Accurate and decoherence-protected adiabatic quantum computation

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PRL **100**, 160506 (2008)
JMP **50**, 102106 (2009)

Workshop on Quantum Algorithms, Computational Models, and Foundations of Quantum Mechanics

Vancouver, July 23, 2010

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Motivation

- Protect Adiabatic Quantum Computation (AQC) against decoherence and control errors

- Find rigorous bounds on fidelity of AQC in presence of coupling to environment

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- Protect Adiabatic Quantum Computation (AQC) against decoherence and control errors
- Find rigorous bounds on fidelity of AQC in presence of coupling to environment
- Main result of this talk:

Theorem: AQC can be performed with a fidelity approaching 1 as a power law in the system size, using only 2-local Hamiltonians, in the presence of 1-local noise, assuming access to (dynamical decoupling) pulses whose width and intervals shrink as a power law in the system size. The power law exponent is linear in the dynamical critical exponent of the closed system.

DAL, "Towards Fault Tolerant Adiabatic Quantum Computation", Phys. Rev. Lett. **100**, 160506 (2008)

In other words:

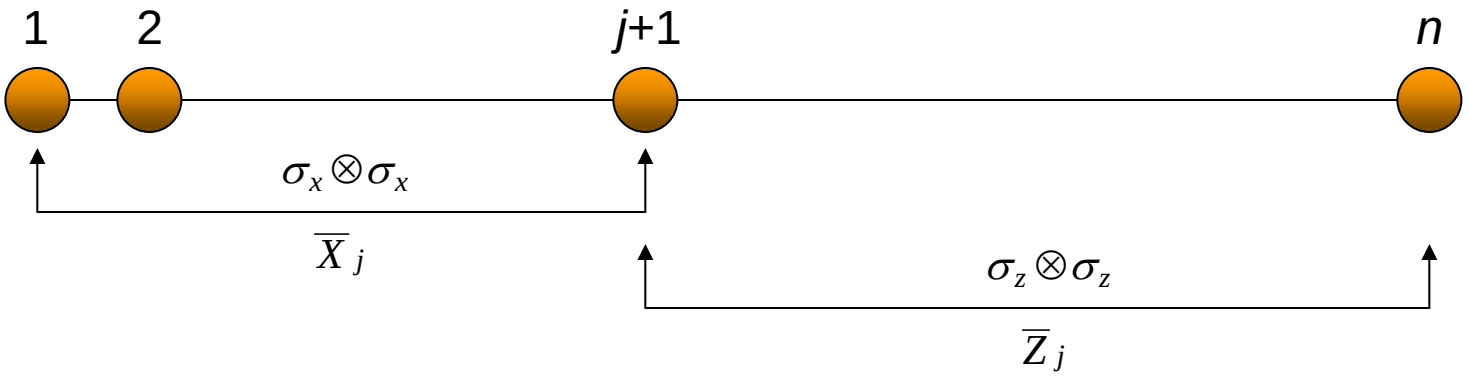
You can get arbitrarily accurate open-system AQC for the following price:

- Design your adiabatic evolution Hamiltonian (2-local) so that it is *analytic* and has N *zero derivatives at the initial and final times*.

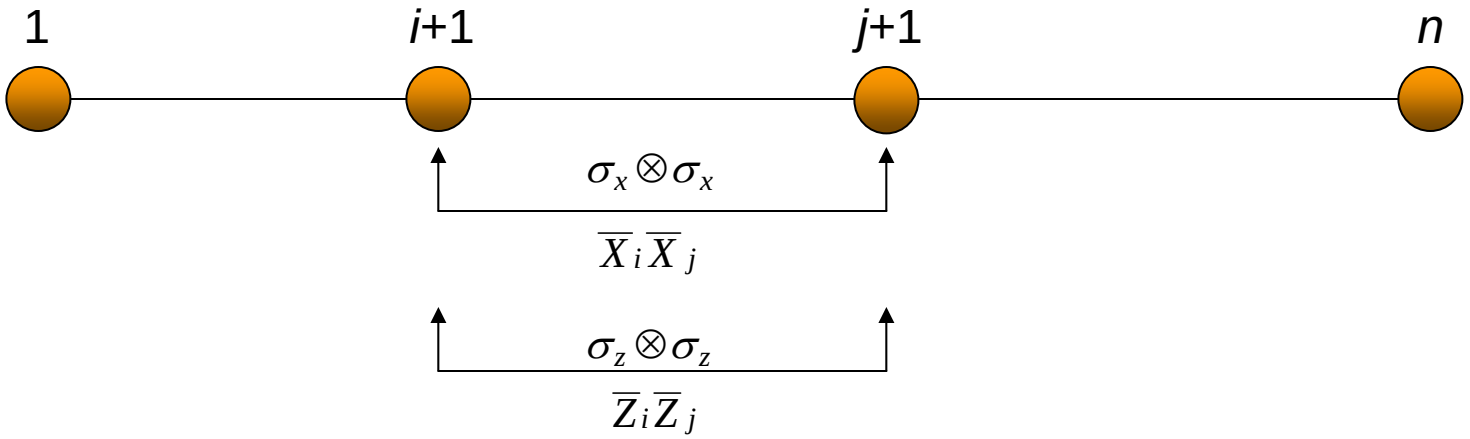
- Apply dynamical decoupling pulses generated by *global* magnetic fields in the x and z directions. The shorter the pulses and pulse intervals, the higher the fidelity. This result doesn't depend on bath temperature.

- To ensure that adiabatic evolution and dynamical decoupling are compatible, encode into a simple (distance-2) quantum error detection code.

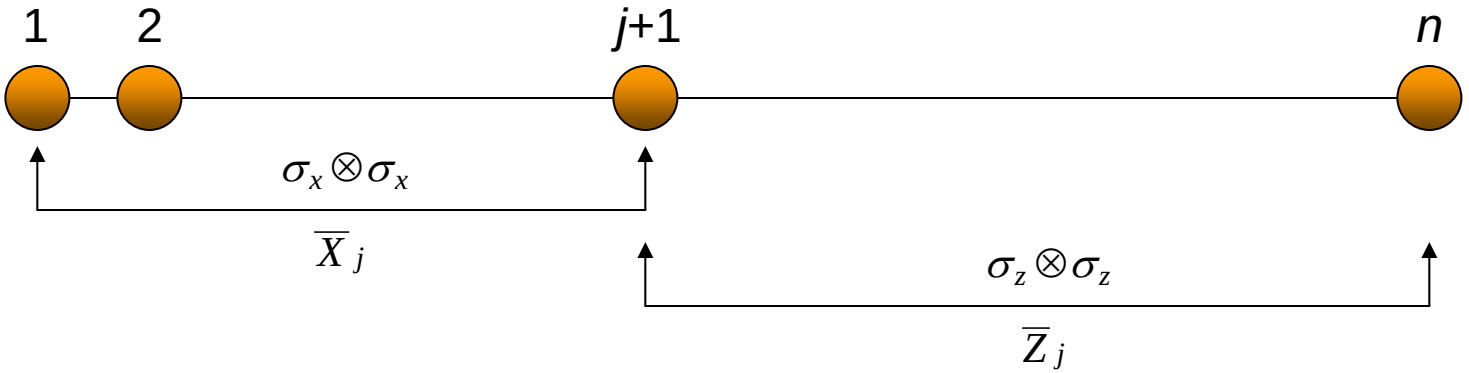
Scheme for dynamical-decoupling-protected **universal** adiabatic QC



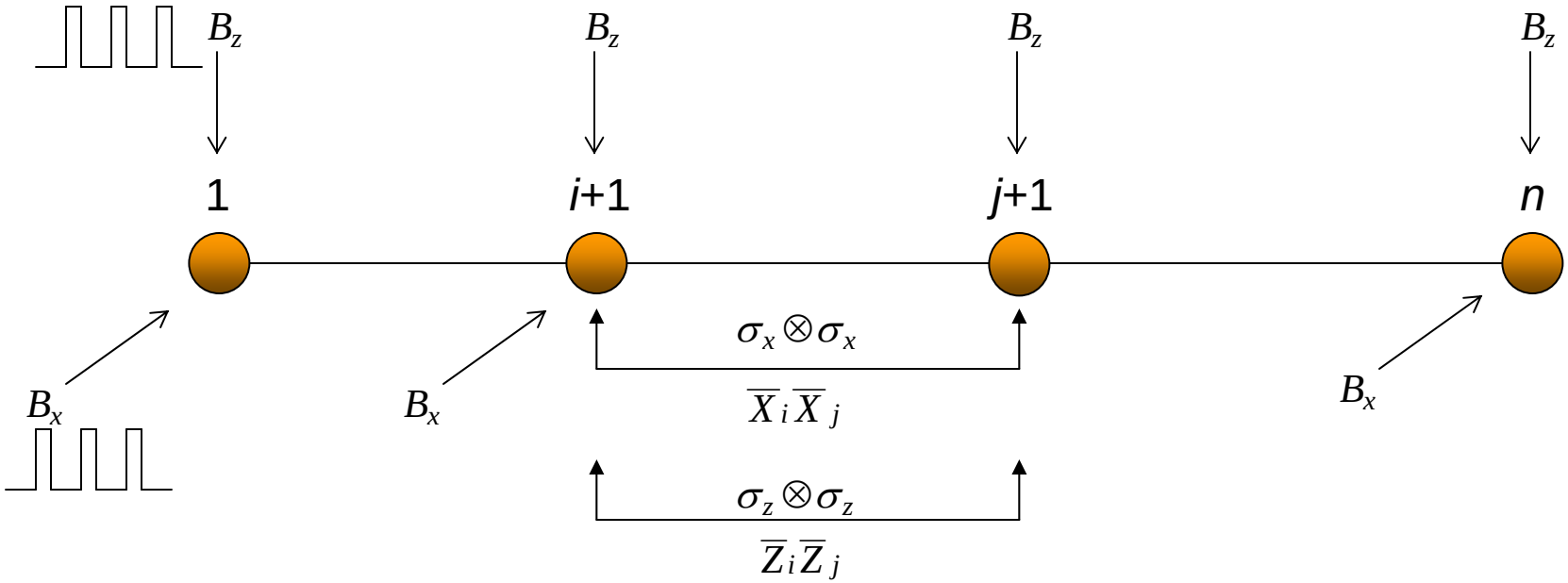
switch interactions adiabatically, locally



Scheme for **dynamical-decoupling-protected** universal adiabatic QC



switch fields rapidly but globally



Proof Strategy (and talk outline)

Let

$$\delta_S \equiv \text{distance between desired ground state and actual system state, at final time } T$$

Lemma 1: the *distance inequality*:

$$\delta_S \leq \underbrace{\text{“decoherence distance” due to } \textit{open} \text{ system non-unitary evolution}}_{d_{\mathsf{D}}} + \underbrace{\text{“adiabatic distance” due to } \textit{closed} \text{ system deviations from adiabaticity}}_{\delta_{\mathsf{ad}}}$$

and show that both these distances can be made arbitrarily small.

How?

- **Lemma 2** — adiabatic distance δ_{ad}: For slow enough evolution, can be made arbitrarily small using *analytic interpolation* and *almost-constant boundary conditions* (or using only C^2 interpolation provided evolving more slowly, for the same error)

- **Lemma 3** (with the help of **Lemmas 4 & 5**) — decoherence distance d_{D}: Can be made arbitrarily small using *dynamical decoupling* (with finite-width pulses and finite pulse intervals)

Tools: Distance Measure and Operator Norm

Distance between states: trace distance

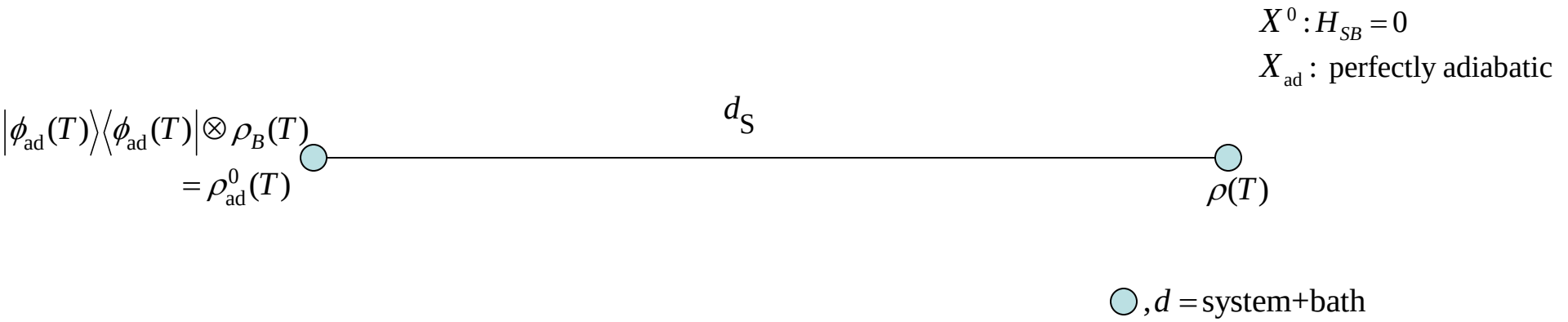
$$\begin{aligned} D[\rho_1, \rho_2] &\equiv \frac{1}{2}\|\rho_1 - \rho_2\|_1 \\ \|A\|_1 &\equiv \mathsf{Tr}|A| = \sum \text{sing.val.}(A), \\ |A| &\equiv \sqrt{A^\dagger A} \end{aligned}$$

When applied to pure states $\rho_i = |\psi_i\rangle\langle\psi_i|$ I'll write $D[\psi_1, \psi_2]$.

Operator norm

$$\begin{aligned} \|A\| &\equiv \sup_{\||\psi\rangle\|=1} \sqrt{\langle\psi|A^\dagger A|\psi\rangle} \\ &= \max \text{sing.val.}(A) \end{aligned}$$

Distances


$X^0 : H_{SB} = 0$
X_{ad} : perfectly adiabatic
d_{S}
$\left|\phi_{\mathrm{ad}}(T)\right\rangle\left\langle\phi_{\mathrm{ad}}(T)\right| \otimes \rho_B(T)$
$= \rho^0_{\mathrm{ad}}(T)$
$\rho(T)$
, d = system+bath

Distances

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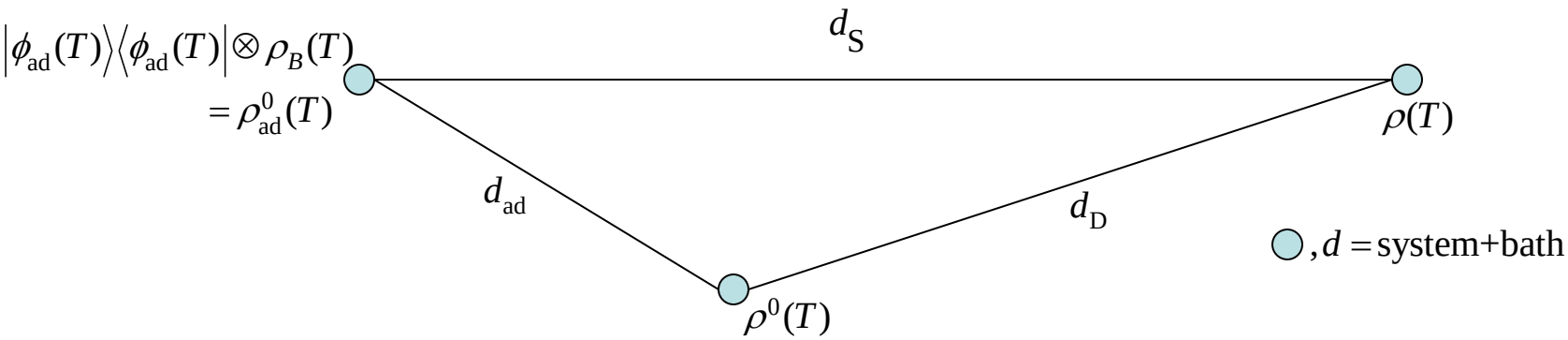

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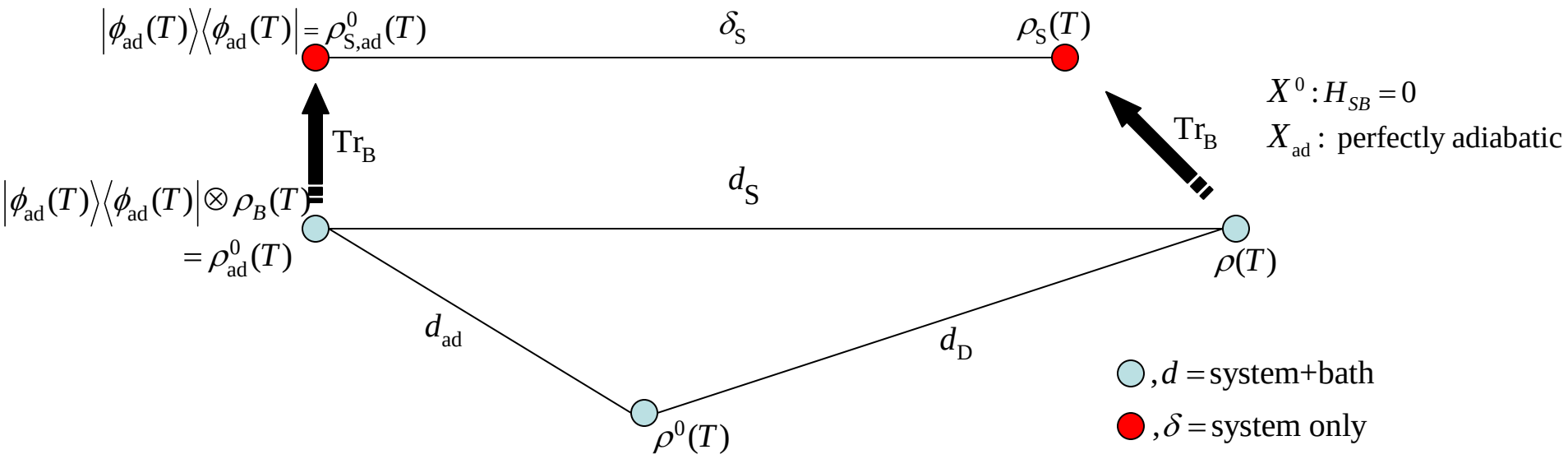
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δ_S
$\rho_S(T)$
Tr_B
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d_S
$\rho(T)$
d_{ad}
d_D
$\rho^0(T)$
, d = system+bath
, δ = system only

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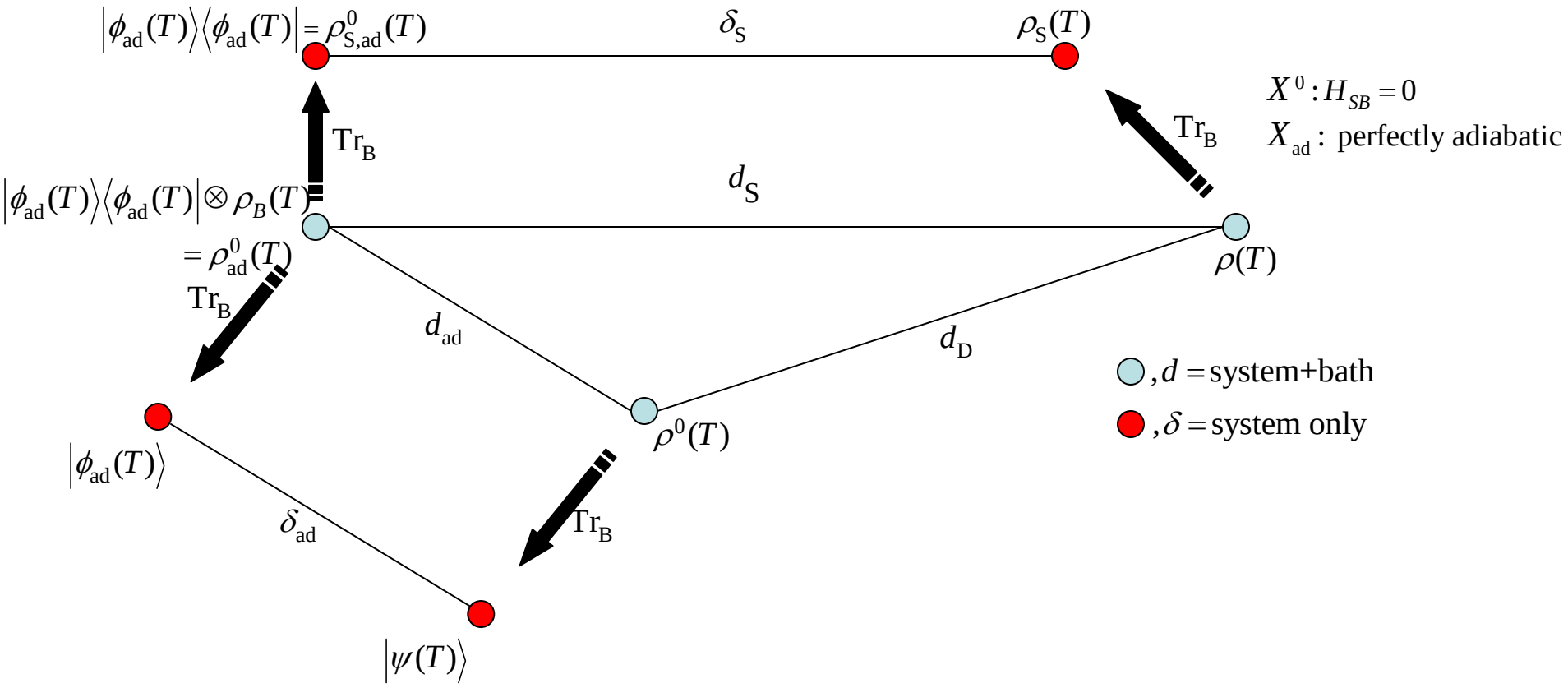
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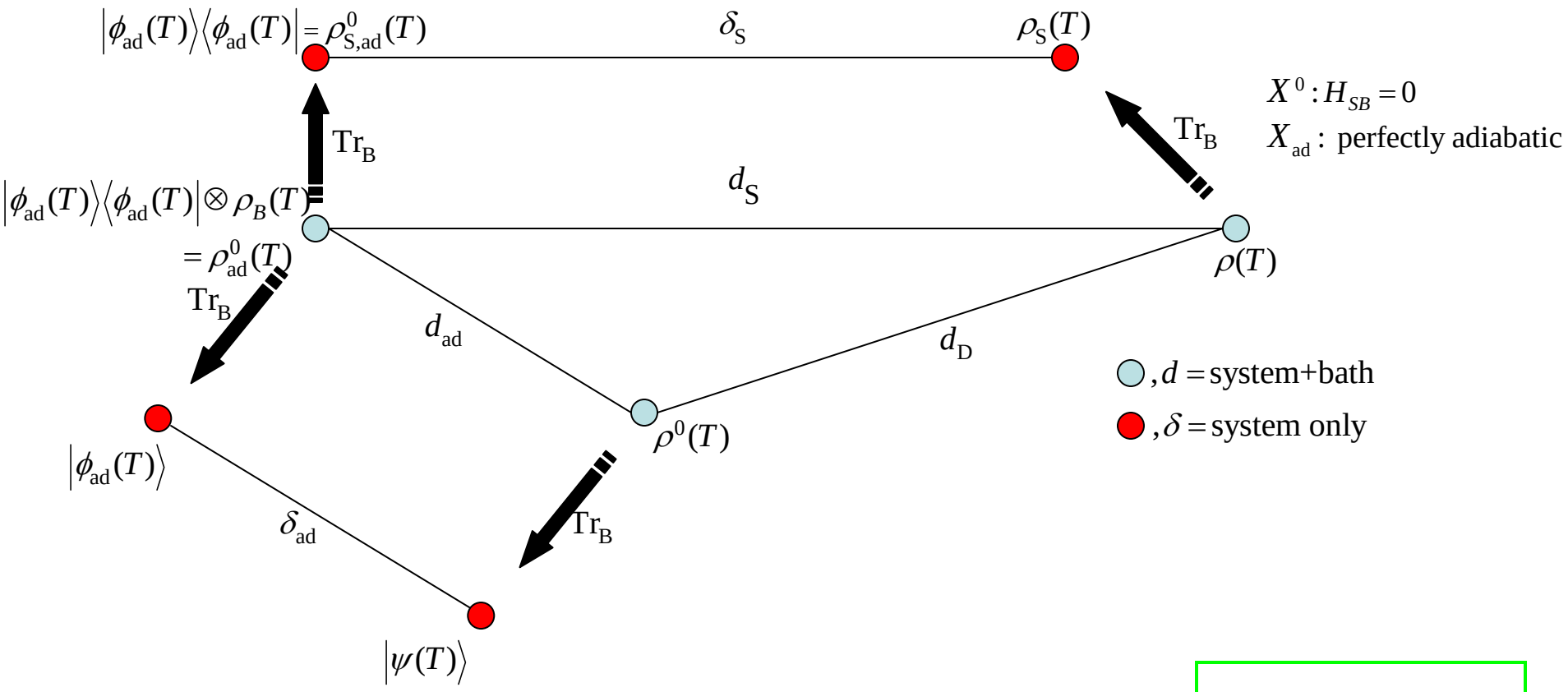
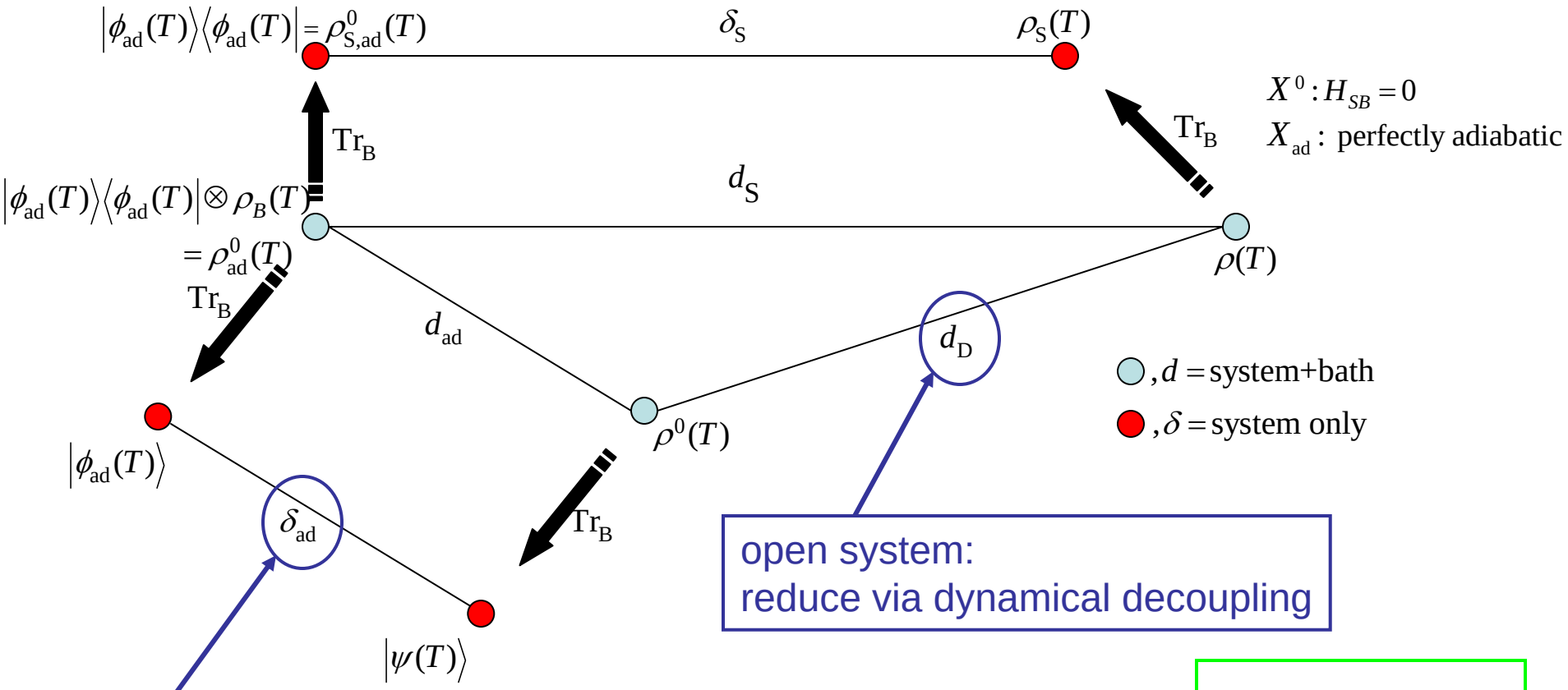
δ_{ad}
Tr_B
$|\psi(T)\rangle$

Distances



$$\delta_S \le \delta_{ad} + d_D$$

Distances



$$\delta_S \leq \delta_{ad} + d_D$$

Part 1: Closed System AQC

Dimensionless time:

$$\tau = t/T \in [0, 1];\ T = \text{final time}$$

True final state $|\psi(1)\rangle$ is the solution of the (rescaled) Schrödinger equation:

$$\frac{d|\psi\rangle}{d\tau} = -iTH_{\mathsf{ad}}|\psi\rangle$$

Goal of AQC: simultaneously minimize $T(n)$ and the error

$$\delta_{\mathsf{ad}} \equiv D[\psi(1), \phi_{\mathsf{ad}}(1)].$$

Questions...

- What determines $T(n)$?

- How to make δ_{ad} small?

Textbook criterion:

$$\text{to have} \quad |\langle\psi(1)|\phi_{\mathsf{ad}}(1)\rangle|^2 \geq 1-\epsilon^2$$

$$\text{need} \quad T \geq \frac{1}{\epsilon}\,\frac{\max_s |\langle \text{excited}(\tau)|\dot{H}_{\mathsf{ad}}(\tau)|\phi_{\mathsf{ad}}(\tau)\rangle|}{\min_\tau \text{gap}^2(\tau)}.$$

Gap dependence on n determines $T(n)$. For efficient AQC algorithms:

$$\Delta(n) \sim \Delta_0 n^{-z}, \;\; z = \text{dynamical critical exponent}$$

Unfortunately this criterion is not quite right...

Adiabatic Distance for Closed Systems

Depending on the differentiability of H_{ad} one can prove different versions of the adiabatic theorem (Lemma 2).

- Jansen, Ruskai, & Seiler [J. Math. Phys. 48, 102111 (2007)]. Assume:
  - the ground state manifold of H_{ad} is gapped.
  - $H_{\mathsf{ad}}(\tau)$ is twice differentiable on $[0,1]$
  - $\dot{H}_{\mathsf{ad}}(0) = \dot{H}_{\mathsf{ad}}(1) = 0$

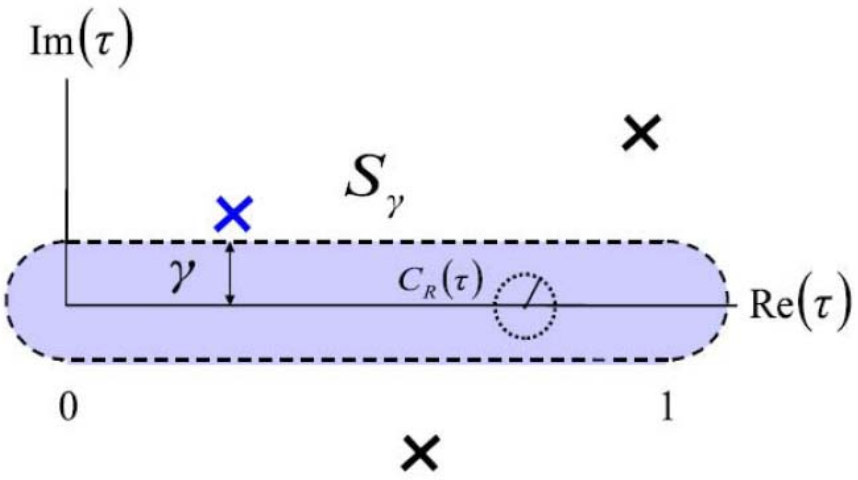
Then (norm is operator norm):

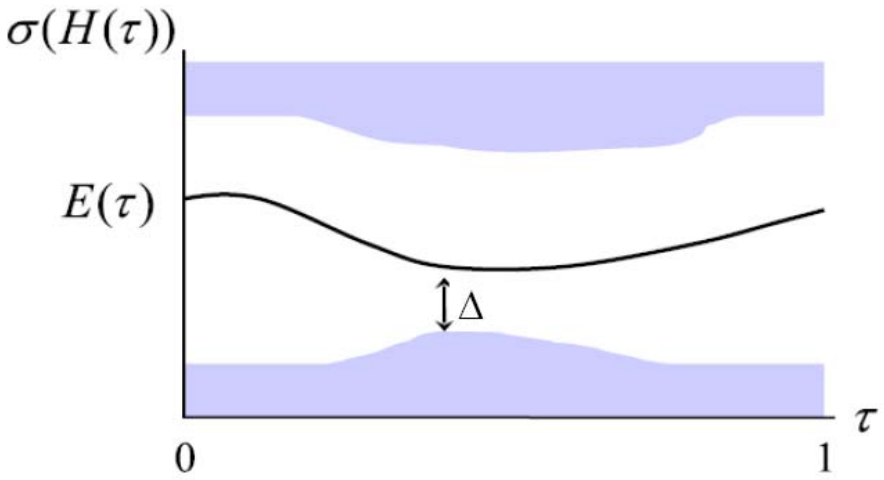
$$T \sim r \frac{\|\dot{H}_{\mathsf{ad}}\|^2}{\Delta^3} \quad \Longrightarrow \quad \delta_{\mathsf{ad}} < r^{-2}$$

DAL, A. Rezakhani, A, Hamma, JMP **50**, 102106 (2009):

Assume $H_{\rm ad}(\tau)$:

- ★ has a non-degenerate and gapped ground state
- ★ is analytic in a strip of width γ around [0,1]
- ★ its first N derivatives vanish at $\tau = 0, 1$





Let $r > 1$. Then

$$T = \frac{r}{\gamma} N \frac{\|\dot{H}_{\mathsf{ad}}\|^2}{\Delta^3} \quad \Longrightarrow \quad \delta_{\mathsf{ad}} < (N+1)^{\gamma+1} r^{-N}$$

$\Longrightarrow$ Closed-system AQC is resilient against control errors which preserve gap but cause $H_{\mathsf{ad}}(s)$ to deviate from its intended path, as long as final Hamiltonian is correct.

This is a form of inherent fault tolerance to control errors not shared by the circuit model!

How to accomplish this? **Design Hamiltonian according to criteria above.**

Part 2: Open System AQC

Joint System-Bath Evolution

$$
\begin{aligned}
H(t) &= \underbrace{H_S(t) \otimes I_B + I_S \otimes H_B}_{H^0} + H_{SB}, \\
H_S(t) &= \underbrace{H_{\mathsf{ad}}(t)}_{\text{implements AQC}} + \underbrace{H_{\mathsf{C}}(t)}_{\text{implements DD}} \\
H_{SB} &= \sum_{\alpha} S_\alpha \otimes B_\alpha
\end{aligned}
$$

[We'll see later how to enforce $[H_{\mathsf{ad}}(t), H_{\mathsf{C}}(t')] = 0 \quad \forall t, t'$.]

Joint system-bath propagator and joint state:

$$U(t) = \mathcal{T}\, \mathsf{exp}[-i \int_0^t H(t')dt'\,]; \qquad \rho(t) = U(t)\rho(0)U(t)^\dagger$$

Decoupled joint system-bath state: $\rho^0(t) = U^0(t)\rho(0)U^0(t)^\dagger$

Decoupling distance: $d_{\mathsf{D}} \equiv D[\rho(T), \rho^{0}(T)]$

$$\delta_S \leq d_{\mathsf{D}} + \delta_{\mathsf{ad}}.$$

Already saw δ_{ad} can be made arbitrarily small using analytic interpolation.

Goal: Simultaneously minimize d_{D} using *dynamical decoupling*.

This is an optimization problem: generically decoherence worsens with increasing T, while closed-system adiabaticity improves.

Dynamical Decoupling (DD) = a sequence of pulses applied to the *system*, sometimes forming a group $\mathcal{G}$, designed to reduce the effective system-bath coupling. Implemented via $H_C(t)$.

The sequence ZOO, in increasing order of performance quality:

PDD = a periodic repetition of a basic sequence

RDD = a random pulse sequence

CDD = a concatenated sequence (recursively structured)

UDD = a sequence optimized to cancel pure qubit dephasing with the smallest possible number of pulses

QDD = a sequence optimized to cancel general qubit decoherence with the smallest possible number of pulses

Dynamical Decoupling Theory

"Symmetrizing group" of pulses $\{g_i\}$ and their inverses are applied in series:

$$(g_N^\dagger \mathbf{f} g_N)\cdots(g_2^\dagger \mathbf{f} g_2)(g_1^\dagger \mathbf{f} g_1) \approx \exp(-i\tau \sum_i g_i^\dagger H_{SB} g_i)$$

$$\mathbf{f} \equiv \exp(-iH_{SB}\tau)$$

first order Magnus expansion

Choose the pulses so that:

$$H_{SB} \longmapsto H_{\text{eff}}^{(1)} \equiv \sum_i g_i^\dagger H_{SB} g_i = 0$$

Dynamical Decoupling Condition

For a qubit the Pauli group $G=\{X,Y,Z,I\}$ (π pulses around all three axes) removes an *arbitrary* H_{SB}:

$$(X\mathbf{f}X)(Y\mathbf{f}Y)(Z\mathbf{f}Z)(I\mathbf{f}I) = X\mathbf{f}Z\mathbf{f}X\mathbf{f}Z\mathbf{f}$$

Periodic DD: periodic repetition of the universal DD pulse sequence

DD Parameters:

$$
\begin{aligned}
K &= \text{no. of pulses} \\
w &= \text{pulse width} \\
\tau &= \text{pulse interval} \\
\tau_c &= K(w+\tau) = \text{cycle time} \\
L &= \text{number of PDD cycles} \\
T &= L\tau_c = \text{total time}
\end{aligned}
$$

Transform to interaction picture defined by $H_{\mathsf{ad}} + H_B$, i.e.,

$$\tilde{U}(t) = U_{\mathsf{ad}}^{\dagger}(t) \otimes U_{B}^{\dagger}(t) U(t)$$

The "effective error Hamiltonian":

$$\tilde{U}(t) \equiv e^{-itH_{\mathsf{eff}}(t)}$$

$H_{\mathsf{eff}}(t)$ can be calculated using Dyson or (better) Magnus expansion.

Note that in decoupled limit:

$$\tilde{U}(t) \overset{H_{SB}\to 0}{\longrightarrow} U_{\mathsf{C}}(t) \overset{t=j\tau_c}{=} I, \text{ i.e., } H_{\mathsf{eff}}(j\tau_c) = 0.$$

Deviation from ideality at final time is thus quantified by

"**error phase**": $\Phi \equiv T \| H_{\mathsf{eff}}(T) \|$

How is the "error phase" related to the "decoupling distance" d_{D}?

"error phase": $\Phi \equiv T\|H_{\text{eff}}(T)\|$

Then distance between desired and actual (**Lemma 4**):

$$\begin{aligned} D[\rho(T), \rho^0(T)] &= d_{\text{D}} \leq \min[1, (e^{\Phi} - 1)/2] \\ &\leq \Phi \quad \text{if } \Phi \leq 1. \end{aligned}$$

Minimize error phase $\Rightarrow$ mininize decoupling distance.

[DAL, P. Zanardi & K. Khodjasteh, Phys. Rev. A 78, 012308 (2008)]

Error Phase Bound for Periodic DD

[K. Khodjasteh & DAL, Phys. Rev. A 78, 012355 (2008)]

Let

$$\begin{aligned} J &\equiv \|H_{SB}\| \\ \beta &\equiv \|H_{\mathsf{ad}} + H_B\| \end{aligned}$$

and assume $J\tau_c < \pi$ (absolute convergence condition of Magnus expansion).

Then (**Lemma 5**):

$$\Phi(T) \leq \underbrace{\frac{JTw}{\tau + w}}_{\substack{\text{error due to} \\ \text{finite pulse width}}} + \underbrace{\frac{(JT)^2}{L} + JT\,\mathsf{min}[1, (\frac{\mathsf{exp}(2\beta\tau_c) - 1}{2\beta\tau_c} - 1)]}_{\substack{\text{error due to terms not removed} \\ \text{by first order DD procedure}}}.$$

Use this to bound decoupling distance.

Joint AQC-DD Optimization for Local Hamiltonians

For local Hamiltonians H_{ad} and H_B:

$$\beta = \|H_{\mathsf{ad}} + H_B\| \sim O(n^2)$$

Recall closed system adiabaticity theorem, ensuring error small if N is large:

$$\begin{aligned} T &= \frac{r}{\gamma} N \frac{\|\dot{H}_{\mathsf{ad}}\|^2}{\Delta^3} \quad \Longrightarrow \quad \delta_{\mathsf{ad}} < (N+1)^{\gamma+1} r^{-N} \\ &\sim \frac{r}{\gamma} N \frac{1}{\Delta_0^3} n^{3z+4} \equiv KL(\tau + w) \end{aligned}$$

where we used gap condition for efficient AQC algorithms:

$$\boxed{\Delta(n) \sim \Delta_0 n^{-z},\ z = \text{dynamical critical exponent}}$$

Given and fixed parameters of the problem are J, Δ_0, and z.

Need to ensure that each of the terms upper bounding $\Phi(T)$ vanishes as a function of n.

This is the case if pulse interval τ and pulse width w scale as

$$\boxed{\tau \sim n^{-(3z+3+\epsilon_1)}/\Delta_0, \qquad w \sim n^{-(6z+5+\epsilon_1+\epsilon_2)}/J}$$

with $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

For then, using **Lemmas 4 & 5** we have proven **Lemma 3**:

$$\boxed{d_{\mathsf{D}} \lesssim n^{-\epsilon_2} + (J/\Delta_0)^2 n^{-(1+\epsilon_1)} + (J/\Delta_0) n^{-\epsilon_1} \overset{n\to\infty}{\longrightarrow} 0}$$

$\Longrightarrow$ *Using PDD with properly chosen parameters we can obtain arbitrarily accurate AQC.*

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$\Longrightarrow$ *Using PDD with properly chosen parameters we can obtain arbitrarily accurate AQC.*

Shortcoming: pulse intervals and widths must shrink with n as a power law...

Could perhaps be remedied by employing *concatenated* DD.

In conclusion we've (almost) proven the Theorem:

> AQC can be performed with a fidelity approaching 1 as a power law in the system size, [using only 2-local Hamiltonians, in the presence of 1-local noise], assuming access to (dynamical decoupling) pulses whose width and intervals shrink as a power law in the system size. The latter power law exponent is linear in the dynamical critical exponent of the closed system.

Part 3: Seamlessly Combining AQC & DD

Need to show how to achieve "non-interference" condition

$$[H_{\mathsf{ad}}(t), H_{\mathsf{C}}(t')] = 0 \quad \forall t, t'.$$

Can be done using encoding.

Example — Stabilizer-normalizer construction:

DD pulses are stabilizer elements. AQC implemented via normalizer elements.

2-Local Universal AQC Resistant Against 1-local Noise

A 2-local Hamiltonian that is universal for AQC (J.D. Biamonte and P.J. Love, arXiv:0704.1287):

$$H_{\mathsf{ad}}^{\mathsf{univ}}(t) = \sum_{i;\alpha\in\{x,z\}} h_i^{\alpha}(t)\sigma_i^{\alpha} + \sum_{i,j;\alpha\in\{x,z\}} J_{ij}^{\alpha}(t)\sigma_i^{\alpha}\sigma_j^{\alpha}.$$

1-local noise (linear decoherence model):

$$H_{SB}^{\mathsf{lin}} = \sum_{\alpha=x,y,z} \sum_{j=1}^{n} \sigma_j^{\alpha} \otimes B_j^{\alpha}$$

Decoupling group that decouples $H_{SB}^{\text{lin}} = \sum_{\alpha=x,y,z} \sum_{j=1}^{n} \sigma_j^\alpha \otimes B_j^\alpha$:

$$\mathcal{G} = \{I, X, Y, Z\};\ X = \bigotimes_{j=1}^{n} \sigma_j^x, \text{ etc.}$$

This requires only *global* pulses.

$\mathcal{G}$ is the stabilizer of an $[[n, n-2, 2]]$ stabilizer code $\mathcal{C}$ (n even, x =even weight binary string):

$$\mathcal{C} = \{|\psi_x\rangle = (|x\rangle + |\text{not } x\rangle)/\sqrt{2}\}$$

E.g. $n = 4$:

$$\begin{aligned} |00\rangle_L &= (|0000\rangle + |1111\rangle)/\sqrt{2},\ |10\rangle_L = (|0011\rangle + |1100\rangle)/\sqrt{2} \\ |01\rangle_L &= (|0101\rangle + |1010\rangle)/\sqrt{2},\ |11\rangle_L = (|1001\rangle + |0110\rangle)/\sqrt{2} \end{aligned}$$

Encoded single-qubit generators for $\mathcal{C} = \{|\psi_x\rangle = (|x\rangle + |\mathsf{not}\, x\rangle)/\sqrt{2}\}$:

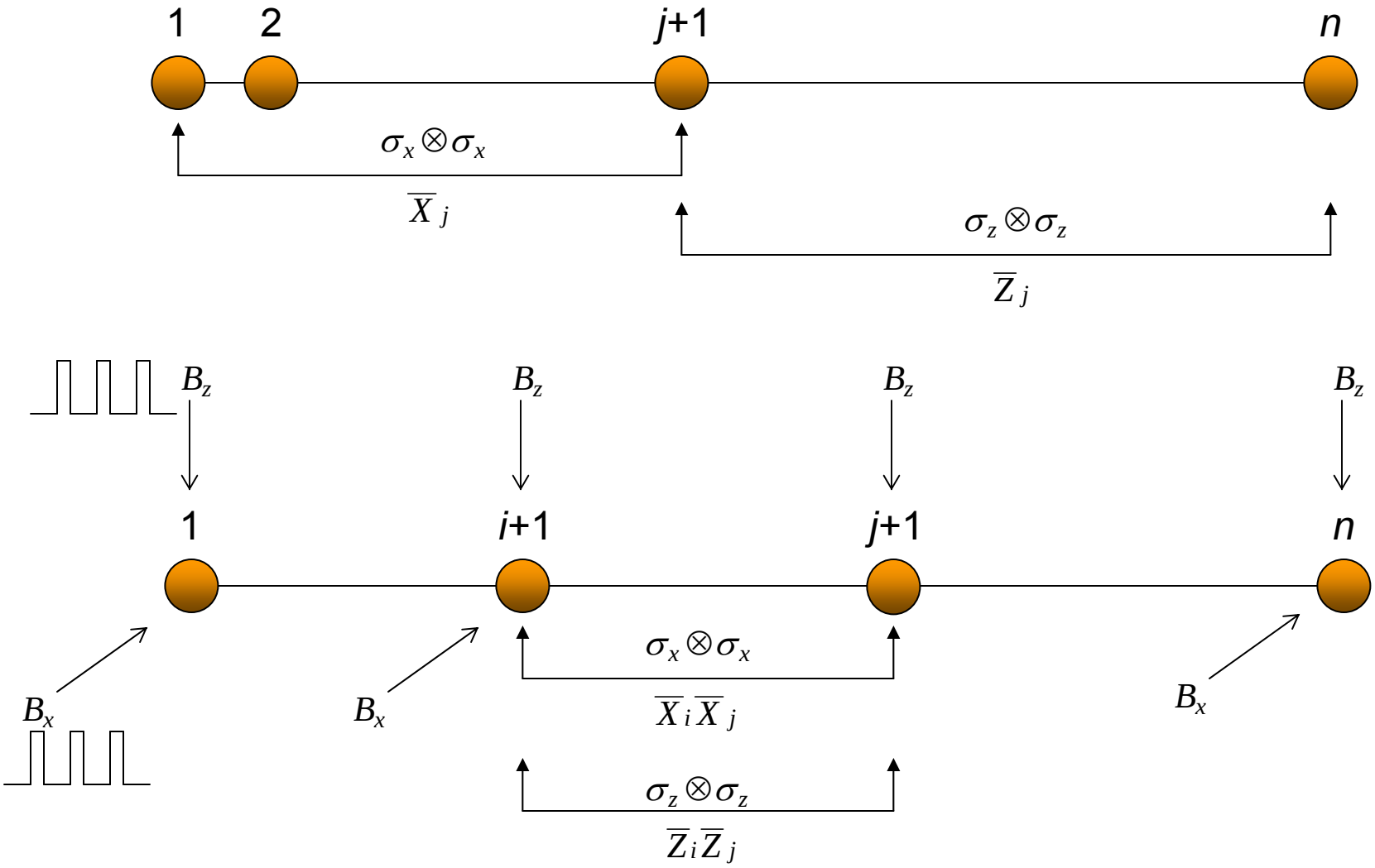
$$\bar{X}_j = \sigma_1^x \sigma_{j+1}^x \qquad \bar{Z}_j = \sigma_{j+1}^z \sigma_n^z$$

Encoded two-qubit generators:

$$\bar{X}_i \bar{X}_j = \sigma_{i+1}^x \sigma_{j+1}^x \qquad \bar{Z}_i \bar{Z}_j = \sigma_{i+1}^z \sigma_{j+1}^z$$

Thus universal AQC can be combined with DD using only 2-local $\sigma_i^x \sigma_j^x$ and $\sigma_i^z \sigma_j^z$ interactions over $\mathcal{C}$.

Universality for the [[*n*,*n*-2,2]] code



Physical examples where X, Z (as pulses for DD) and $\sigma_i^x \sigma_j^x, \sigma_i^z \sigma_j^z$ (as Hamiltonians for AQC) are available and controllable:

- Capacitively coupled flux qubits (D.V. Averin and C. Bruder, Phys. Rev. Lett. **91**, 057003 (2003))

- Spin models implemented with polar molecules (A. Micheli, G. Brennen, and P. Zoller, Nature Phys. **2**, 341 (2006))

Conclusions

- Theorem: AQC can be performed with a fidelity approaching 1 as a power law in the system size n, using only 2-local Hamiltonians, in the presence of 1-local noise, assuming access to (dynamical decoupling) pulses whose width and intervals shrink as a power law in the system size. The latter power law exponent is linear in the dynamical critical exponent of the closed system.

Conclusions

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- Open questions:

  - Can a similar result be shown with n-independent pulse width and interval?

  - The distance bound is rather crude because of use of the triangle inequality;

    can it be tightened by directly treating adiabatic evolution in the open system?

  - A fault-tolerance threshold for AQC?

Phys. Rev. Lett. **100**, 160506 (2008)

Odds and Ends

Energy gap stabilization

- Jordan, Farhi & Shor (PRA **74**, 052322 (2006)) introduced a scheme that can protect AQC against 1-local noise using a energy gap against local excitations.

- Such gaps can be engineered into the Hamiltonians using error detecting codes

- However, the resulting universal Hamiltonians are at least 4-local

- Their scheme is fully compatible with the DD scheme presented here

Control Error

- Suppose final Hamiltonian is slightly off:

$$H(T) = H_0(T) + \lambda V$$

- Resulting ground state is:

$$|\tilde{\psi}_0\rangle = |\psi_0\rangle + \lambda \sum_{k \neq 0} |\psi_k\rangle \frac{\langle \psi_k | V | \psi_0 \rangle}{E_0 - E_k} + \mathcal{O}(\lambda^2)$$

gap protection

Why not Use Standard Quantum Error Correction?

Not known how to embed fault-tolerant QEC into adiabatic evolution:

- Requires feedback, which may break adiabaticity (A. Allahverdyan & G. Mahler, arXiv:0804.1643)

- Embedded FT-QEC circuit will face different error model than what it was designed for

We used dynamical decoupling (open-loop, feedback-free) to deal with coupling to environment, and showed it can deal with the relevant error model.

And, we used an analytic interpolation to boost fidelity of closed-system adiabatic evolution.

Is AQC "Better" than Other QC Paradigms?

No:

· Computationally equivalent to circuit model (Aharonov et al., :quant-ph/0405098; A. Mizel, DAL, M. Mitchell, PRL 99, 070502 (2007))

· Doesn't have a fault tolerance theory to back it up

Maybe:

· Perhaps more easily implemented in certain systems, in particular solid state (quantum dots, superconducting)

· Inherently protected against leakage if $k_B T$ is smaller than gap

· Inherently protected against path deformations: only end points matter

Open System Evolution

Consider the uncoupled setting $H_{SB} = 0$, denoted by the superscript 0.

$$\text{ideal adiabatic state: } \rho^0_{S,\mathsf{ad}}(t) = |\phi_{\mathsf{ad}}(t)\rangle\langle\phi_{\mathsf{ad}}(t)|$$

$$\begin{aligned} \text{actual state under } H_{\mathsf{ad}}(t) &: \quad \rho^0_S(t) = |\psi(t)\rangle\langle\psi(t)| \\ \text{state under } H_{\mathsf{C}}(t) &: \quad \rho^0_{\mathsf{C}}(t) \\ \text{state under } H_B(t) &: \quad \rho^0_B(t) \end{aligned}$$

$$\begin{aligned} \text{decoupled joint state:} &\quad \rho^0(t) = \rho^0_S(t) \otimes \rho^0_{\mathsf{C}}(t) \otimes \rho^0_B(t) \\ \text{ideal adiabatic joint state:} &\quad \rho^0_{\mathsf{ad}}(t) \equiv \rho^0_{S,\mathsf{ad}}(t) \otimes \rho^0_{\mathsf{C}}(t) \otimes \rho^0_B(t) \end{aligned}$$

Let d (δ) denote distances in the joint (system) Hilbert space.

$$\begin{aligned}
\text{target distance:} \qquad & \delta_S \equiv D[\rho_S(T), \rho^0_{S,\mathsf{ad}}(T)] \\
\text{decoupling distance:} \qquad & d_{\mathsf{D}} \equiv D[\rho(T), \rho^0(T)] \\
\text{adiabatic distance:} \qquad & d_{\mathsf{ad}} \equiv D[\rho^0(T), \rho^0_{\mathsf{ad}}(T)] = \delta_{\mathsf{ad}}
\end{aligned}$$

Proof of the Distance Inequality

Partial trace can only decrease distance:

$$\underbrace{D[\rho_S(T), \rho^0_{S,\mathsf{ad}}(T)]}_{\delta_S} \leq D[\rho(T), \rho^0_{\mathsf{ad}}(T)]$$

Triangle inequality:

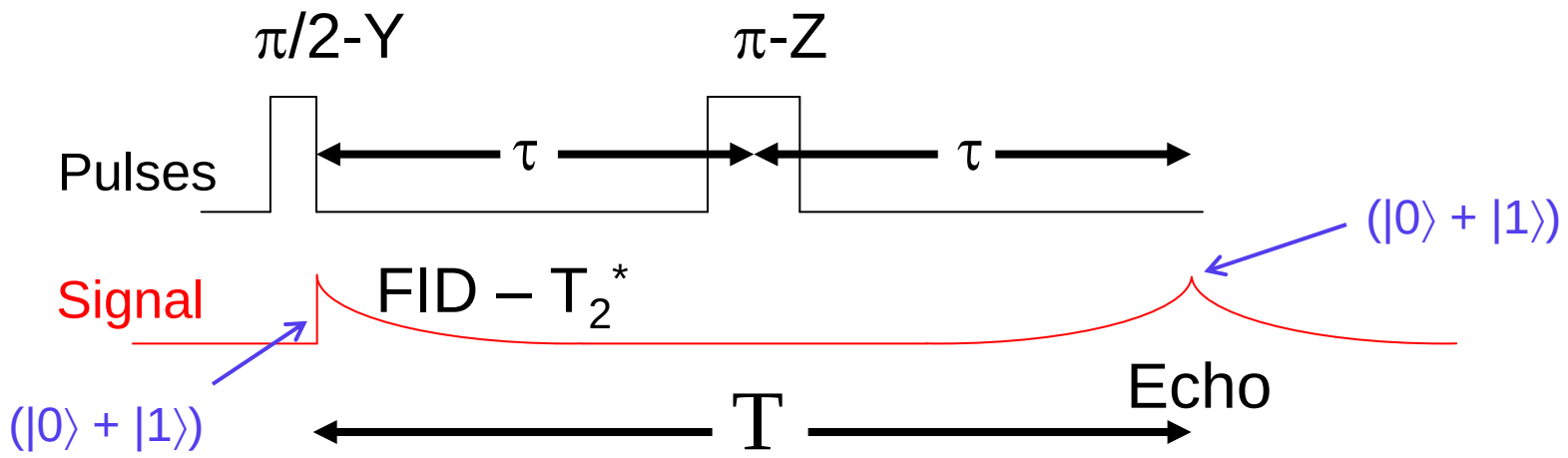
$$D[\rho(T), \rho^0_{\mathsf{ad}}(T)] \leq \underbrace{D[\rho(T), \rho^0(T)]}_{d_{\mathsf{D}}} + \underbrace{D[\rho^0(T), \rho^0_{\mathsf{ad}}(T)]}_{\delta_{\mathsf{ad}}}$$

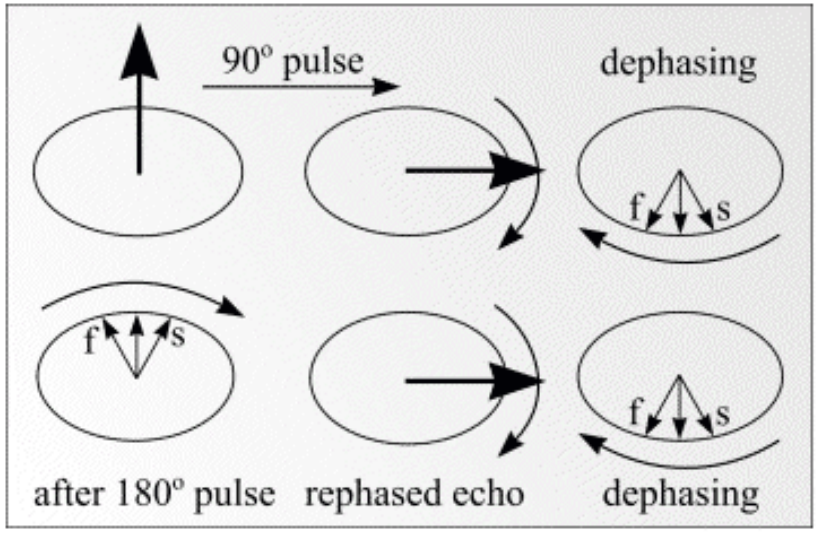
$\Longrightarrow$ desired distance inequality (**Lemma 1**):

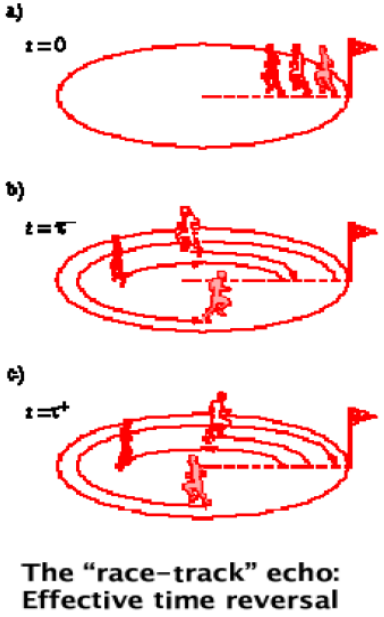
$$\delta_S \leq d_{\mathsf{D}} + \delta_{\mathsf{ad}}.$$

Dynamical Decoupling

Inspired by Hahn spin echo (1950)







The "race-track" echo:
Effective time reversal

Dynamical Decoupling (DD) = a sequence of pulses applied to the *system*, sometimes forming a group $\mathcal{G}$, designed to reduce the effective system-bath coupling. Implemented via $H_C(t)$.

The sequence ZOO, in increasing order of performance quality:

PDD = a periodic repetition of a basic sequence

RDD = a random pulse sequence

CDD = a concatenated sequence (recursively structured)

UDD = a sequence optimized to cancel pure qubit dephasing with the smallest possible number of pulses

QDD = a sequence optimized to cancel general qubit decoherence with the smallest possible number of pulses

Subsystem Code Construction for $[H_{\mathsf{ad}}(t), H_{\mathsf{C}}(t')] = 0$

The decoupling group $\mathcal{G}$ induces a decomposition of the system Hilbert space H_S via its group algebra $C\mathcal{G}$ and its commutant $C\mathcal{G}'$:

$$\mathcal{H}_S \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J},$$

$$\mathbb{C}\mathcal{G} \cong \bigoplus_J I_{n_J} \otimes M_{d_J}, \quad \mathbb{C}\mathcal{G}' \cong \bigoplus_J M_{n_J} \otimes I_{d_J}.$$

$n_J =$ multiplicity of irrep J; $d_J =$ dimension

Adiabatic state is encoded into a left factor $C_J \equiv \mathbb{C}^{n_J}$: an n_J-dimensional code C_J storing $\log n_J$ qubits. AQC is enacted via $\mathbb{C}\mathcal{G}'$.

DD pulses act on the right factors, enacted via the elements of $\mathbb{C}\mathcal{G}$.

DD pulses project each S_α in system-bath Hamiltonian to $\bigoplus_J \lambda_{J,\alpha} I_{n_J} \otimes I_{d_J}$.

Non-interference condition is satisfied because $[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}'] = 0$.