Twists in Topological Codes

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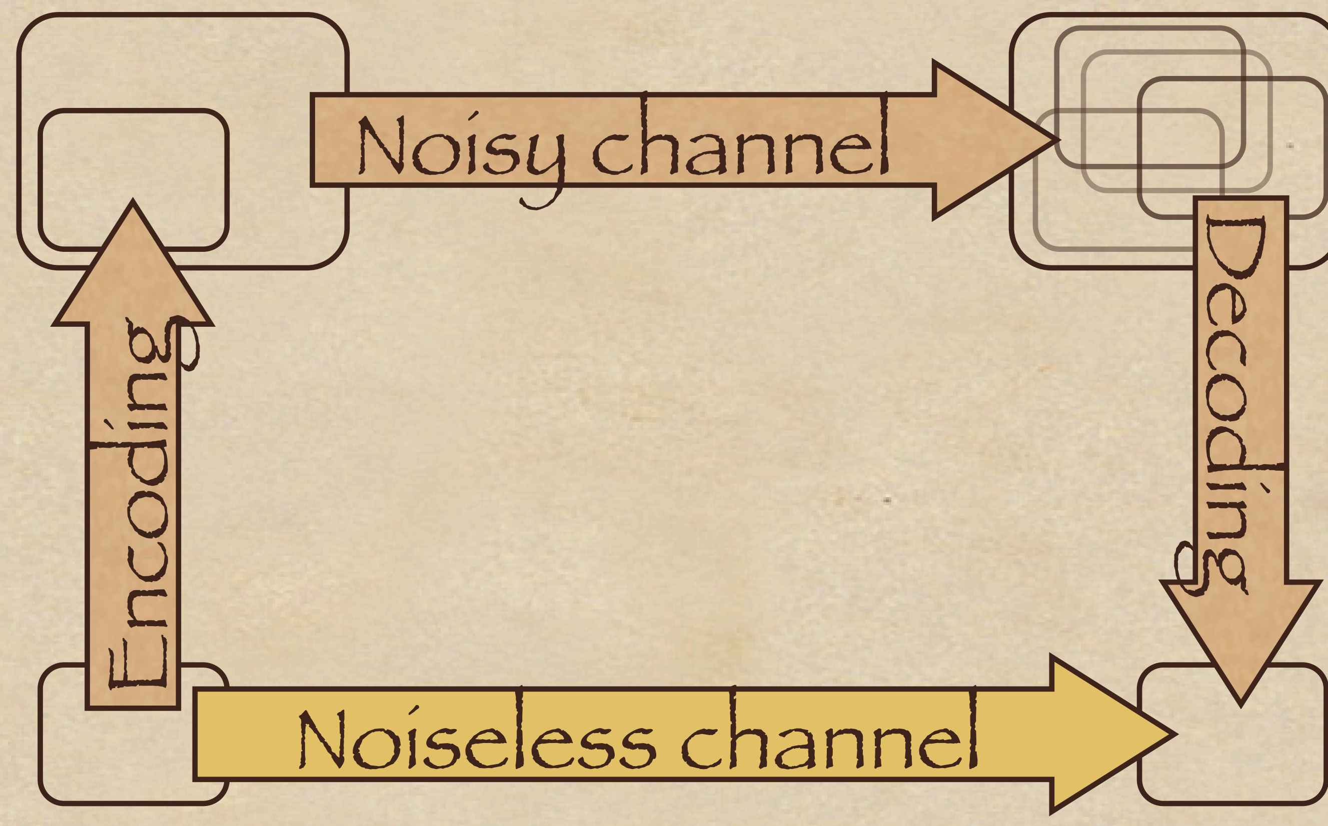
Outline

- Motivation
- Anyons and twists
- Twists in toric codes
- Twists in topological subsystem codes

Motivation

- Quantum error correction attempts to create faithful quantum channels from noisy ones



- Typically this involves encoding in a subspace

Motivation

- The code subspace can be defined in terms of commuting observables: check operators (CO)

$$C_i|\psi\rangle = c_i|\psi\rangle$$

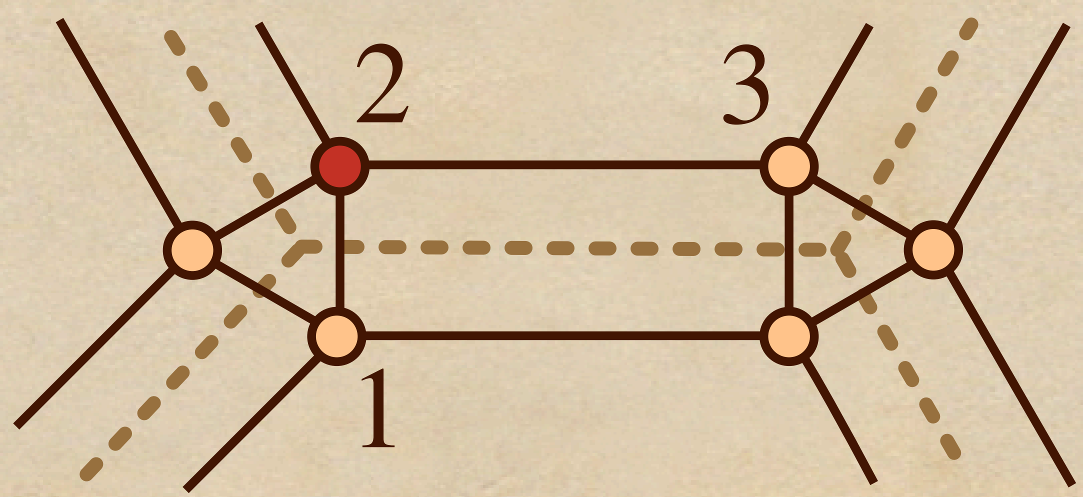
- Errors typically change CO values:

CO measurement → error syndrome →

→ compute most probable error →correct

Motivation

- In some settings locality is crucial
- Topological codes have geometrically local COs
- Topological subsystem codes (TSC): Just 2-local measurements!



$Z_1 Z_2$

$Y_2 X_3$

Motivation

- Fault-tolerance in topological memories: repeatedly measure error syndrome → → keep track of errors
- How to compute?
  - Transversal gates (color codes)
  - Boundaries & code deformation
- NOT possible for TSCs!!!

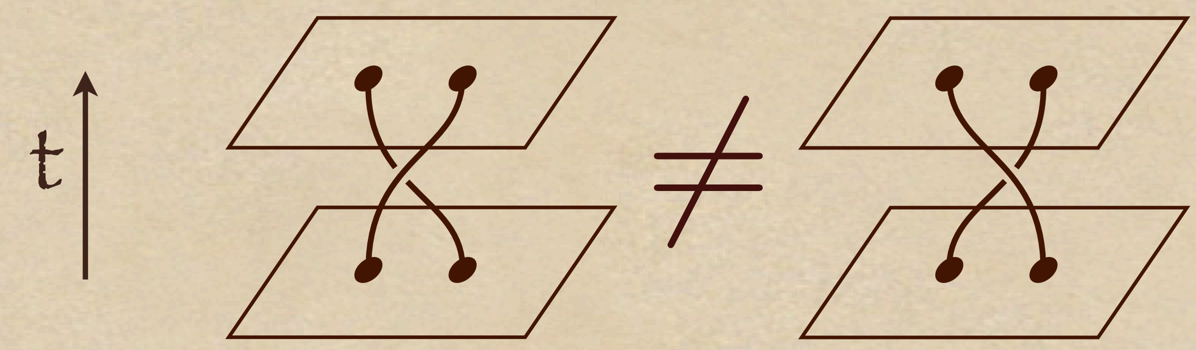
Motivation

Topological codes	VS	Topological order
Code subspace	$\leftrightarrow$	Ground subspace
Error correction	$\leftrightarrow$	Energy gap
Error syndrome	$\leftrightarrow$	Excitation configuration

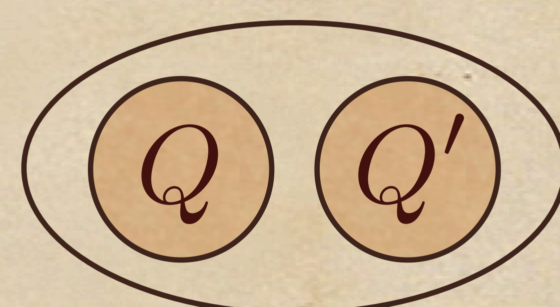
- In 2D excitations are very special: ANYONS.
- Using anyon symmetries → All CLIFFORD gates by code deformation on TSCs!!!

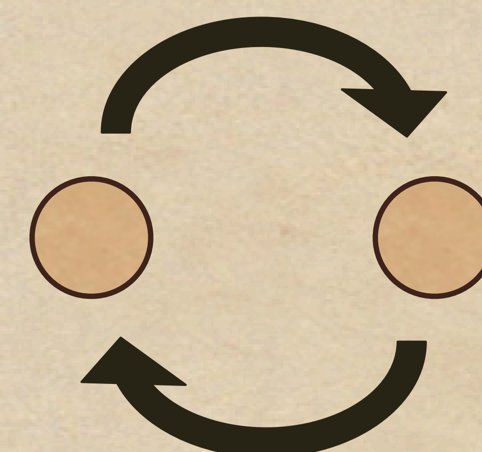
Anyons

- 2D: statistics beyond bosons and fermions



- Ingredients of an anyon model:



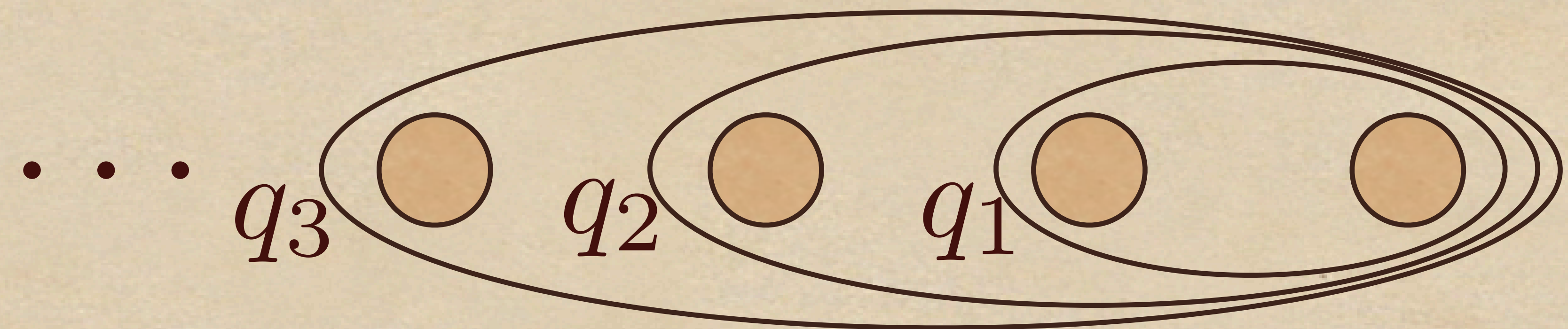


$Q \in \{a, b, \dots\}$ $\quad$ $Q \times Q' = q_1 + q_2 + \dots$ $\quad$ $|\psi\rangle \longrightarrow ?$

Top. charges $\quad$ Fusion rules $\quad$ Braiding rules

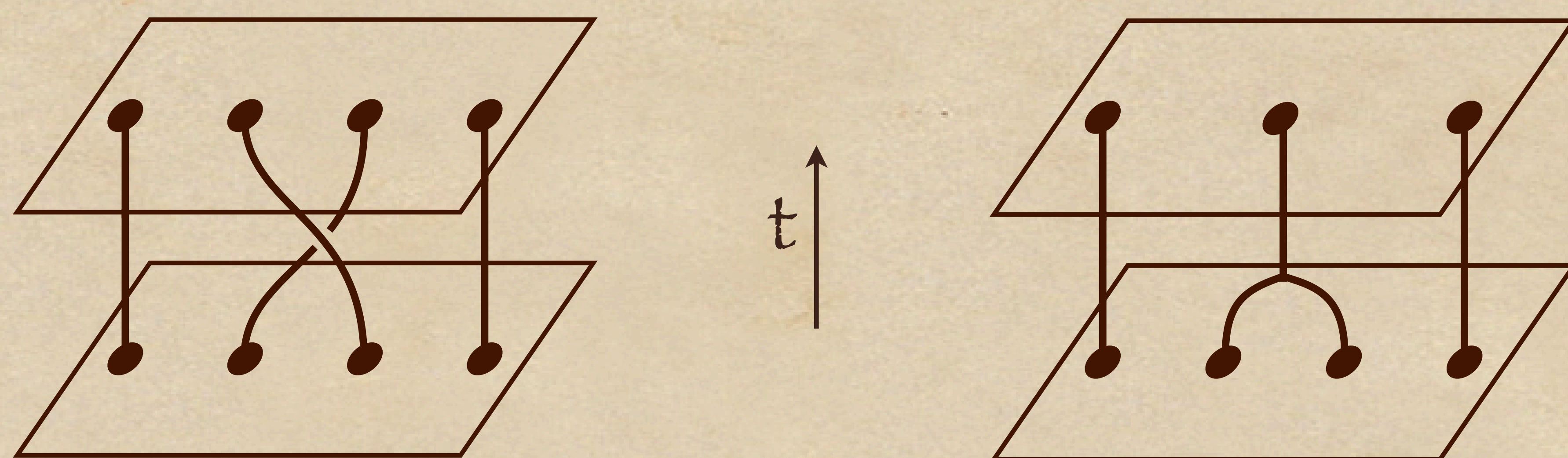
Topological QC

- Qubits encoded on fusion channels



Compute = Braid

Measure = Fuse



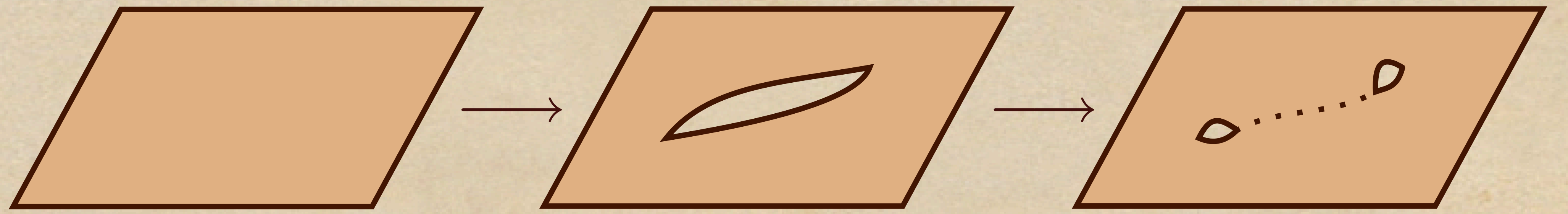
- BUT: In abelian models a x b = c → trivial code

Symmetries

- Anyon symmetry: charge permutation producing an equivalent anyon model

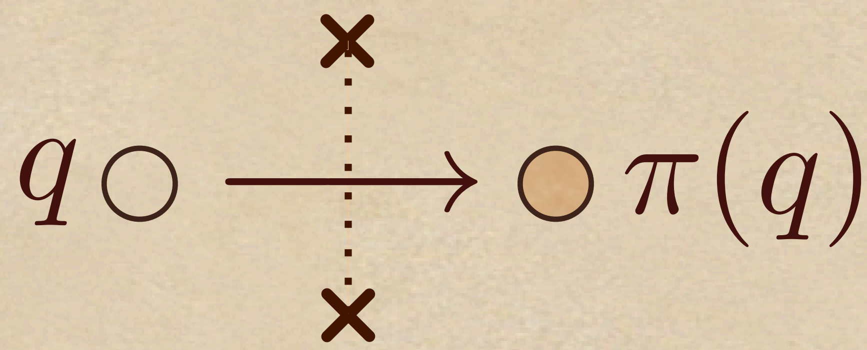
$$q \longrightarrow \pi(q)$$

- Imagine 'cutting' the anyons' 2D world and gluing it again up to a symmetry

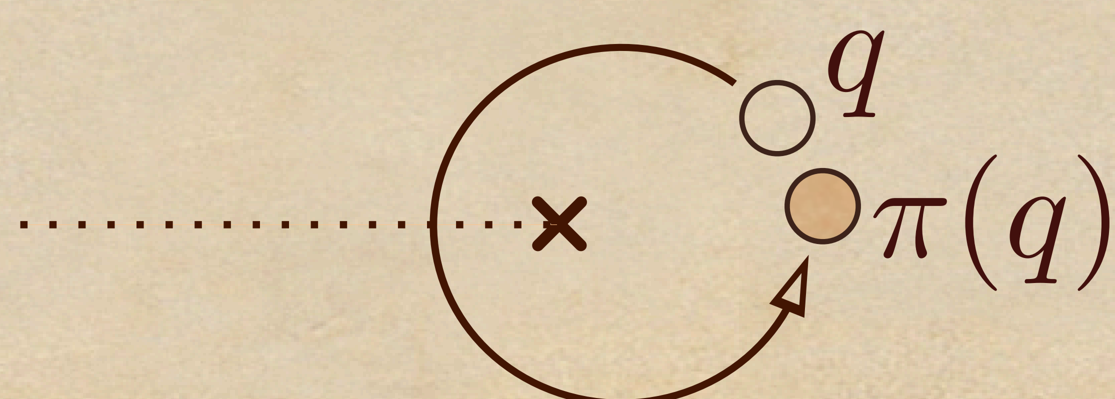


Twists

- Across the cut, charges change:



- Topologically, the cut location is unphysical.
- Endpoints are meaningful: under monodromy they permute charges → TWISTS

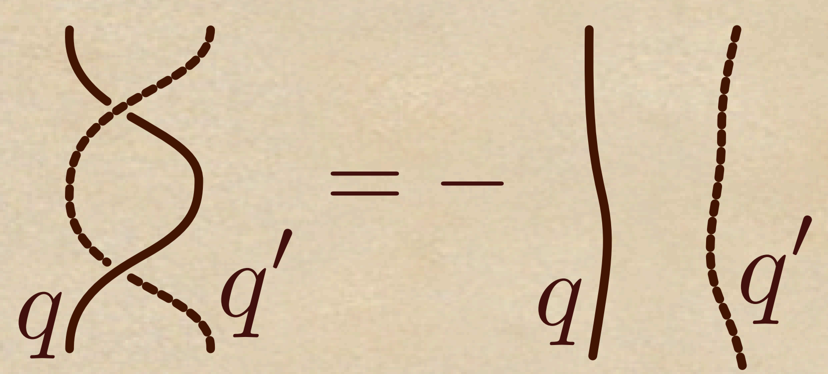


Twists

- Example: quantum double of Z_2 (toric code)
- Charges: $\{\mathbf{1}, e, m, \epsilon\}$
- Fusion: $e \times m = \epsilon \qquad e \times \epsilon = m \qquad m \times \epsilon = e$

$$e \times e = m \times m = \epsilon \times \epsilon = \mathbf{1}$$

- Braiding: $e, m \rightarrow$ bosons $\qquad \epsilon \rightarrow$ fermion

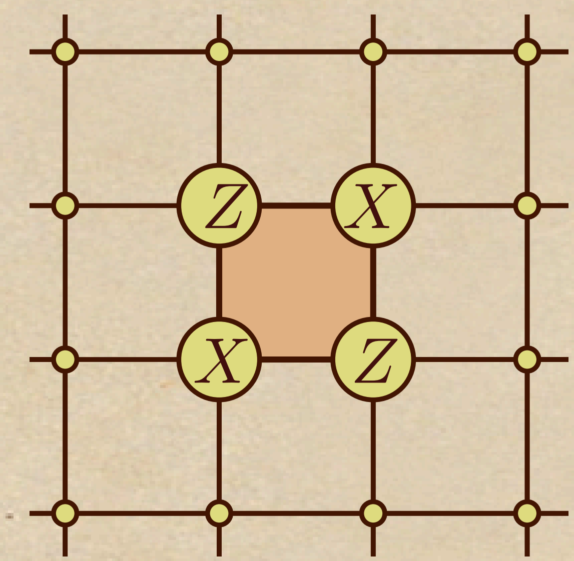


$$\mathbf{1} \neq q \neq q' \neq \mathbf{1}$$

- Nontrivial symmetry: $e \leftrightarrow m \qquad$ realization?

Toric code

- Qubits form a square lattice
- 4-local check operators at plaquettes



$$C_k := X_k Z_{k+\mathbf{i}} X_{k+\mathbf{i}+\mathbf{j}} Z_{k+\mathbf{j}}$$

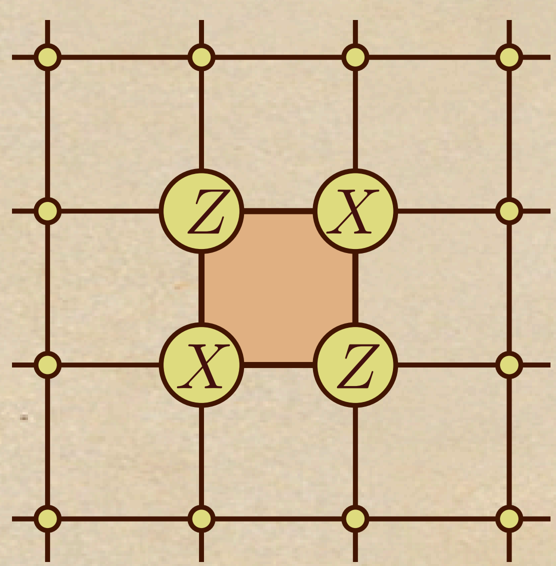
- Hamiltonian:

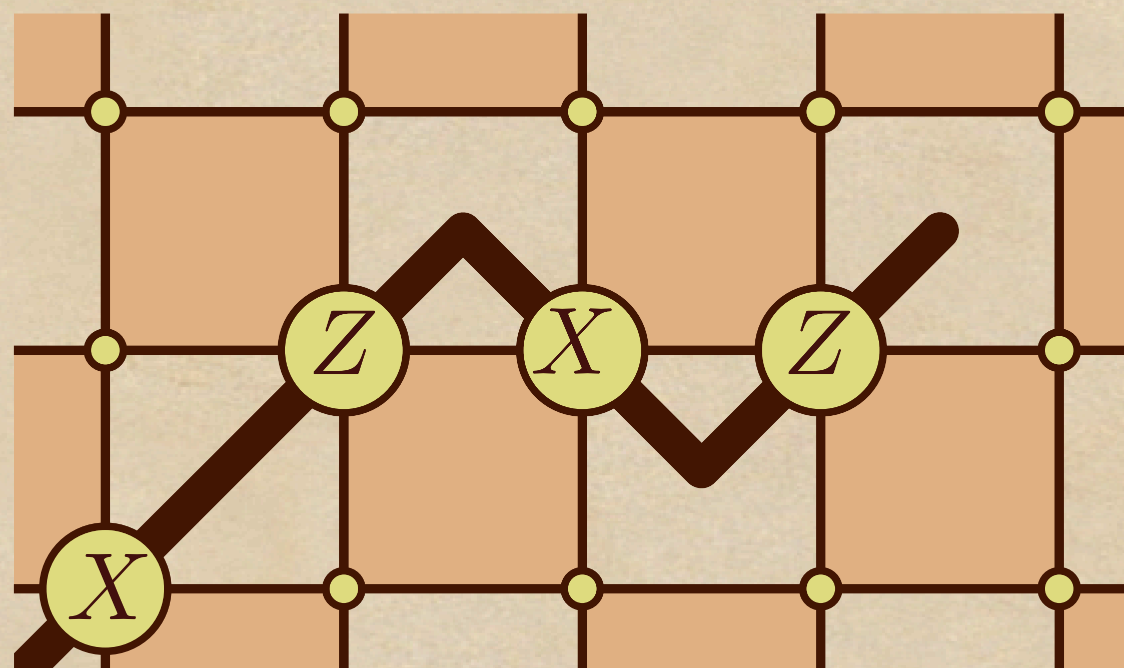
$$H := -\sum_k C_k$$

- Excitations live in plaquettes

Toric code

- String operators create/destroy excitations at their endpoints

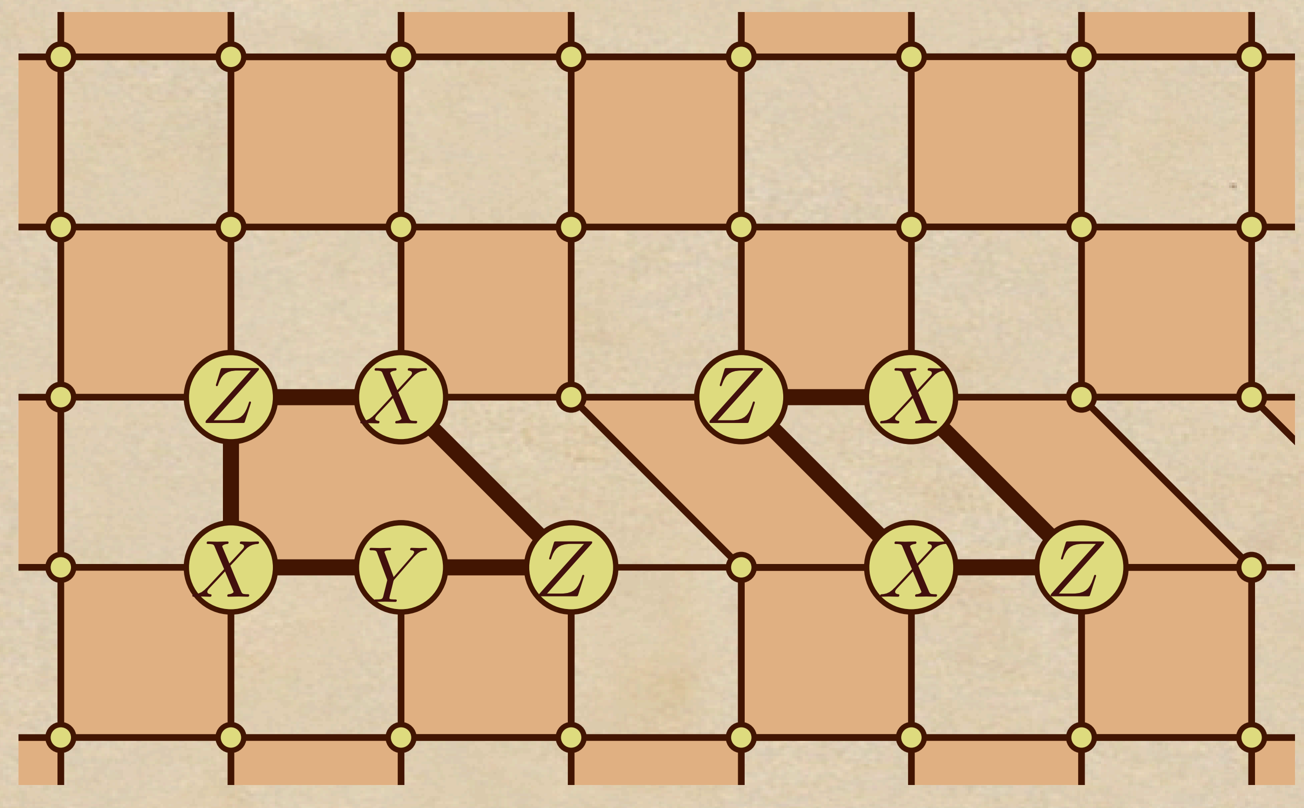




- Two types of excitations: e (light) and m (dark)

Toric code twists

- To get twists, we simply add dislocations



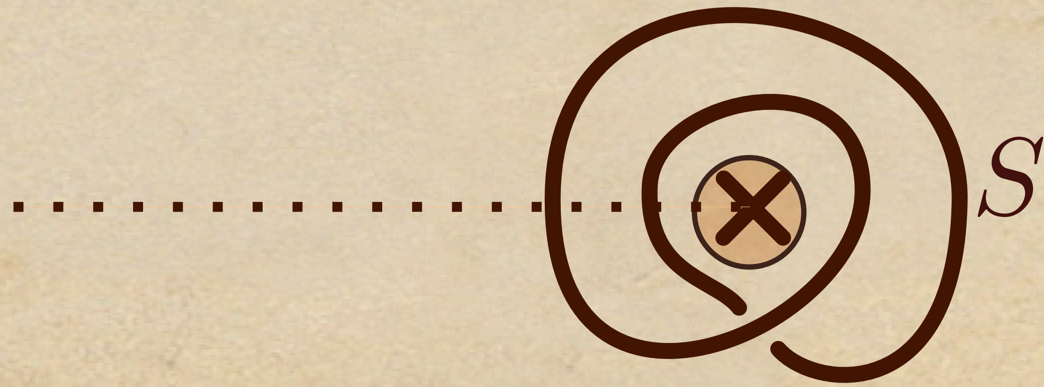
- Twists can be locally created in PAIRS only

Generalized charges

- EVEN number of twists → 4 possible charges

	$\mathbf{1}$	e	m	ϵ
S_e	+1	+1	-1	-1
S_m	+1	-1	+1	-1

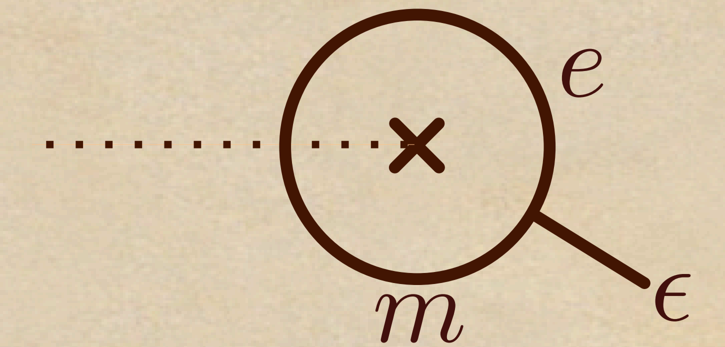
- ODD number of twists → 2 possible charges



	σ_+	σ_-
S	+i	-i

Fusion rules

- Twists are sinks for fermions:



- Non-abelian fusion rules!

$$\sigma_\pm \times \sigma_\pm = 1 + \epsilon \qquad \sigma_\pm \times \sigma_\mp = e + m$$

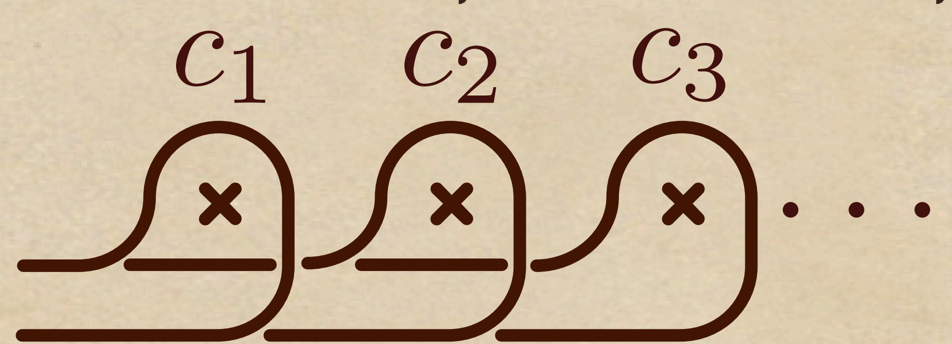
$$\sigma_\pm \times \epsilon = \sigma_\pm \qquad \sigma_\pm \times e = \sigma_\pm \times m = \sigma_\mp$$

- We recover Ising rules:

$$\sigma \times \sigma = \mathbf{1} + \psi \qquad \sigma \times \psi = \sigma \qquad \psi \times \psi = \mathbf{1}$$

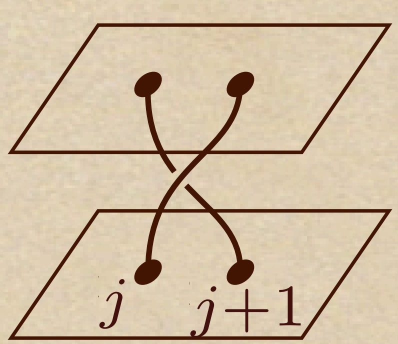
Majorana operators

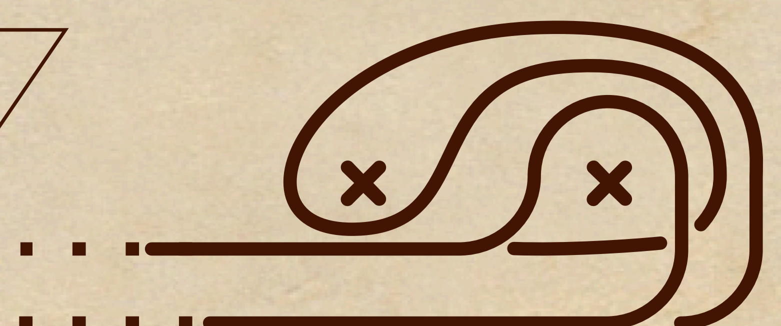
- All closed string ops can be expressed in terms of a set of open string ops → Majorana operators



$$c_j c_k + c_k c_j = 2\delta_{jk}$$

- Braiding is also Ising-like → 1-qubit Clifford gates



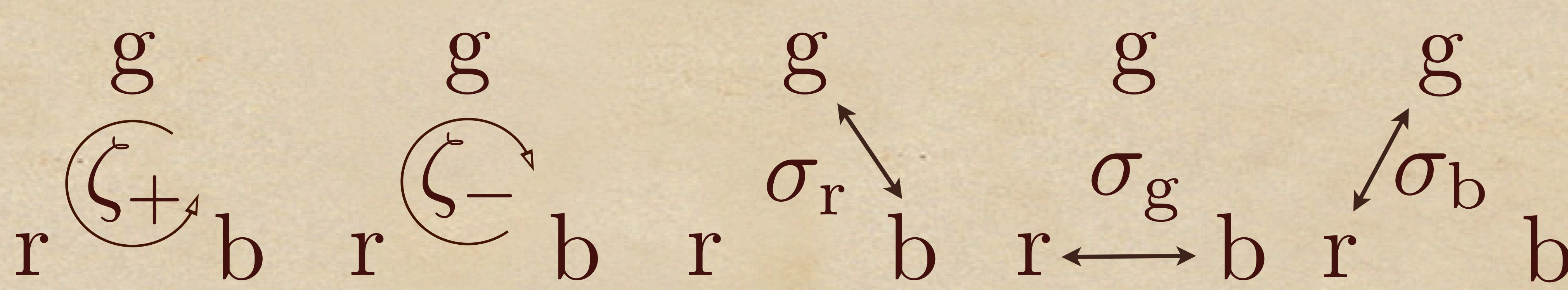


$$c_j \to c_{j+1}$$

$$c_{j+1} \to -c_j$$

TSC twists

- Charges: $\{\mathbf{1}, \mathrm{r}, \mathrm{g}, \mathrm{b}\}$
- Like in toric code, but all are fermions
- Symmetries: any permutation of fermions



- Non-commutative fusion rules! (unlike anyons)

TSC twists

- Transpositions → 2 possible charges
- As in toric codes, we fix one
- Colored Majorana operators:

The i-th twist is σ_{c_i} and i<j

$$k_i k_j = \begin{cases} k_j k_i & \text{if } c_i = \zeta_+(c_j), \\ -k_i k_j & \text{otherwise.} \end{cases}$$

TSC twists

- Braiding is now more interesting

$$k_j \to k_{j+1}$$

$$k_{j+1} \to \begin{cases} -k_j & \text{if } c_j = c_{j+1}, \\ i k_j k_{j+1} & \text{if } c_j = \zeta_-(c_{j+1}), \\ -k_j k_{j+1} & \text{otherwise.} \end{cases}$$

- This gives the whole Clifford group!!!

Conclusions

- Topological codes stand out for their locality
- TSCs only require 2-local measurements
- Twists reflect anyon symmetries
- Twist are a tool to improve topological codes
- With twists, via code deformation, we can implement all Clifford gates on TSCs

PRL 105.030403 / arXiv:1004.1838

arXiv:1006.5260