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For How Long is it Possible to Quantum Compute?

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QAMF WORKSHOP

Vancouver, UBC - July 23-25, 2010

Overview

- Motivation and previous work
- Hamiltonian formulation of a fault-tolerant quantum processor
- Results

Motivation: *Protect Quantum Information from Decoherence*

Some standard strategies:

1) Decoherence free-subspaces

2) Topological systems

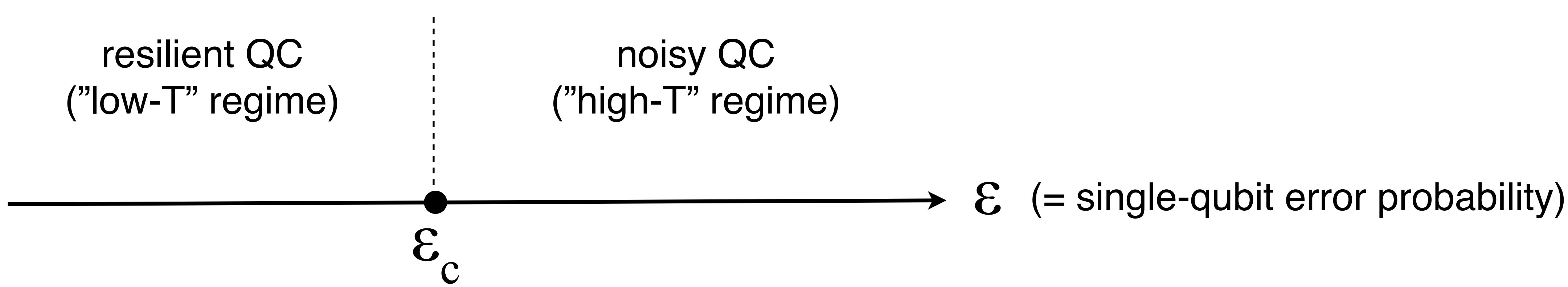
3) Dynamical decoupling

4) Quantum error correction → *likely the most versatile and universal approach*

- encode information
- measure ancilla qubits
- correct for errors

Major result: The "threshold theorem"

> *Provided that the noise level is below a certain critical value, quantum computation can be proceed "indefinitely"*



Possible Pitfalls of the Standard Fault-Tolerant QEC Theory:

1) The theorem is derived using stochastic error models (not Hamiltonians).

2) It implicitly assumes that perturbation theory works and converges.

3) Correlated environments and memory effects are not considered.

We avoided some of these problems in our previous work:

- Phys. Rev. Lett. **97**, 040501 (2006); **98**, 040501 (2007).

- Phys. Rev. A **78**, 012314 (2008).

- Hamiltonian, microscopic error models.
- Probabilistic calculations taking into account correlations.
- Description of how correlated environments affect the threshold theorem.

Quick review: Correlated environments and the Threshold Theorem

Novais, EM, Baranger [PRL **98**, 040501 (2007); PRA **78** 012314 (2008)]

Hamiltonian

$$H = H_{\text{computer}} + H_{\text{bath}} + V \longleftarrow$$ qubit-bath interaction: $$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha \, f_\alpha(\mathbf{x}) \, \sigma_\alpha(\mathbf{x})$$

- H_{bath}: environment's "free" Hamiltonian (gapless)
- $\sigma_\alpha(\mathbf{x})$: qubits
- $f_\alpha(\mathbf{x})$: environmental degrees of freedom

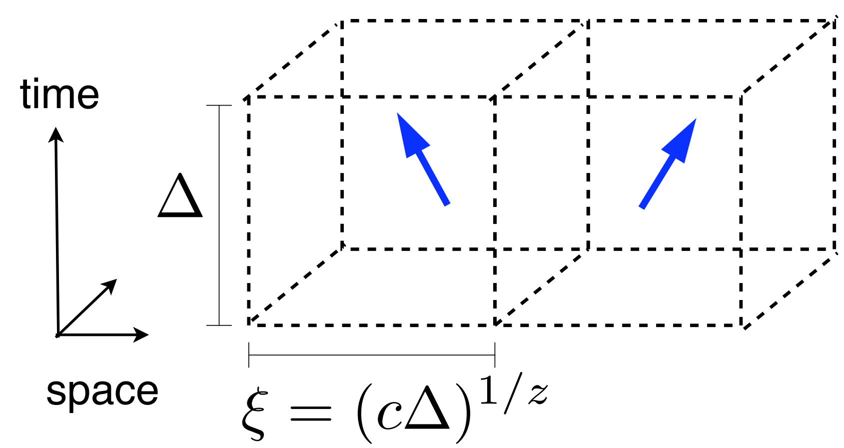
Three important parameters (critical environment)

- spatial dimension D
- bath mode velocity c
- dynamical exponent z

$$\langle f_\alpha(\mathbf{x}_1, t_1) \, f_\alpha(\mathbf{x}_2, t_2) \rangle_{\text{env}} \sim O\left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{2\delta_\alpha}}, \frac{1}{|t_1 - t_2|^{2\delta_\alpha / z}} \right)$$

Main hypothesis

only one qubit per hypercube



- defines a single-qubit error probability ϵ

- maps into an impurity problem for short times.

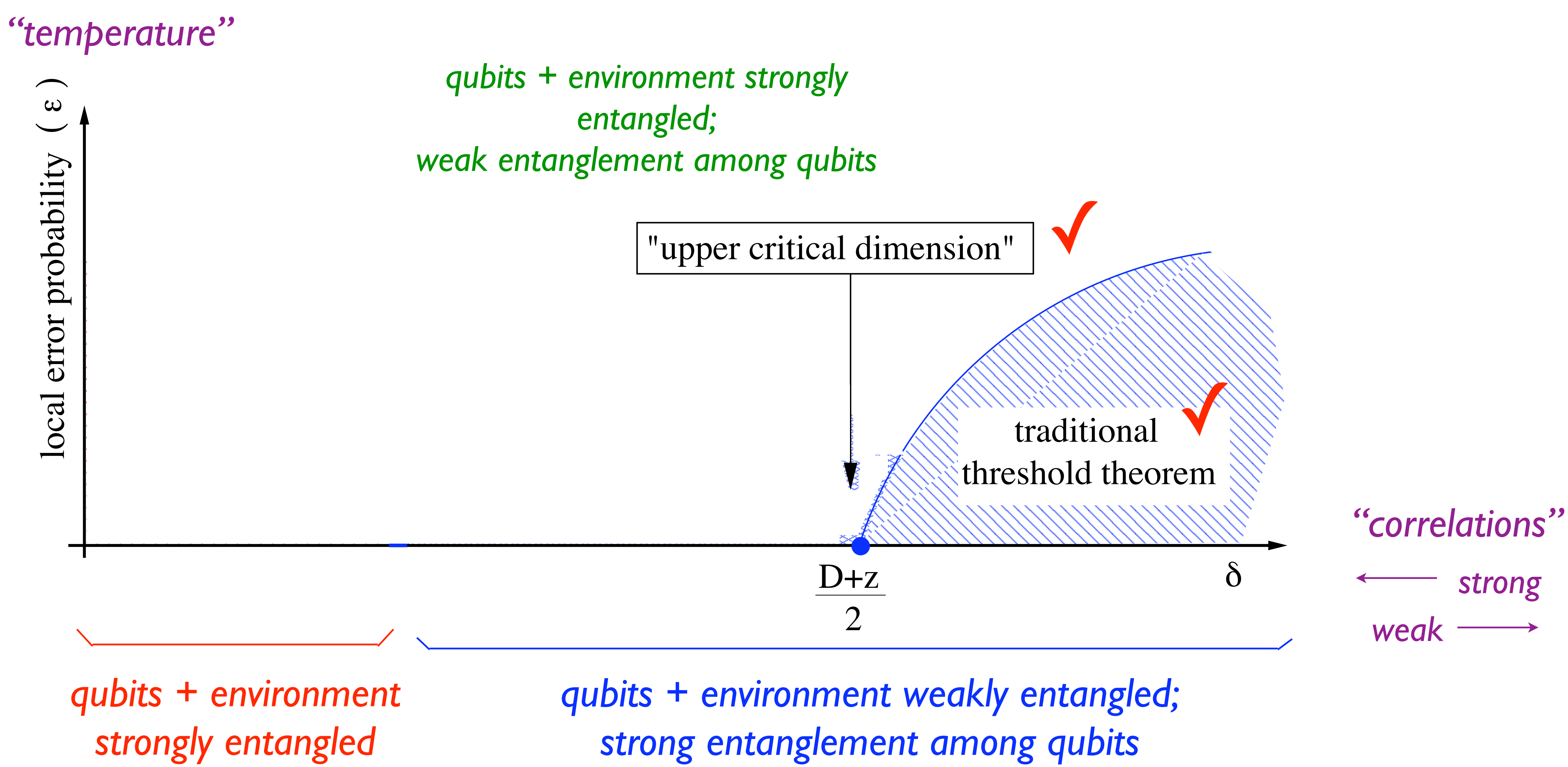
Δ = period of QEC cycle

Main result of previous work:

Novais, EM, Baranger [PRL **98**, 040501 (2007); PRA **78** 012314 (2008)]

$D + z < 2\delta_\alpha$ ⇒ the effect of long-range correlations are small; QEC and resilient quantum computing are ok.

$D + z > 2\delta_\alpha$ ⇒ correlations grow unbounded; threshold theorem derivation breaks down.



Current Work:

Novais, EM, Baranger [arXiv: 1004.3247]

1) Change the question: Given a certain desired error tolerance, for long can we compute using QEC?

2) Avoid the hypercube assumption.

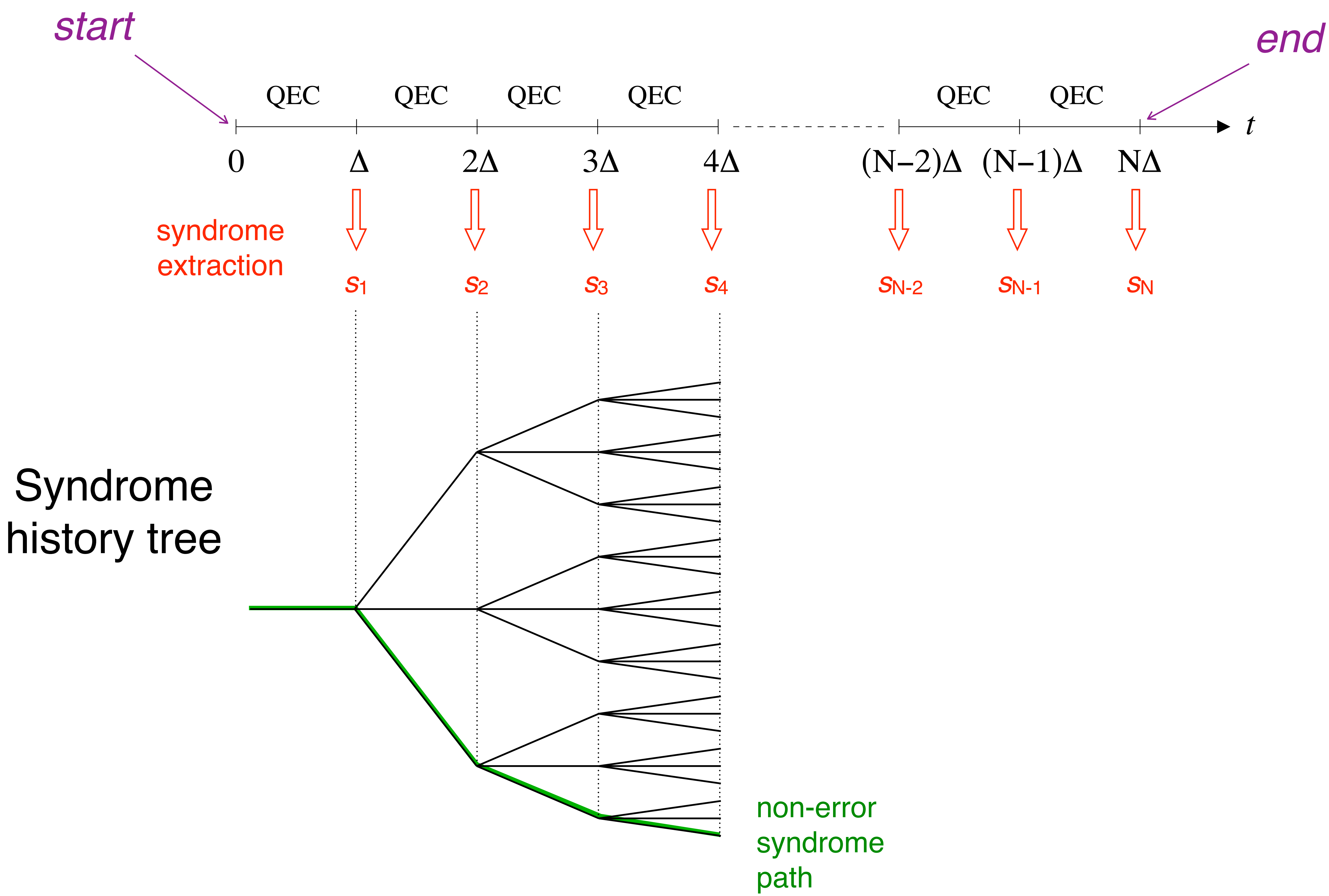
"friendly" assumptions

- standard quantum error correction code (circuit model QC).
- state preparation, gates, measurements all done perfectly.
- quantum evolution with only non-error syndromes.

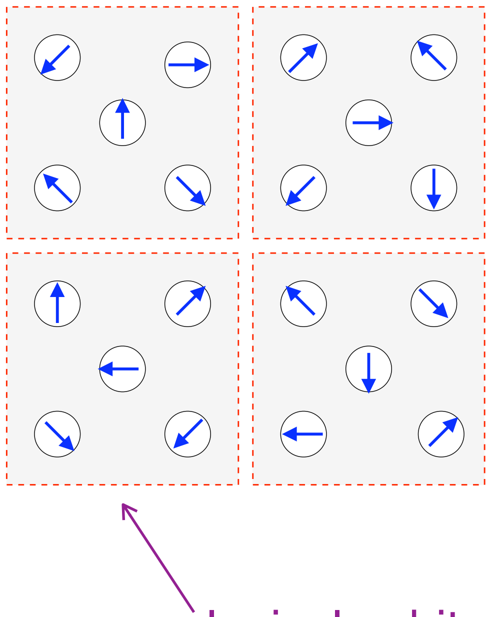
"unfriendly" assumptions

- gapless bath: power-law correlations (non exponential)
  (phonons, EM fluctuations, etc.)

Time-Evolution under QEC:



Taking into account the QEC code structure:



Example: 5-qubit code (corrects for one-qubit errors)

$$|\bar{0}\rangle = \frac{1}{4}\left[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \right.$$
$$\left. - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\right]$$

$$|\bar{1}\rangle = \frac{1}{4}\left[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \right.$$
$$\left. - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\right]$$

stabilizers and logical operations

$$g_1 = xzzx1$$
$$g_2 = 1xzzx$$
$$g_3 = x1xzz$$
$$g_4 = zx1xz$$

$$\bar{X} = xxxxx$$
$$\bar{Y} = yyyyy$$
$$\bar{Z} = zzzzz$$

$4^5 = 1024$ error operators

16 possible syndromes

but only 64 error operators can lead to a non-error syndrome

	$\bar{1}$	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
$\bar{1}$	11111	$xxxxx$	$yyyyy$	$zzzzz$
g_1	$xzzx1$	$1yy1x$	$zxxzy$	$y11yz$
g_2	$1xzzx$	$x1yy1$	$yzxxz$	$zy11y$
g_3	$x1xzz$	$1x1yy$	$zyzxx$	$yzy11$
g_4	$zx1xz$	$y1x1y$	$xzyzx$	$1yzy1$
h_1	$xy1yx$	$1zxz1$	$z1y1z$	$yxzxy$
h_2	$1zyyz$	$xyzzy$	$yx11x$	$z1xx1$
h_3	$yyz1z$	$zzyxy$	$11xyx$	$xx1z1$
h_4	$xxy1y$	$11zxz$	$zz1y1$	$yyxzx$
h_5	$z1zyy$	$yxyzz$	$xyx11$	$1z1xx$
h_6	$yxxy1$	$z11zx$	$1zz1y$	$xyyxz$
h_7	$1yxxy$	$xz11z$	$y1zz1$	$zxyyx$
h_8	$zyyz1$	$yzzyx$	$x11xy$	$1xx1z$
h_9	$y1yxx$	$zxz11$	$1y1zz$	$xzxyy$
h_{10}	$yz1zy$	$zyxyz$	$1xyx1$	$x1z1x$
h_{11}	$zzx1x$	$yy1x1$	$xxzyz$	$11yzy$

Model for the critical environment:

physical qubits

$$H_I = \sum_{\mathbf{x}} \sum_{\alpha=\{x,z\}} \lambda_\alpha \, f^\alpha(\mathbf{x}) \, \sigma^\alpha(\mathbf{x})$$

qubit-bath interaction

bosonic bath

$$f^\alpha(\mathbf{x}) = \sum_{\mathbf{k}\neq 0} \left(g_{\alpha,|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}} a^\dagger_{\alpha,\mathbf{k}} + g^*_{\alpha,|\mathbf{k}|} e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\alpha,\mathbf{k}} \right)$$

$$H_B = \sum_{\alpha=\{x,z\}} \sum_{\mathbf{k}\neq 0} \omega_{\alpha,|\mathbf{k}|} \, a^\dagger_{\alpha,\mathbf{k}} a_{\alpha,\mathbf{k}} \qquad \left[a_{\alpha,\mathbf{k}}, a^\dagger_{\beta,\mathbf{k}'} \right] = \delta_{\alpha\beta} \, \delta_{\mathbf{k},\mathbf{k}'}$$

form factor: $|g_{\alpha,|\mathbf{k}|}|^2 \sim |\mathbf{k}|^{2s_\alpha}$

dispersion relation: $\omega_{\alpha,|\mathbf{k}|} \sim |\mathbf{k}|^{z_\alpha}$

spectral function

$$J_\alpha(\omega) \sim \lambda_\alpha^2 \, \omega^{(D+2s_\alpha - z_\alpha)/z_\alpha}$$

$$J(\omega) = \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \, \delta(\omega - \omega_k)$$

Time evolution: First QEC cycle

system+bath evolution operator

$$U(\Delta,0) = T_t\, e^{-i\int_0^\Delta dt\, H_I(t)} = 1 - i\int_0^\Delta dt\, H_I(t) - \frac{1}{2}\int_0^\Delta dt \int_0^t dt'\, H_I(t)\, H_I(t') + \ldots$$

lowest order perturbation theory (short times / small coupling)

$$U(\Delta,0) \approx 1 - i \sum_{\alpha=x,z}\sum_{\mathbf{x}} \lambda_\alpha\, \Delta\, f^\alpha(\mathbf{x},0)\, \sigma^\alpha_{\mathbf{x}}$$

$\boxed{\lambda_\alpha\,\Delta \ll 1}$ expansion parameter

syndrome extraction and error correction operation:

$$U(\Delta,0) \longrightarrow v_s(\Delta,0)$$

evolution operator for *logical qubits* after a non-error syndrome

$$\bar{v}_0(\Delta,0) \approx \bar{1} + i\Delta^3 \sum_{\mathbf{x}} \sum_{\alpha,\beta=\{x,z\}} \sum_{i,j,k} \eta^{\alpha\beta}_{ijk}\, \lambda_\alpha \lambda_\beta^2\, f^\alpha(\mathbf{x}_i,0)\, f^\beta(\mathbf{x}_j,0)\, f^\beta(\mathbf{x}_k,0)\, \bar{\sigma}^\alpha_{\mathbf{x}} + O(\lambda^5)$$

numerical coefficients

$O(\lambda^2)$ renormalization dropped

Determining the expansion coefficients:

operators leading to a non-error syndrome

	$\bar{1}$	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
$\bar{1}$	11111	$xxxxx$	$yyyyy$	$zzzzz$
g_1	$xzzx1$	$1yy1x$	$zxxzy$	$y11yz$
g_2	$1xzzx$	$x1yy1$	$yzxxz$	$zy11y$
g_3	$x1xzz$	$1x1yy$	$zyzxx$	$yzy11$
g_4	$zx1xz$	$y1x1y$	$xzyzx$	$1yzy1$
h_1	$xy1yx$	$1zxz1$	$z1y1z$	$yxzxy$
h_2	$1zyyz$	$xyzzy$	$yx11x$	$z1xx1$
h_3	$yyz1z$	$zzyxy$	$11xyx$	$xx1z1$
h_4	$xxy1y$	$11zxz$	$zz1y1$	$yyxzx$
h_5	$z1zyy$	$yxyzz$	$xyx11$	$1z1xx$
h_6	$yxxy1$	$z11zx$	$1zz1y$	$xyyxz$
h_7	$1yxxy$	$xz11z$	$y1zz1$	$zxyyx$
h_8	$zyyz1$	$yzzyx$	$x11xy$	$1xx1z$
h_9	$y1yxx$	$zxz11$	$1y1zz$	$xzxyy$
h_{10}	$yz1zy$	$zyxyz$	$1xyx1$	$x1z1x$
h_{11}	$zzx1x$	$yy1x1$	$xxzyz$	$11yzy$

$$\begin{aligned}
\eta^{xy}_{1524} = \eta^{xy}_{1235} = \eta^{xy}_{2314} = \eta^{xy}_{3425} = \eta^{xy}_{4513} &= 1 \\
\eta^{xz}_{1423} = \eta^{xz}_{2534} = \eta^{xz}_{1345} = \eta^{xz}_{2415} = \eta^{xz}_{3512} &= 1 \\
\eta^{yz}_{3425} = \eta^{yz}_{1235} = \eta^{yz}_{4513} = \eta^{yz}_{2314} = \eta^{yz}_{1524} &= 1
\end{aligned}$$

(all other coefficients vanish)

Notice: 3 or 5 insertions of the interaction only
(one per physical qubit, at most)

Time evolution (Cont.):

rearrangement (separating *intra* versus *inter* logical qubit correlations)

$$\bar{v}_0(\Delta,0) \approx \bar{1} + i\Delta \sum_{\mathbf{x}} \sum_{\alpha=\{x,z\}} (\lambda^*_\alpha + \Gamma_\alpha) f^\beta(\mathbf{x},0)\, \bar{\sigma}^\alpha_{\mathbf{x}}$$

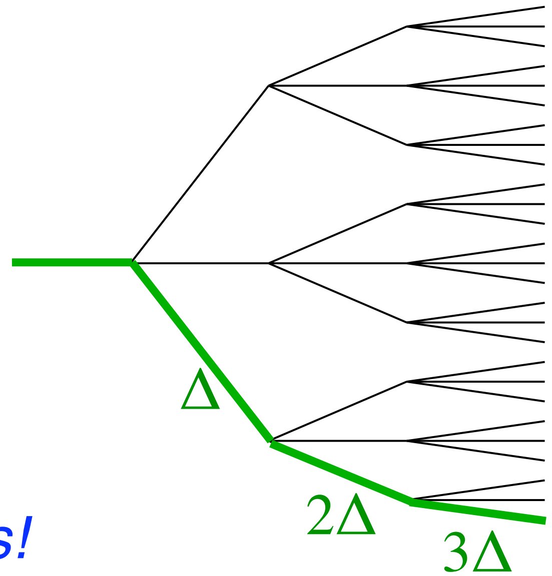
effective coupling constant $\lambda^*_\alpha = \lambda_\alpha \sum_{\mathbf{x}} \sum_{\beta=\{x,z\}} \sum_{i,j,k} \eta^{\alpha\beta}_{ijk} (\lambda_\beta \Delta)^2 \sum_{\mathbf{k}\neq 0} |g_{\alpha,|\mathbf{k}|}|^2 \, e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}$

repeat for subsequent cycles

$$\bar{U}(N\Delta,0) = \bar{v}_0(N\Delta,(N-1)\Delta)\, \bar{v}_0((N-1)\Delta,(N-2)\Delta) \ldots \bar{v}_0(\Delta,0)$$

quantum evolution of logical qubit after *N* QEC cycles:

$$\bar{U}(T=N\Delta,0) \approx T_t\, e^{i\int_0^T dt \sum_{\mathbf{x}} \sum_{\alpha=\{x,z\}} \lambda^*_\alpha\, f^\alpha(\mathbf{x},t)\, \bar{\sigma}^\alpha_{\mathbf{x}}}$$



Same functional form as for physical (unprotected) qubits!
Main effect of QEC: renormalization of coupling constant

Upper bound to computation time

- Trace distance as a measure of information degradation

$$D\big(\rho_R(T) - \rho_{\mathrm{ideal}}\big) = \frac{1}{2}\mathrm{Tr}\,\|\rho_R(T) - \rho_{\mathrm{ideal}}\| \qquad \|A\| = \sqrt{A^\dagger A}$$

- Information lost by a single logical qubit

$$D\big(\rho_R(T) - \rho_{\mathrm{ideal}}\big) = \sqrt{|\delta\sigma^+(T)|^2 + \frac{1}{4}[\delta\sigma^z(T)]^2} \qquad \delta\sigma^\alpha(T) = \langle\sigma^\alpha(T)\rangle - \langle\sigma^\alpha\rangle$$

- Given a certain D_{thresh}, one can compute the available time T_{max} by solving

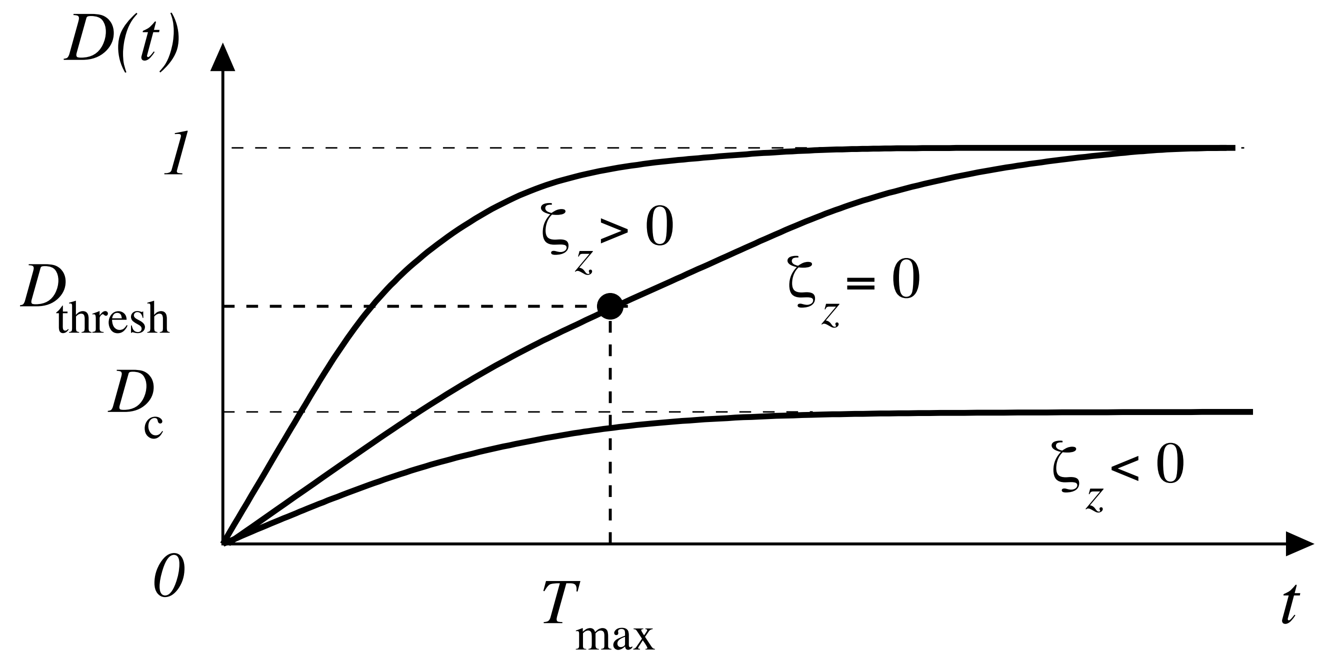
$$D\big(\rho_R(T_{\mathrm{max}}) - \rho_{\mathrm{ideal}}\big) = D_{\mathrm{thresh}}$$

Example: *isolated logical qubit, no transversal coupling (λ_x =0)*

Long-time behavior controlled by the exponent

$$\zeta_z \equiv 2(z_z - s_z) - \mathrm{D}$$

bath spatial dimension



$$T_{\max} \approx \Delta \begin{cases} \infty & \zeta_z < 0 \\ \exp\left[C\, D_{\mathrm{thresh}} \left(\frac{\omega_0}{\lambda_z^*}\right)^2\right] & \zeta_z = 0 \\ \frac{D_{\mathrm{thresh}}^{z_z/\zeta_z}}{\omega_0 \Delta} \left(\frac{\omega_0}{\lambda_z^*}\right)^{2z_z/\zeta_z} & 0 < \zeta_z < 2z_z \\ \left(\frac{2\pi}{k_0 L}\right)^{(\zeta_z - 2z_z)} \frac{\sqrt{D_{\mathrm{thresh}}}}{\lambda_z^* \Delta} & \zeta_z > 2z_z \end{cases}$$

"superohmic" ($\zeta_z < 0$)

"ohmic" ($\zeta_z = 0$)

"subohmic" ($0 < \zeta_z < 2z_z$ and $\zeta_z > 2z_z$)

Another example: *logical qubit array (M qubits, array dimension D_q)*

$\zeta_\alpha \equiv 2(z_\alpha - s_\alpha) - \mathrm{D}$ ⟵ if intra-logical qubit correlations dominate *(sparse array)*

$\zeta_\alpha \equiv 2(z_\alpha - s_\alpha) - \mathrm{D} + \mathrm{D}_q$ ⟵ if inter-logical qubit correlations dominate *(dense array)*

$$T_{\max} \approx \Delta \times \begin{cases} \infty & \zeta_\alpha < 0 \\ \exp\left[B\,\frac{D_{\mathrm{thresh}}}{M(\lambda^*_\alpha/\omega_0)}\right] & \zeta_\alpha = 0 \\ \frac{1}{\omega_0\Delta}\left[\frac{D_{\mathrm{thresh}}}{M(\lambda^*_\alpha/\omega)}\right]^{z_\alpha/\zeta_\alpha} & 0 < \zeta_\alpha < z_\alpha \\ \left(\frac{2\pi}{k_0 L}\right)^{\zeta_\alpha - z_\alpha} \frac{D_{\mathrm{thresh}}}{M(\lambda^*_\alpha\Delta)} & \zeta_\alpha > z_\alpha \end{cases}$$

best case ($\zeta_\alpha < 0$)

marginal case ($\zeta_\alpha = 0$)

worst case ($\zeta_\alpha > z_\alpha$)

compromise between array size, code effectiveness, and amplitude of coupling to environment

M large ⇔ λ^* small

Notice: Different layers of concatenation only alter λ_α^* (higher power in the bare λ_α)

Case 1: 2D sparse logical qubit array; level-0 concatenation
GaAs double-quantum dot charge physical qubits
bath: piezoelectric acoustic phonons (3D)

$J(\omega) \sim \omega^3$

$$s_z = 1/2$$
$$z_z = 1$$
$$D = 3$$

$\rightarrow$ $\zeta_z = 0$ $\rightarrow$ $$T_{\max} \approx \Delta \exp\left[B \frac{D_{\mathrm{thresh}}}{M(\lambda_z^*/\omega_0)}\right]$$

marginal case

Case 2: 1D dense logical qubit array; level-0 concatenation
superconductor charge qubits
bath: electromagnetic (gate voltage) fluctuations (effectively 1D)

$J(\omega) \sim \omega$

$$s_z = 1/2$$
$$z_z = 1$$
$$D = 1$$
$$D_q = 1$$

$\rightarrow$ $\zeta_z = 1$ $\rightarrow$ $$T_{\max} \approx \omega_0 \left[\frac{D_{\mathrm{thresh}}}{M(\lambda_z^*/\omega_0)}\right]$$

"bad" case

Summary

- We developed a Hamiltonian formulation of QEC in the presence of critical environments. Details of error code and concatenation level are incorporated. The non-Markovian dynamics of the bath is taken into account.

- The effective, long-time evolution operator of the *logical* qubits has the same functional form of the original operator for *physical* qubits, but with a renormalized coupling constant.

- Given a certain final state tolerance, we developed an estimate of the maximum computational time for a non-error syndrome history; it is straightforward to estimate for other histories (will yield shorter computational times).

Funding:

Resources:

- Phys. Rev. Lett. **97**, 040501 (2006); **98**, 040501 (2007).
- Phys. Rev. A **78**, 012314 (2008).
- arXiv:1004.3247
- links to publications in http://www.physics.ucf.edu/~mucciolo/

THANKS!