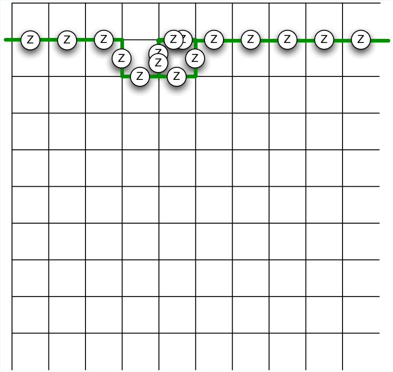

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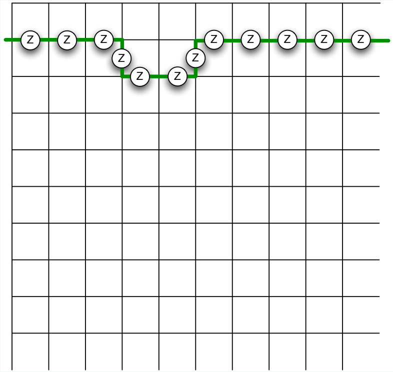

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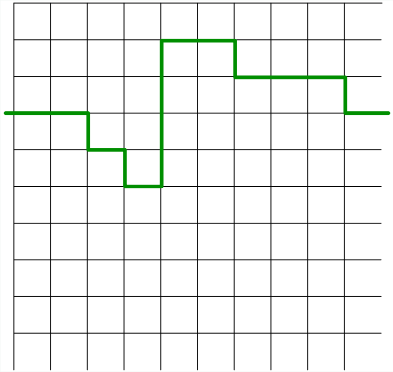


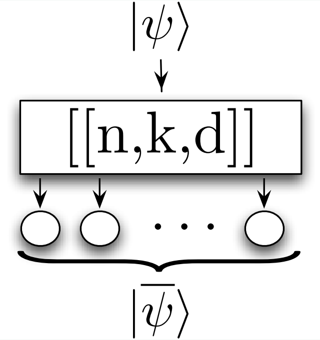
Summary

Stabilizer $\leftrightarrow$ Topology

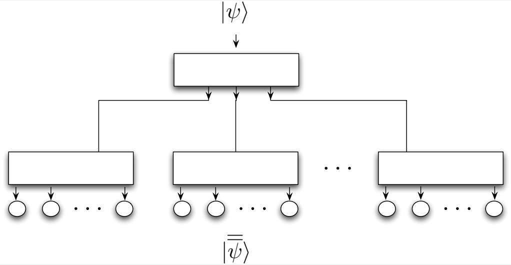
- Every element of the stabilizer is a trivial cycle and vice-versa
- Every logical operator is a non-trivial cycle and vice-versa
- $\Rightarrow$ Topological equivalence classes

Concatenated Codes

Codes



Concatenated Codes

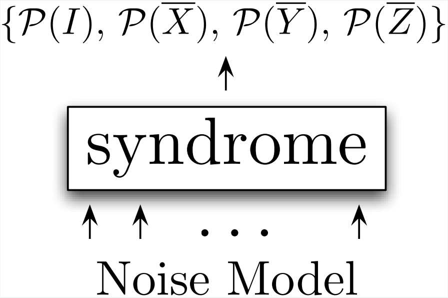


- k layers of encoding $\longrightarrow n^k$ qubits
- Error rate decays doubly exponentially : $k \sim \log \epsilon$

Efficient Optimal Decoder

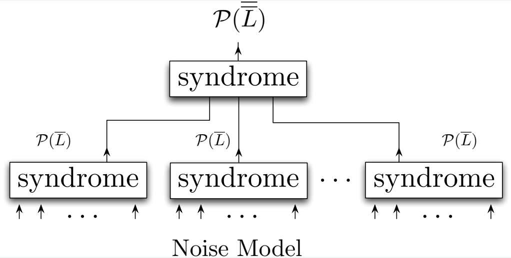
David Poulin, Phys. Rev. A 74, 052333 (2006)

Optimal (Soft) Decoder



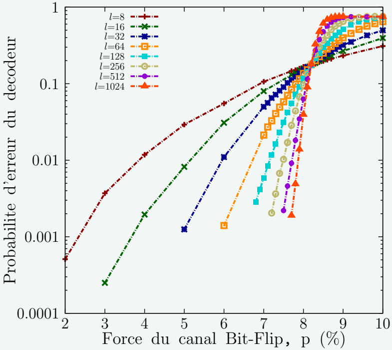
- Exponential in n, but n is constant
- Distillates error probability on the logical qubits

Recursive Decoder



- k layers with at most n^k codes
- Complexity : $\mathcal{O}(n^k k)$

Threshold



- The threshold is the noise strength under which it is useful to encode

Previous Method

- PMA : perfect matching algorithm (Preskill, Landahl et al.)

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Our solution (Phys. Rev. Lett. 104, 050504 (2010))

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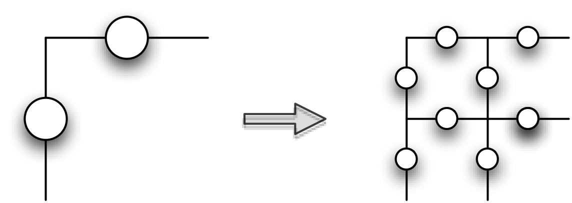
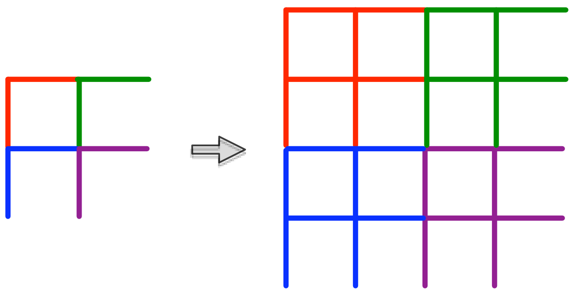
- More resiliant to noise : threshold of $\sim 16.5\%$ under depolarizing noise
- Not limited to toric code (e.g. color codes : triplet of defects)

Subcode



- Imagine we had a surface encoding taking 2 qubits into 8

Toric Code : A concatenation ?



- We could recurse on this encoding the build a bigger surface code

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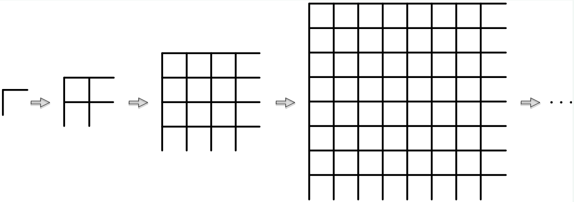

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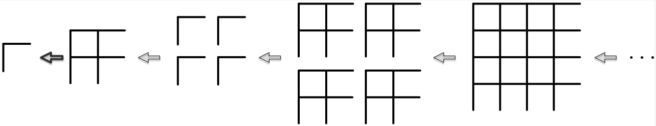
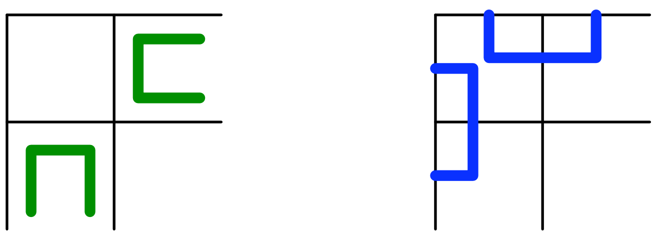
Decoding



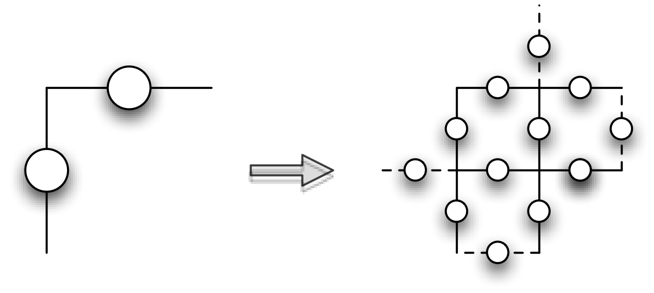
- If the toric code is just a concatenated code, then we know how to decode it efficiently !

Incomplete Stabilizers



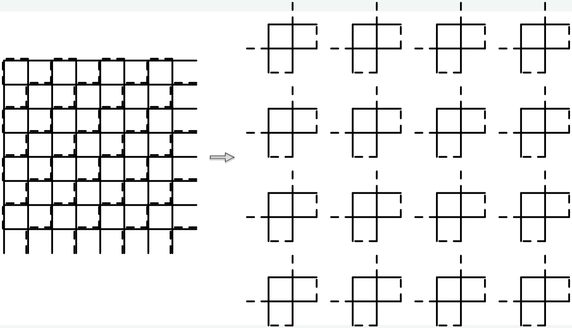
- Some of the stabilizers are incomplete

Closing the Stabilizers



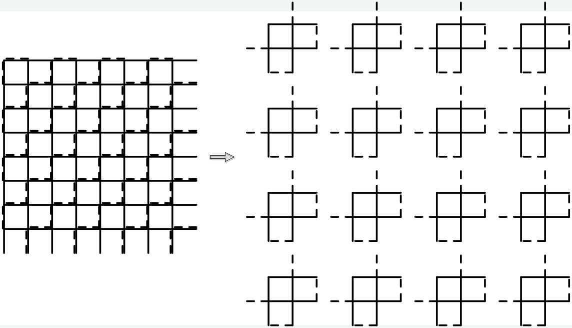
- We complete the stabilizer by adding qubits to the subcode

Closing the Stabilizers



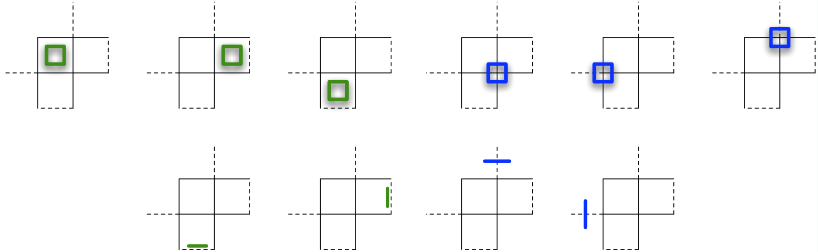
- By adding these qubits the construction is no more a concatenation

Concatenated code decoder



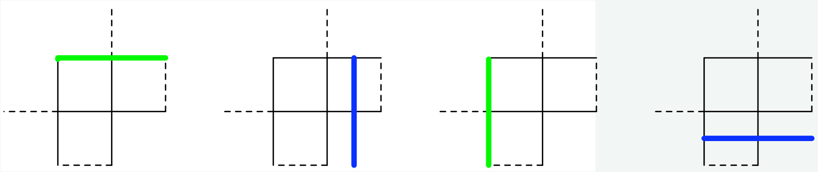
- Even though shared qubits correspond to the same physical entity, we are going to treat them as two diffrent qubits with the same noise model
- Main approximation : Decode with the concatenated code decoder anyway

Characterizing the subcode



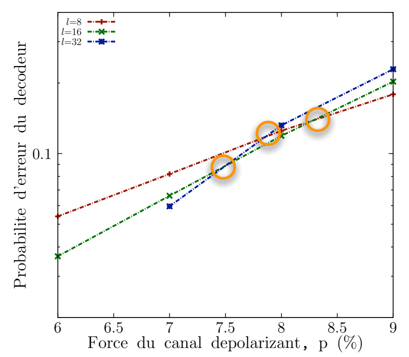
- SubCode stabilizer generators : 10

Characterizing the subcode



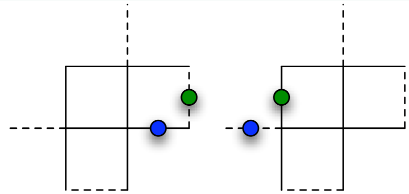
- SubCode stabilizer generators : 10
- $\Rightarrow$ 2 logical qubits

Results



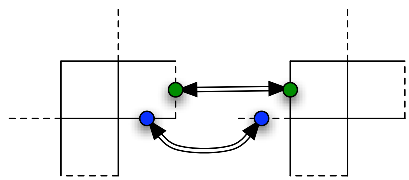
- Is there a threshold at all ? At best, these are size effects

Inconsistency



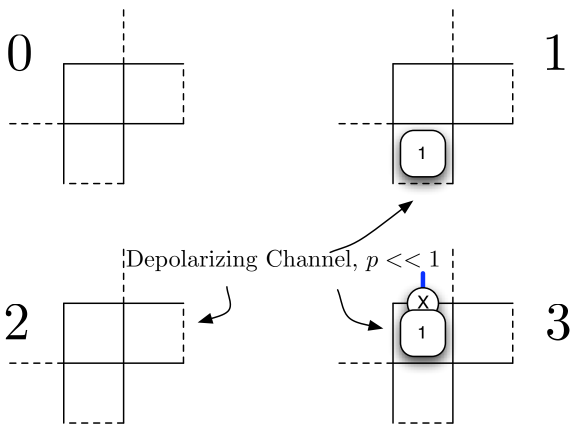
- By treating shared qubits as independent ones, we introduce inconsistencies
- A compromise between this and exact decoding would be to enforce consistency

Generalized Belief Propagation (Jonathan S. Yedidia)



- Self-consistency constraints on shared qubits
- Neighboring unit cells exchange messages
  $\longrightarrow$ Belief propagation
- Compromise on shared qubits

Intuition about GBP



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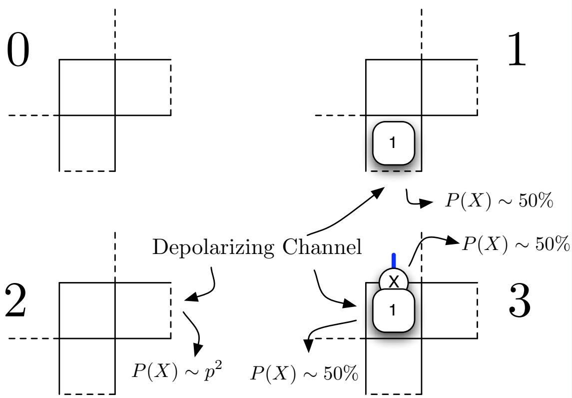

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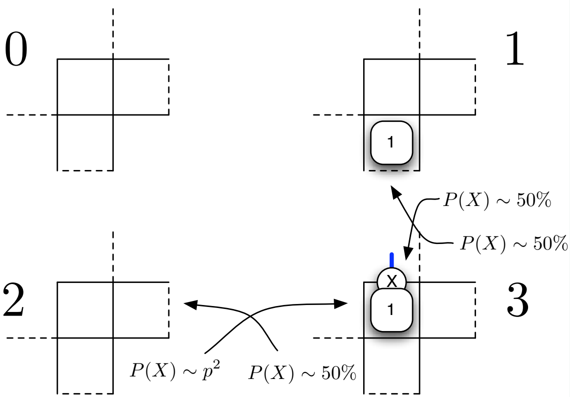

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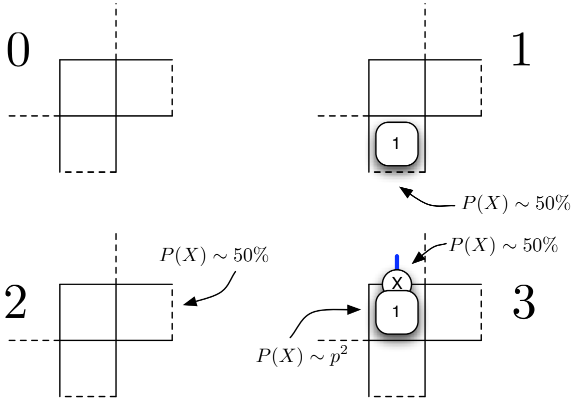

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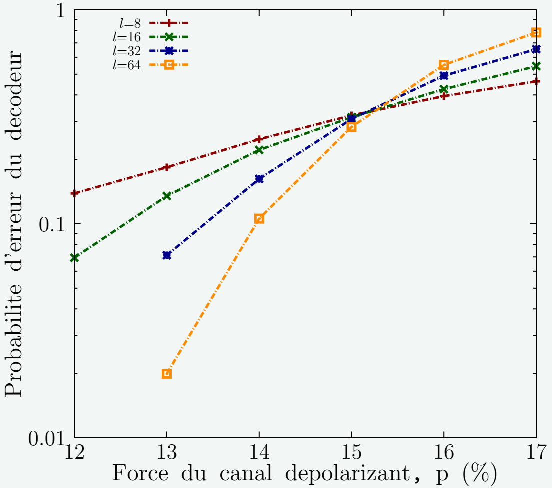
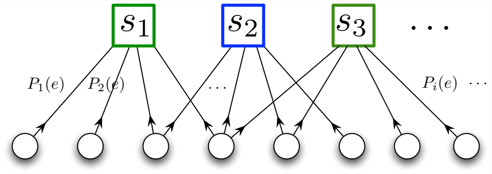

1
0.1
0.01
12
13
14
15
16
17
l=8
l=16
l=32
l=64
Probabilite d'erreur du decodeur
Force du canal depolarizant, p (%)

Preliminary Physical Decoding



- BP on the bare stabilizers and qubits

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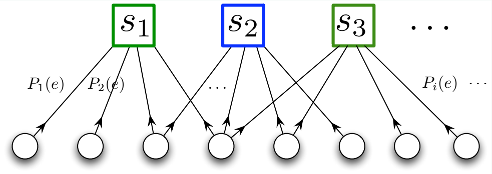

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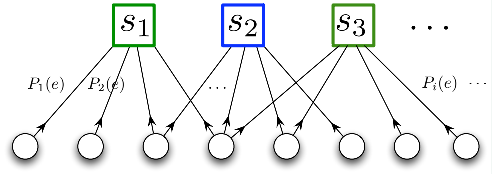
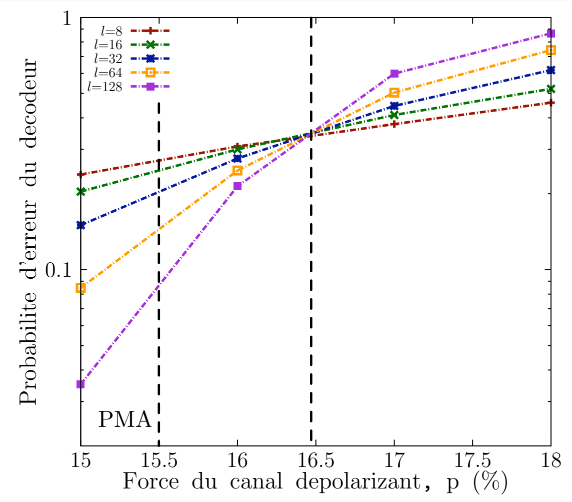
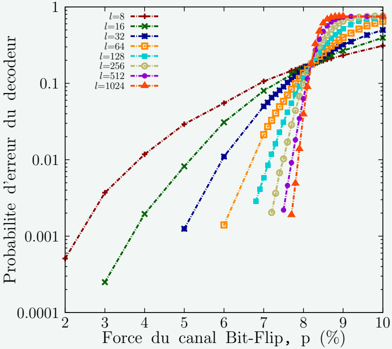


- BP on the bare stabilizers and qubits
- Accounts correlations between X and Z introduced by Y
- Its output is the input to the concatenated decoder

Results


1
0.1
l=8
l=16
l=32
l=64
l=128
PMA
15
15.5
16
16.5
17
17.5
18
Probabilite d'erreur du decodeur
Force du canal depolarizant, p (%)

Results



- Unit cell (2×1) + decoding the 2 types of defects independently $\Rightarrow \ell = 1024$ lattice

Conclusion

- Topological codes use highly non-local operators to encode information
- We proposed an efficient ($\mathcal{O}(\log \ell)$ time) to decode them $\rightarrow$ Concatenated codes, GBP
- More resiliant to noise under depolarizing noise than known methods (16.5% vs. 15.5%)
- It enabled decoding of color codes $p_{th} \sim 8.7\%$ (Héctor Bombin)

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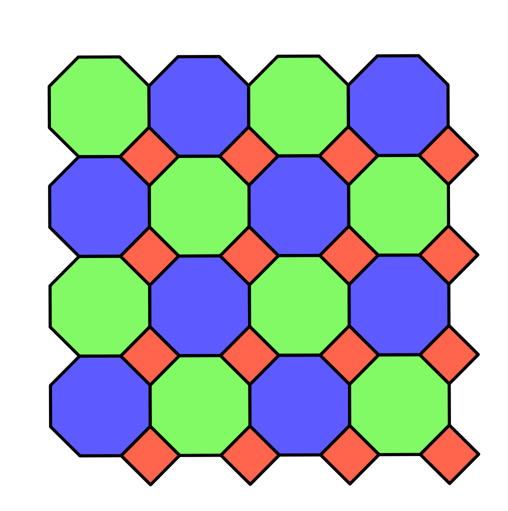
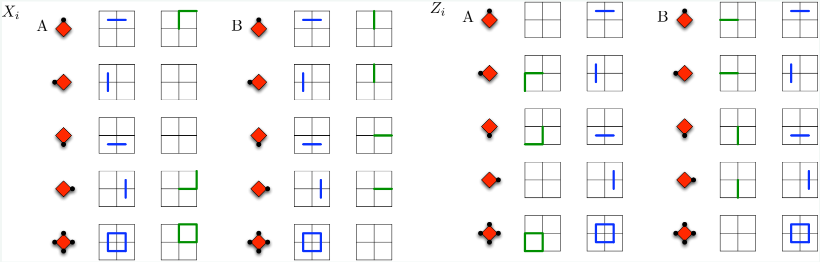
Work in progress : Color Codes



Color Codes : Mapping



Color Codes : Results