The relevance of quasi-probability representations to quantum information theory

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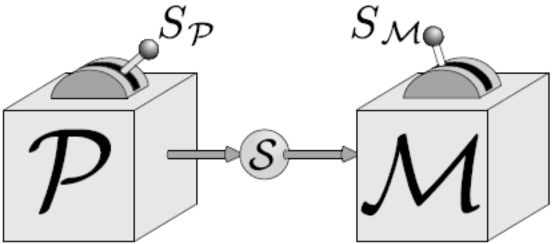
Big question

The long term goal of the proposed research is to develop new tools to study the differences in the quantum and classical aspects of quantum dynamical maps that are essential for quantum computation and quantum information theory. These tools should provide new insights into the necessary and sufficient conditions for quantum information to provide an advantage over classical information.

Are quasi-probability representations good tools?

Operational models



$$(\mathcal{P}, \mathcal{M}, K, \{\Pr(k|P \wedge M)\})$$

- R. W. Spekkens, *Contextuality for preparations, transformations, and unsharp measurements*, Physical Review A 71, 052108+ (2005).
- N. Harrigan and T. Rudolph, *Ontological models and the interpretation of contextuality*, arXiv:0709.4266 (2007).

Quantum modeling

Quantum theory is an instance of an operational model:

- $P \in \mathcal{P} \mapsto \rho_P \in \mathbb{D}(\mathcal{H})$.
- $(M, k) \in \mathcal{M} \times K \mapsto E_{M,k} \in \mathbb{E}(\mathcal{H})$.
- $\Pr(k|P \wedge M) = \mathrm{Tr}(\rho_p E_{M,k})$.

Ontological models

$$(\mathcal{P}, \mathcal{M}, K, \Lambda, \{\Pr(\lambda|P)\}, \{\Pr(k|M \wedge \lambda)\})$$

- Ontic space: Λ.
- $P \mapsto \Pr(\lambda|P)$.
- $(M, k) \mapsto \Pr(k|M \wedge \lambda)$.
- $\Pr(k|P \wedge M) = \sum_\lambda \Pr(k|M \wedge \lambda) \Pr(\lambda|P)$.

- R. W. Spekkens, *Contextuality for preparations, transformations, and unsharp measurements*, Physical Review A 71, 052108+ (2005).
- N. Harrigan and T. Rudolph, *Ontological models and the interpretation of contextuality*, arXiv:0709.4266 (2007).
- N. Harrigan, T. Rudolph, and S. Aaronson, *Representing probabilistic data via ontological models*, arXiv: 0709.1149 (2007).

Non-contextuality

A property of an ontological model:

- Preparation non-contextual if

$$\Pr(k|P \wedge M) = \Pr(k|P' \wedge M) \Rightarrow \Pr(\lambda|P) = \Pr(\lambda|P').$$

- Measurement non-contextual if

$$\Pr(k|P \wedge M) = \Pr(k|P \wedge M') \Rightarrow \Pr(k|M \wedge \lambda) = \Pr(k|M' \wedge \lambda).$$

- The model is called a *non-contextual ontological model* if it is both preparation and measurement non-contextual.

- R. W. Spekkens, *Contextuality for preparations, transformations, and unsharp measurements*, Physical Review A 71, 052108+ (2005).

Beltrametti-Bugajski model

A non-contextual ontological model of pure state quantum models:

- $\Lambda = \mathcal{H}$.
- $\Pr(\lambda|P) = \delta(\lambda - \psi_P)$.
- $\Pr(k|M \wedge \lambda) = \mathrm{Tr}[|\lambda\rangle\langle\lambda|E_{M,k}]$.
- $\Pr(k|P \wedge M) = \sum_\lambda \Pr(k|M \wedge \lambda) \Pr(\lambda|P)$.

- N. Harrigan and T. Rudolph, *Ontological models and the interpretation of contextuality*, arXiv:0709.4266 (2007).

Classical representations

Theorem

Suppose the ontological model $(\mathcal{P}, \mathcal{M}, K, \Lambda, \{\Pr(\lambda|P)\}, \{\Pr(k|M \wedge \lambda)\})$ *of the full quantum theory is preparation non-contextual. Then, there exists a affine mapping* $\mu : \mathrm{Ran}(\rho) \to Prob(\Lambda)$ *satisfying* $\mu(\rho_P) = \Pr(\lambda|P)$.

Similarly there exists an affine map $\xi(E_{M,k}) = \Pr(k|M \wedge \lambda)$ together with μ satisfy:

- $\mu_\rho(\lambda) \in [0, 1]$ and $\sum_\lambda \mu_\rho(\lambda) = 1$,
- $\xi_E(\lambda) \in [0, 1]$ and $\xi_{\mathbb{1}}(\lambda) = 1$,
- $\mathrm{Tr}(\rho E) = \sum_\lambda \mu_\rho(\lambda)\xi_E(\lambda)$.

- C. Ferrie and J. Emerson, *Framed Hilbert space: hanging the quasi-probability pictures of quantum theory*, New Journal of Physics 11, 063040+ (2009).

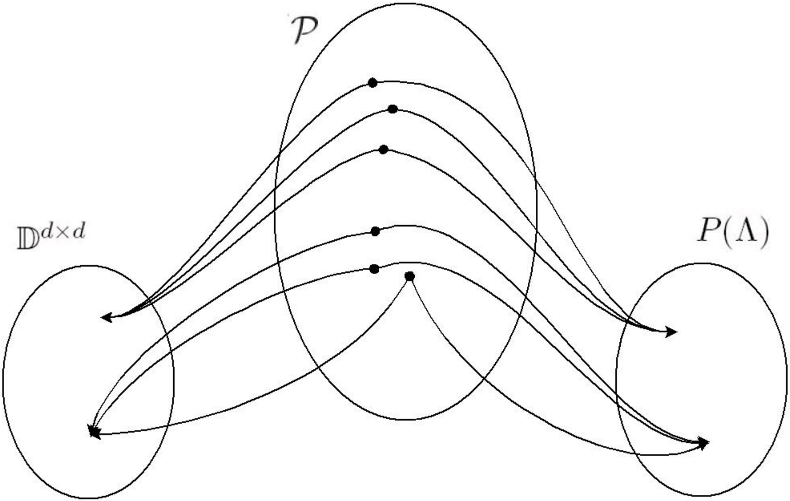
Negativity, non-classicality and contextuality

Theorem

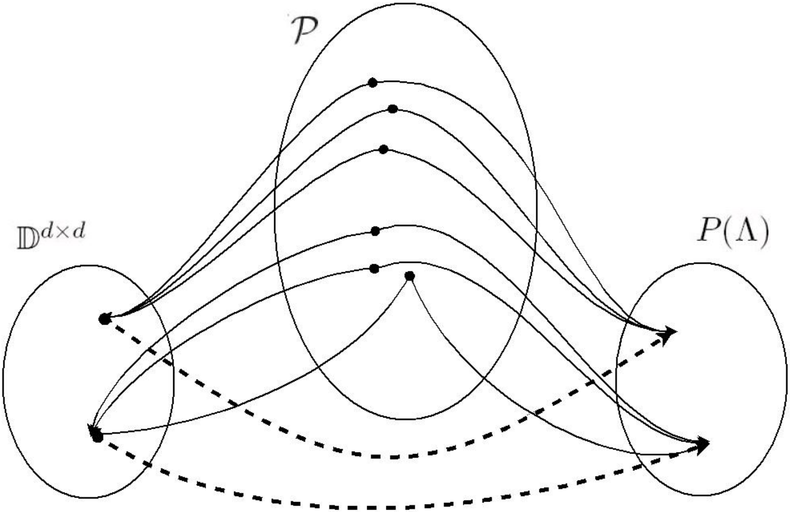
A classical representation of quantum theory does not exist.

- R. W. Spekkens, *Negativity and Contextuality are Equivalent Notions of Nonclassicality*, Physical Review Letters 101, 020401, (2008).
- C. Ferrie and J. Emerson, *Frame representations of quantum mechanics and the necessity of negativity in quasi-probability representations*, Journal of Physics A: Mathematical and Theoretical 41, 352001, (2008).
- C. Ferrie and J. Emerson, *Framed Hilbert space: hanging the quasi-probability pictures of quantum theory*, New Journal of Physics 11, 063040+ (2009).
- C. Ferrie and R. Morris, *Framing Hilbert space: building the quasi-probability representations to infinity*, arXiv:0910.3198 (2009).

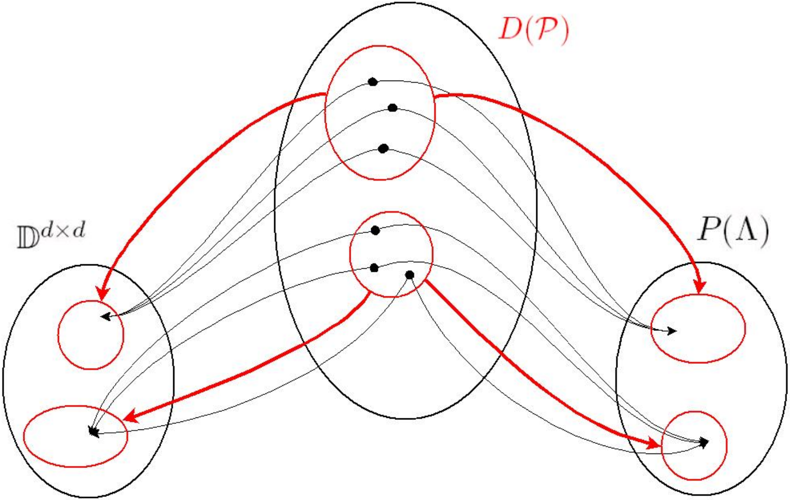
Non-contextuality



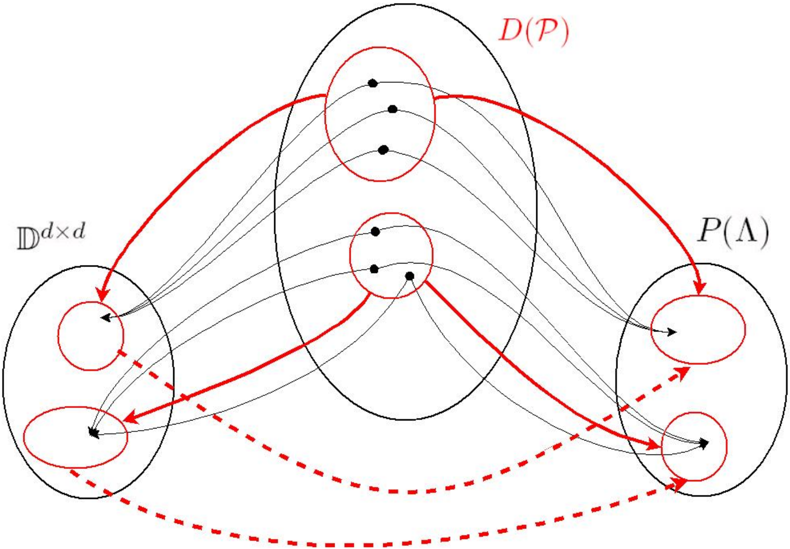
Non-contextuality



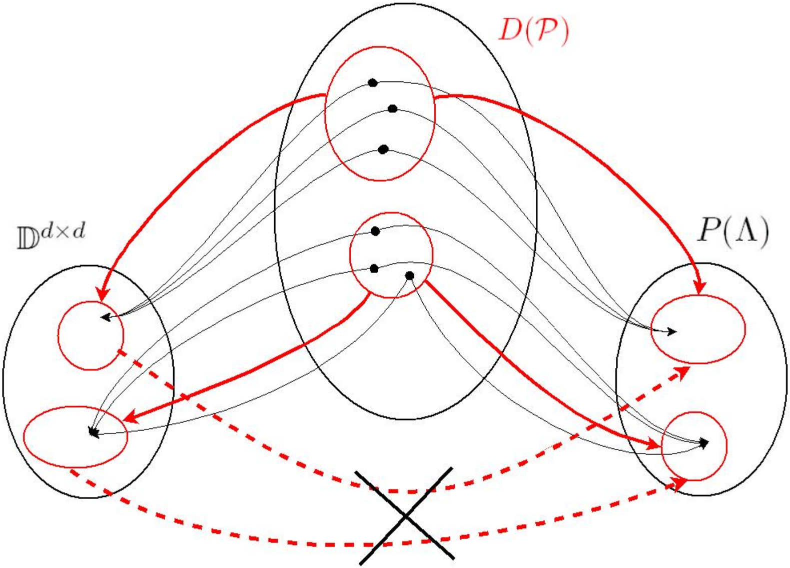
Non-non-contextuality



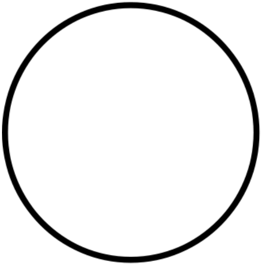
Non-non-contextuality

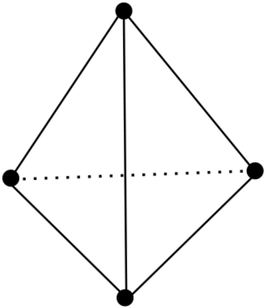


Non-non-contextuality



1000 word proof





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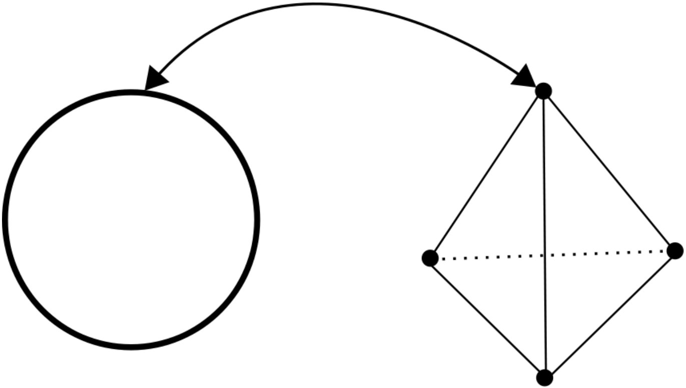

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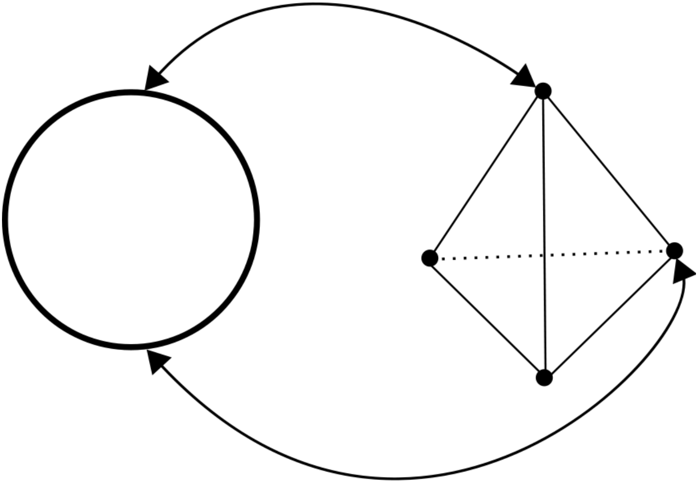

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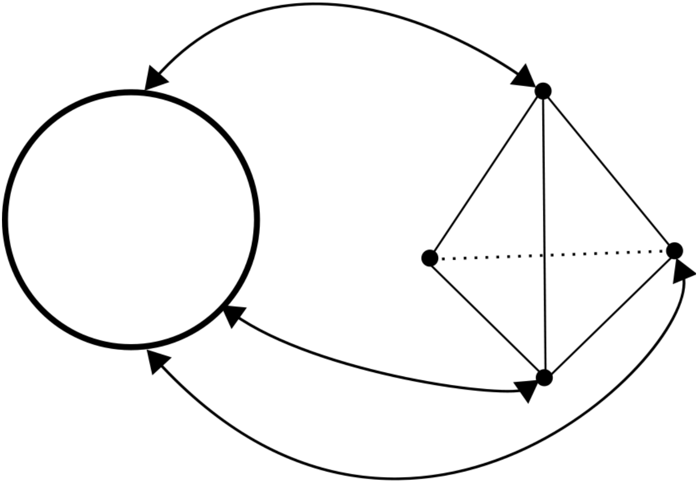

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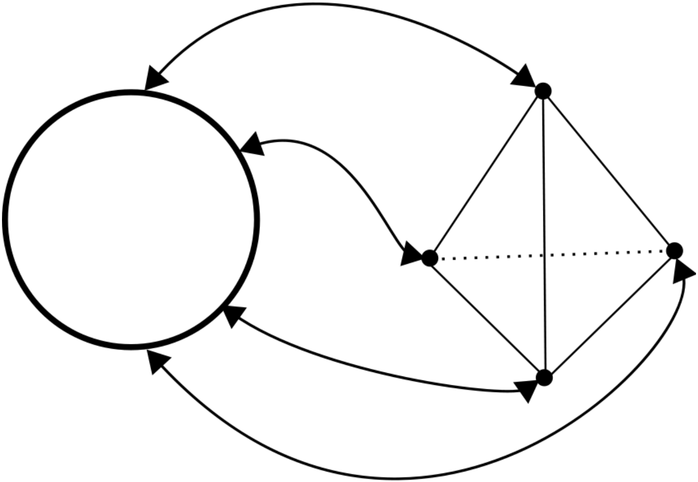

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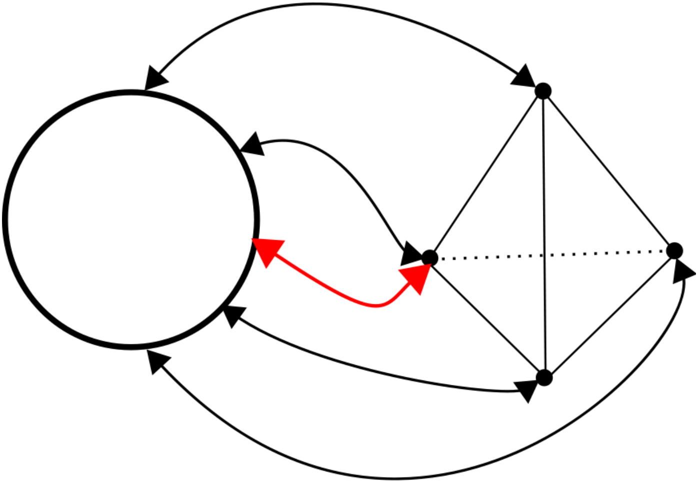

Quasi-probability representations

Take a classical representation:

- $\mu_\rho(\lambda) \in [0,1]$ and $\sum_\lambda \mu_\rho(\lambda) = 1$,
- $\xi_E(\lambda) \in [0,1]$ and $\xi_{\mathbb{1}}(\lambda) = 1$,
- $\mathrm{Tr}(\rho E) = \sum_\lambda \mu_\rho(\lambda)\xi_E(\lambda)$.

and relax it:

A *quasi-probability representation of quantum theory* is a pair of affine mappings μ and ξ which satisfy:

- $\mu_\rho(\lambda) \in \mathbb{R}$ and $\sum_\lambda \mu_\rho(\lambda) = 1$,
- $\xi_E(\lambda) \in \mathbb{R}$ and $\xi_{\mathbb{1}}(\lambda) = 1$,
- $\mathrm{Tr}(\rho E) = \sum_\lambda \mu_\rho(\lambda)\xi_E(\lambda)$.

- C. Ferrie and J. Emerson, *Framed Hilbert space: hanging the quasi-probability pictures of quantum theory*, New Journal of Physics 11, 063040+ (2009).

Quasi-probability functions

Table 1. Finite quasi-probability representations.

Author(s)	Year	Valid dimensions	Phase space	Index field	Redundancy	Quantum theory scope
Stratonovich [20]	1957	Any	Sphere	Polar coordinates	Continuous	States
Wootters [4]	1987	Prime[a]	$d \times d$ lattice	$\mathbb{Z}_d$	No	Standard
Cohendet *et al* [16]	1987	Odd	$d \times d$ lattice	$\mathbb{Z}_d$	No	States
Leonhardt [17]	1995	Even[b]	$2d \times 2d$ lattice	$\mathbb{Z}_{2d}$	Four-fold	States
Heiss and Weigert [19]	2000	Any	Sphere[c]	Arbitrary	No	States
Hardy [23]	2001	Any	None	n/a	No	Operational
Havel [24]	2003	Any	None	n/a	No	States
Gibbons *et al* [5]	2004	Power of prime	$d \times d$ lattice	$\mathbb{F}_d$	No	States
Ruzzi *et al* [25]	2005	Odd	$d \times d$ lattice	$\mathbb{Z}_d$	No	States
Chaturvedi *et al* [26]	2006	Any	$d \times d$ lattice	$\mathbb{Z}_d$	No	States
Gross [27]	2006	Odd	$d \times d$ lattice	$\mathbb{Z}_d$	No	States

[a] Wootters' original discrete Wigner function [4] is usually understood to be valid for prime dimension but, as discussed in section 2.2, is easily extended to any dimension by combining prime dimensional phase spaces.
[b] Leonhardt [17] also defines a discrete Wigner function valid for odd dimensional, which is equivalent to the other odd dimensional cases [16].
[c] The phase space of Heiss and Weigert is any subset of d points on the sphere that can be indexed arbitrarily.

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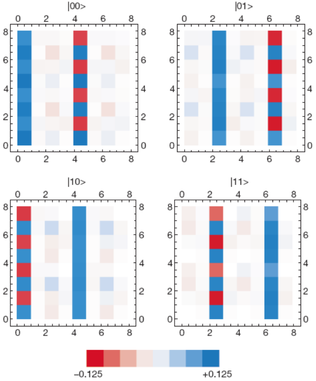
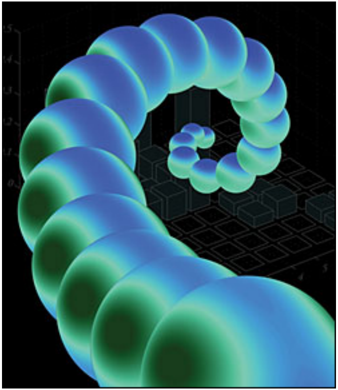
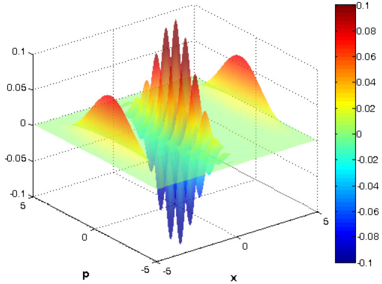
Examples

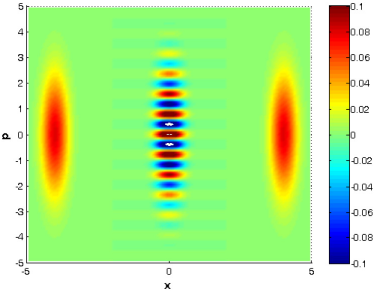




- C. Miquel *et al*, *Interpretation of tomography and spectroscopy as dual forms of quantum computation*, Nature 418, 59 (2002).
- L. K. Shalm, R. B. A. Adamson and A. M. Steinberg, *Squeezing and over-squeezing of triphotons*, Nature 457, 67 (2009).

Negativity as quantumness





Negativity as quantumness

Negativity as quantumness

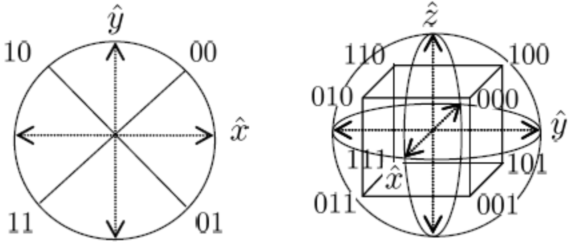
Negative about negativity

- A state with negativity in one representation is positive in another (and vice versa).
- Negativity *per se* is not a good notion of quantumness.

$\Rightarrow$ States aren't "quantum", it's what you do with them!

- Non-locality lead to idea of entanglement as a resource.
- Can negativity do the same?

Yes!



- Negativity and contextuality are "equivalent".
- Preparation contextuality leads to the idea of random access codes as a resource.

- R. W. Spekkens, *Negativity and Contextuality are Equivalent Notions of Nonclassicality*, Physical Review Letters 101, 020401, (2008).
- R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner, and G. J. Pryde, *Preparation Contextuality Powers Parity-Oblivious Multiplexing*, Physical Review Letters 102, 010401+ (2009).

Half full or half empty?

The good:

- Visualize quantum states removed of complex numbers.
- Exploit the contextuality connection.

The bad:

- Attempt to gain intuition from using a *discrete* phase space.

The ugly:

- Define "quantumness" as negativity in a *particular* representation.

Thank you!