Why the quantum?
Insights from classical theories with a statistical restriction

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QAMF workshop, UBC

Classical statistical theory

+

fundamental restriction on statistical distributions

$\Downarrow$

A large part of quantum theory

In the sense of reproducing the operational predictions

Classical statistical theory

+

fundamental restriction on statistical distributions

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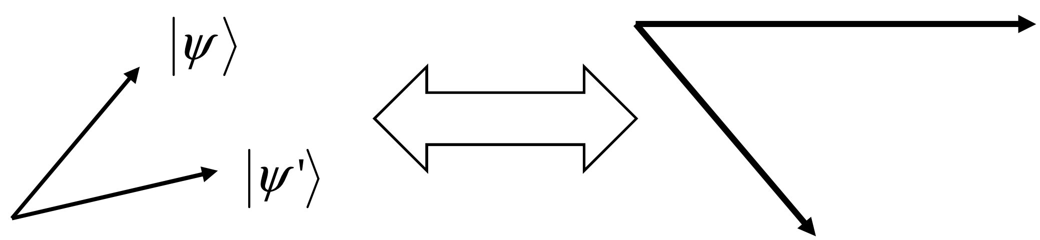
A large part of quantum theory

In the sense of reproducing the operational predictions



i.e. quantum states emerge as statistical distributions (epistemic states)

Classical theory	Statistical theory for the classical theory	Restricted Statistical theory for the classical theory
Mechanics	Liouville mechanics	Restricted Liouville mechanics = Gaussian quantum mechanics
Bits	Statistical theory of bits	Restricted statistical theory of bits $\simeq$ Stabilizer theory for qubits
Trits	Statistical theory of trits	Restricted statistical theory of trits = Stabilizer theory for qutrits

These theories include:

- **Most basic quantum phenomena**

e.g. noncommutativity, Interference, coherent superposition, collapse, complementary bases, no-cloning, …

- **Most quantum information-processing tasks**

e.g. teleportation, key distribution, quantum error correction, improvements in metrology, dense coding, …

- **A large part of entanglement theory**

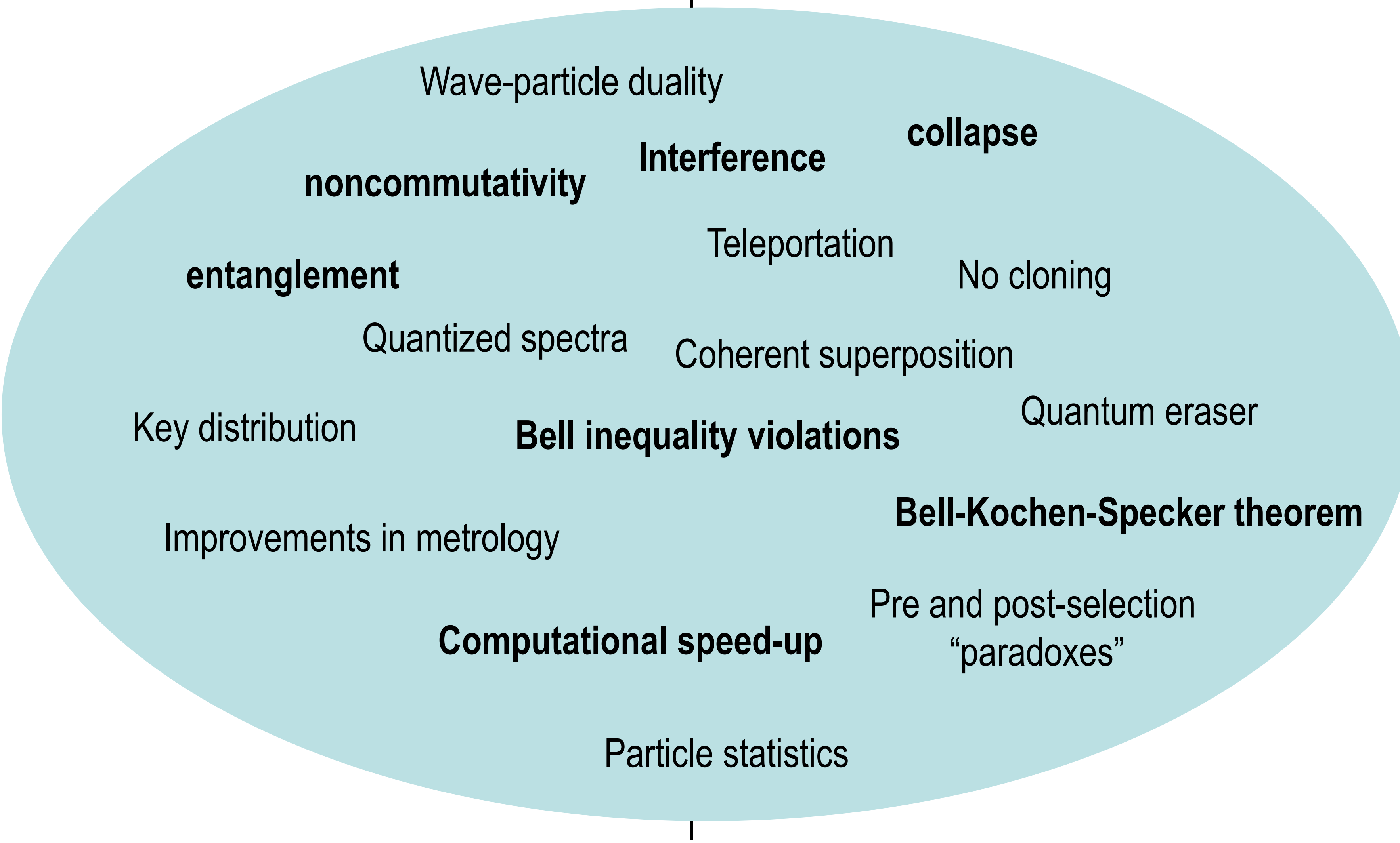
e.g. monogamy, distillation, deterministic and probabilistic single copy entanglement transformation, catalysis, …

- **A large part of the formalism of quantum theory**

e.g. Choi-Jamiolkowski isomorphism, Naimark extension, Stinespring dilation, multiple convex decompositions of states, …

Categorizing quantum phenomena

Those arising in a restricted statistical classical theory | **Those not arising in a restricted statistical classical theory**



Categorizing quantum phenomena

Those arising in a restricted statistical classical theory	**Those not arising in a restricted statistical classical theory**
Interference	**Bell inequality violations**
Noncommutativity	**Computational speed-up**
Entanglement	**Bell-Kochen-Specker theorem**
Collapse	Certain aspects of items on the left
Wave-particle duality	Others…
Teleportation	
No cloning	
Key distribution	
Improvements in metrology	
Quantum eraser	
Coherent superposition	
Pre and post-selection "paradoxes"	
Others…	

Quantized spectra?
Particle statistics?
Others…

Categorizing quantum phenomena

Those arising in a restricted statistical classical theory

Interference
Noncommutativity
Entanglement
Collapse
Wave-particle duality
Teleportation
No cloning

Not so strange after all!

Improvements in metrology
Quantum eraser
Coherent superposition
Pre and post-selection "paradoxes"
Others…

Those not arising in a restricted statistical classical theory

Bell inequality violations
Computational speed-up
Bell-Kochen-Specker theorem
Certain aspects of items on the left
Others…

Still surprising!
Find more!
Focus on these

Quantized spectra?
Particle statistics?
Others…

A research program

Speculative possibility for an axiomatization of quantum theory

Principle 1: There is a fundamental restriction on observers capacities to know and control the systems around them

Principle 2: **???** (Some change to the classical picture of the world)

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Classical theory	Statistical theory for the classical theory	Restricted Statistical theory for the classical theory
Mechanics	Liouville mechanics	Restricted Liouville mechanics = **Quadrature** quantum mechanics
Bits	Statistical theory of bits	Restricted statistical theory of bits $\simeq$ Stabilizer theory for qubits
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Classical complementarity as a statistical restriction with broad applicability

Joint work with Olaf Schreiber

Building upon:
RS, quant-ph/0401052 [Phys. Rev. A 75, 032110 (2007)]
S. van Enk, arxiv:0705.2742 [Found. Phys. 37, 1447 (2007)]
D. Gross, quant-ph/0602001 [J. Math. Phys. 47, 122107 (2006)]
S. Bartlett, T. Rudolph, RS, unpublished

A fact about operational quantum theory:

Jointly-measurable observables	=	a commuting set of observables (relative to matrix commutator)

This suggests a restriction on a classical statistical theory:

Jointly-knowable variables	=	a commuting set of variables (relative to Poisson bracket)

Continuous degrees of freedom

Configuration space: $\mathbb{R}^n \ni (x_1, x_2, \ldots, x_n)$

Phase space: - $\equiv \mathbb{R}^{2n} \ni (x_1, p_1, x_2, p_2, \ldots, x_n, p_n) \equiv m$

Functionals on phase space: F : - $\rightarrow \mathbb{R}$

$$X_k(m) = x_k$$
$$P_k(m) = p_k$$

Poisson bracket of functionals:

$$[F, G](m) \equiv \sum_{i=1}^{n} \left(\frac{\partial F}{\partial X_i}\frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial X_i}\right)(m)$$

The linear functionals / canonical variables are:

$$F = a_1 X_1 + b_1 P_1 + \cdots + a_n X_n + b_n P_n \qquad a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$$
$$G = c_1 X_1 + d_1 P_1 + \cdots + c_n X_n + d_n P_n \qquad c_1, d_1, \ldots, c_n, d_n \in \mathbb{R}$$

$[F, G](m) \equiv \sum_{i=1}^{n} (a_i d_i - b_i c_i)$ Independent of *m*

Discrete degrees of freedom $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$

Configuration space: $(\mathbb{Z}_d)^n \ni (x_1, x_2, \ldots, x_n)$

Phase space: - $\equiv (\mathbb{Z}_d)^{2n} \ni (x_1, p_1, x_2, p_2, \ldots, x_n, p_n) \equiv m$

Functionals on phase space: F : - $\rightarrow \mathbb{Z}_d$

$$X_k(m) = x_k$$
$$P_k(m) = p_k$$

Poisson bracket of functionals:

$$[F,G](m) \equiv \sum_{i=1}^{n} \; (F[m + e_{x_i}] - F[m])(G[m + e_{p_i}] - G[m])$$
$$-(F[m + e_{p_i}] - F[m])(G[m + e_{q_i}] - G[m])$$

The linear functionals / canonical variables are:

$$F = a_1 X_1 + b_1 P_1 + \cdots + a_n X_n + b_n P_n \qquad a_1, b_1, \ldots, a_n, b_n \in \mathbb{Z}_d$$
$$G = c_1 X_1 + d_1 P_1 + \cdots + c_n X_n + d_n P_n \qquad c_1, d_1, \ldots, c_n, d_n \in \mathbb{Z}_d$$

$$[F,G](m) = \sum_{i=1}^{n} [F(e_{x_i}) G(e_{p_i}) - F(e_{p_i}) G(e_{x_i})]$$
$$= \sum_{i=1}^{n} (a_i d_i - b_i c_i)$$ Independent of *m*

A canonically conjugate pair $[F, G] = 1$

e.g. $\{X_1, P_1\}, \{X_2, P_2\}$, and $\{X_1 + X_2, P_1 + P_2\}$

A commuting pair $[F, G] = 0$

e.g. $\{X_1, X_2\}, \{X_1, P_2\}$, and $\{X_1 - X_2, P_1 + P_2\}$

The principle of classical complementarity:

An observer can only have knowledge of the values of a commuting set of canonical variables and is maximally ignorant otherwise.

Symplectic geometry

Symplectic inner product $\omega : - \times - \to \mathbb{R}$

$$\omega(m, m') = m^T J m' \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 & & & \cdots \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Thus

$$\omega(m, m') = \sum_i \left(q_i p'_i - p_i q'_i \right)$$

The linear functionals

$$F = \sum_i (a_i X_i + b_i P_i)$$

form a dual space $\Omega^* \equiv (\mathbb{R})^{2n} \ni (a_1, b_1, \ldots, a_n, b_n)$

$\{X_1, P_1, \ldots, X_n, P_n\}$ is dual to $\{e_{x_1}, e_{p_1}, \ldots, e_{x_n}, e_{p_n}\}$

$$\begin{aligned} \omega(F, G) &= \sum_{i=1}^{n} (a_i d_i - b_i c_i) \\ &= [F, G] \end{aligned}$$

Poisson bracket of functionals = symplectic inner product of vectors

Valid epistemic states:

These are specified by:

A set of known variables $\mathcal{V}$

$$\forall F, G \in \mathcal{V} : [F, G] = 0$$

A valuation of the known variables

$$v : \mathcal{V} \to \mathbb{R}(\mathbb{Z}_d)$$

Example:

$$\mathcal{V} = \{X_1, P_2\}$$

$$v(X_1) = 2, v(P_2) = 2$$

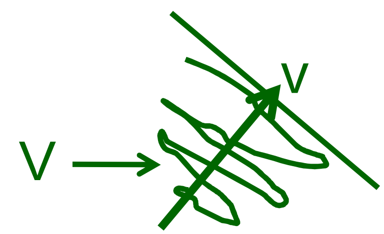
Equivalently,

An isotropic subspace $V \subseteq \Omega^*$

$$\forall F, G \in V : \omega(F, G) = 0$$

A valuation vector $v \in V^* \subseteq \Omega$

$$v : \forall F \in V, \quad F^T v = v(F)$$



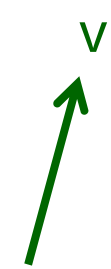
The ontic states consistent with the epistemic state (V,v) are

$$\{m \in - \mid \forall F \in \mathcal{V} : F(m) = v(F)\}$$
$$= \{m \in - \mid \forall F \in V : F^T m = F^T v\}$$
$$= \{m \in - \mid P_V m = v\}$$
$$\equiv V^{\perp} + v$$

(Dirac-delta / Kronecker delta)

The associated distribution is

$$p_{V,v}(m) = \frac{1}{N}\delta_{V^{\perp}+v}(m)$$



Example

$$\mathcal{V} = \{X_1, X_2\}$$
$$v(X_1) = 1, v(X_2) = 2$$

$$V^{\perp} + v = \{m \in - \mid X_1(m) = 1,\ X_2(m) = 2\}$$
$$= \{(1, s, 2, t) \mid s, t \in \mathbb{R}\}$$

“Heisenberg picture” and “Schrodinger picture”

Valid reversible transformations:

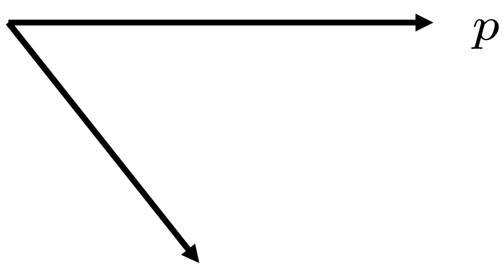
Those that preserve the Poisson bracket / symplectic inner product:
The group of symplectic affine transformations (Clifford group)

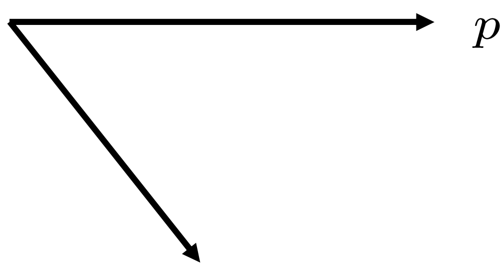
for $m \in \Omega$

$$m \mapsto Sm + a$$

where $[Su, Sv] = [u, v]$ Symplectic

and $a \in \Omega$ Affine (Heisenberg-Weyl)



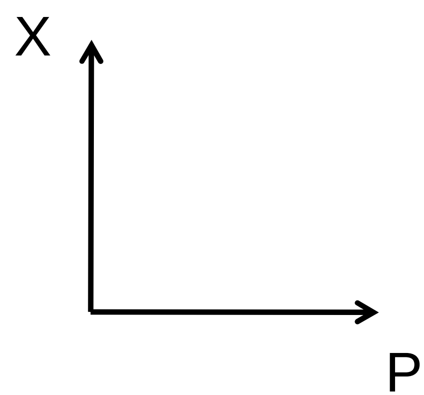


Valid reproducible measurements:

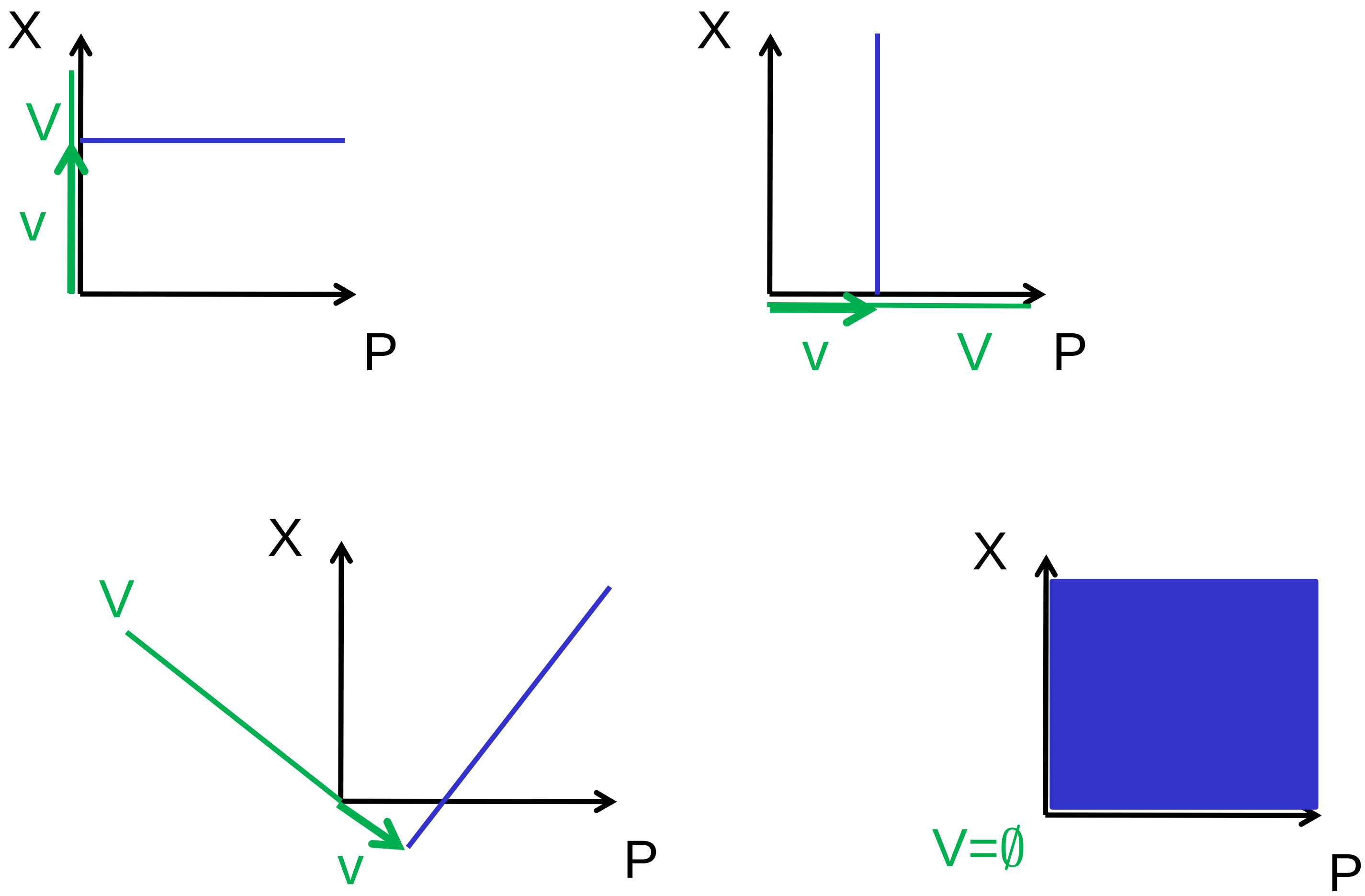
Any commuting set of canonical variables

Restricted Liouville mechanics

- = $\mathbb{R}^{2n}$



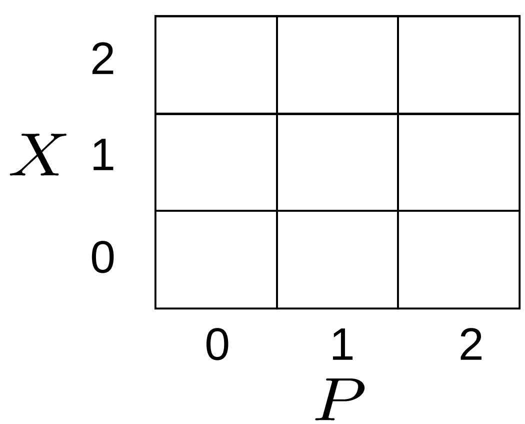
Valid epistemic states for a single degree of freedom



Valid epistemic states for a pair of degrees of freedom

Restricted statistical theory of **trits**

$$- \;=\; (\mathbb{Z}_3)^{2n}$$

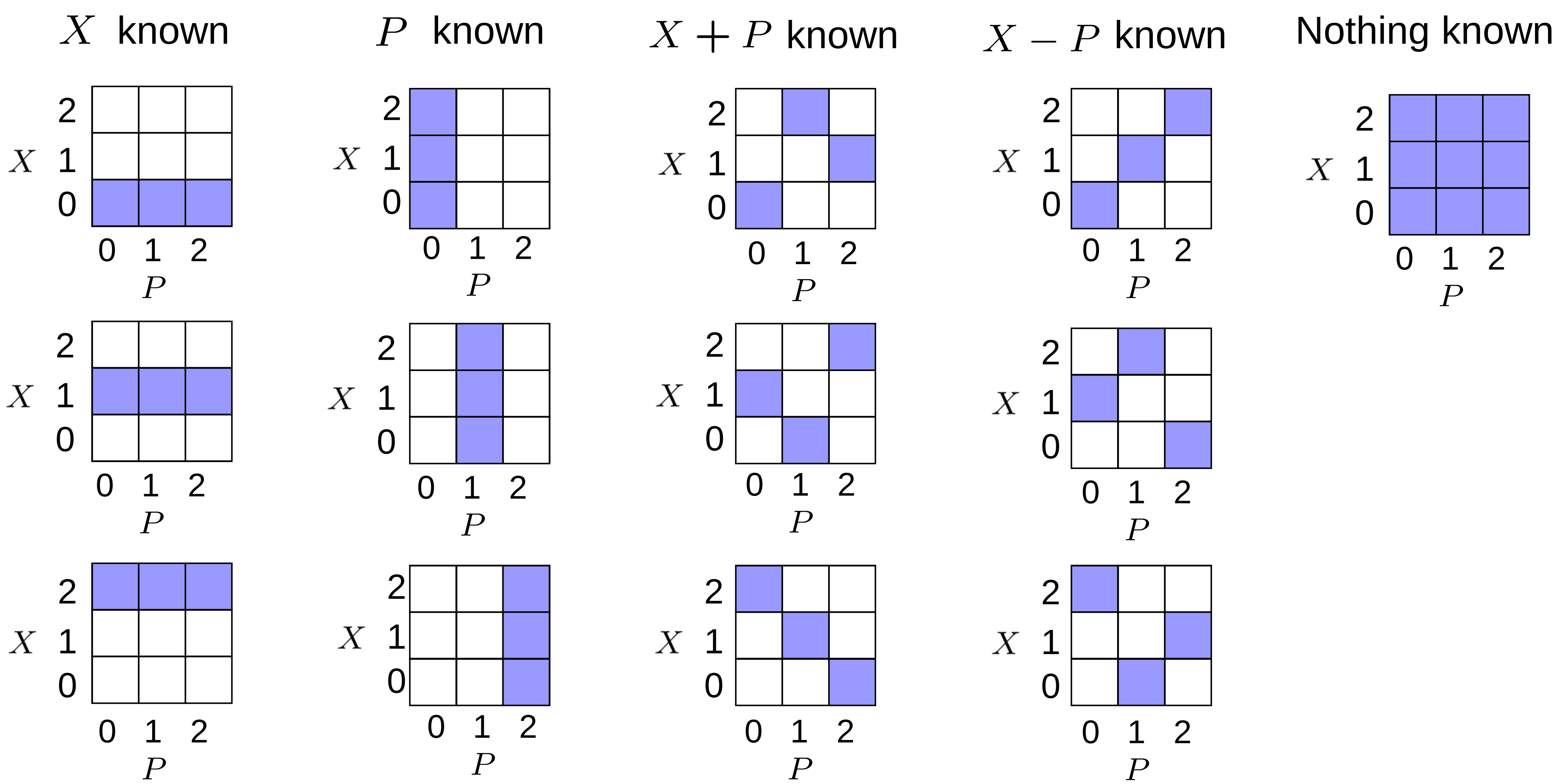


Valid epistemic states for a single trit

Canonical variables $aX + bP$

$X, \quad P, \quad X+P, \quad X-P(=X+2P)$

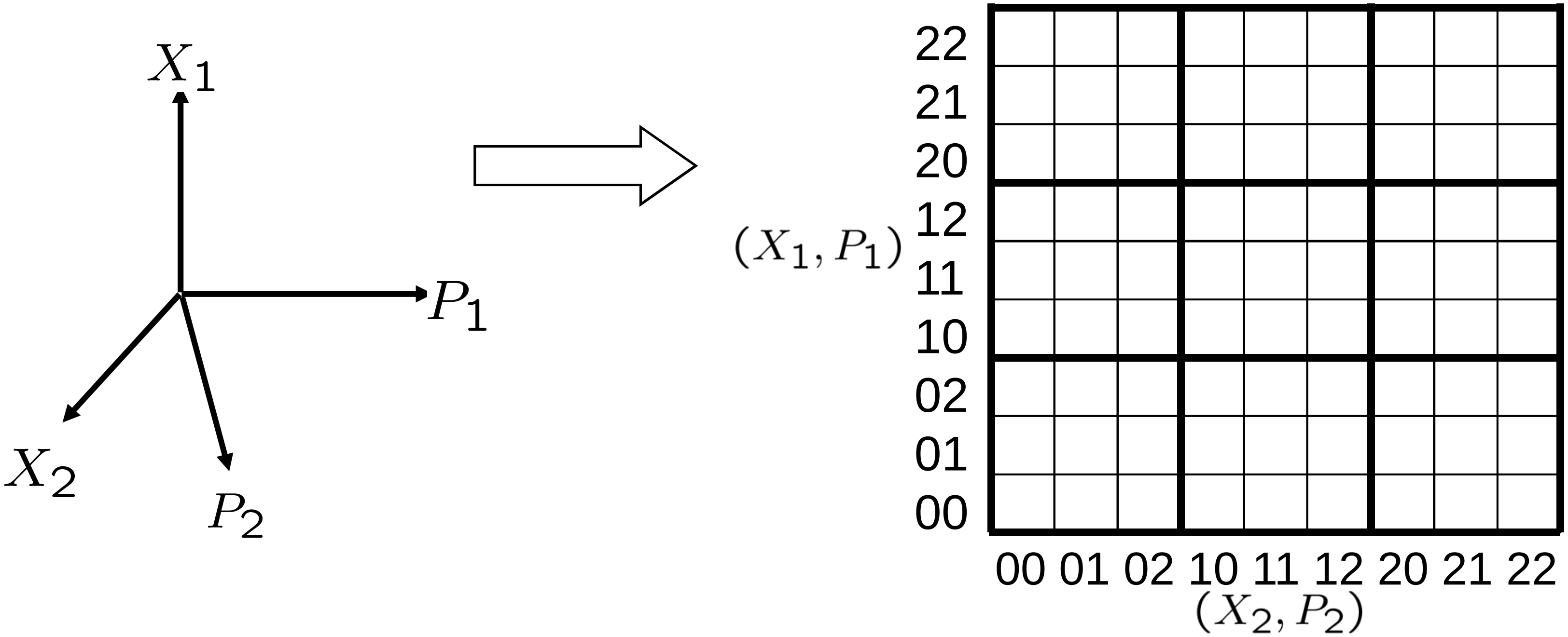
Commuting sets:
The singleton sets
the empty set



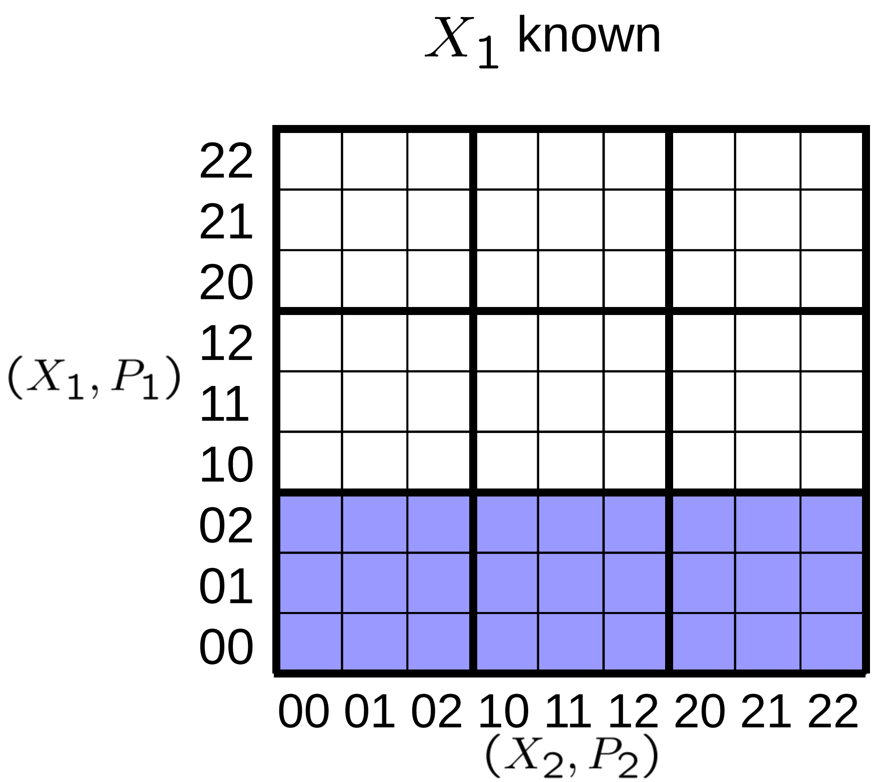
Valid epistemic states for a pair of trits

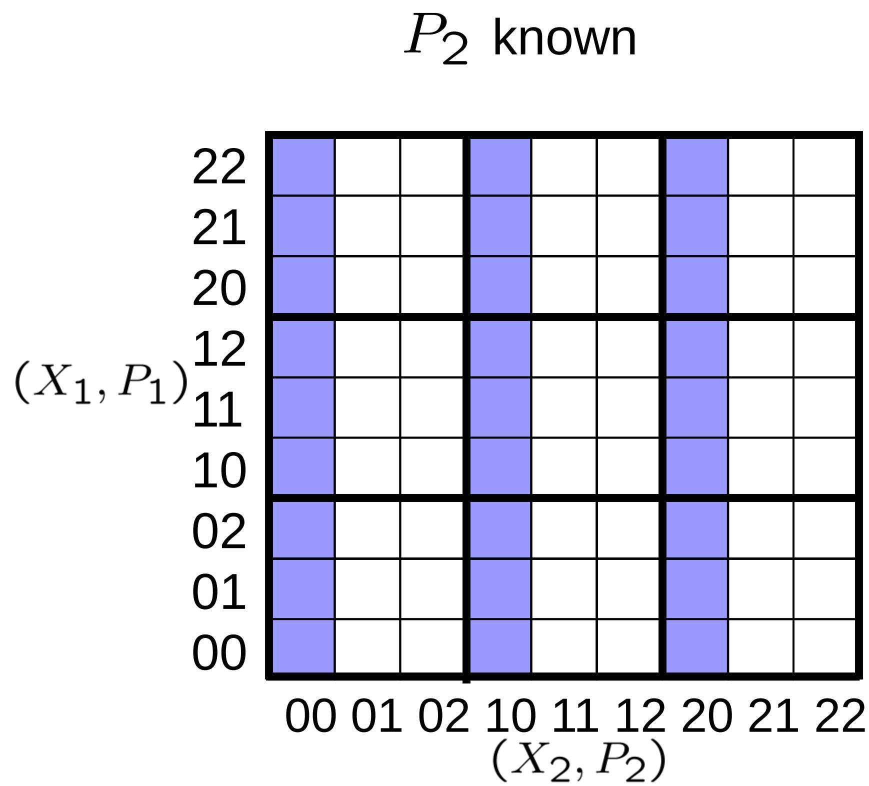
Canonical variables $a_1 X_1 + b_1 P_1 + a_1 X_2 + b_2 P_2$ $\quad a_1, b_1, a_2, b_2 \in \mathbb{Z}_3$

How to represent this graphically

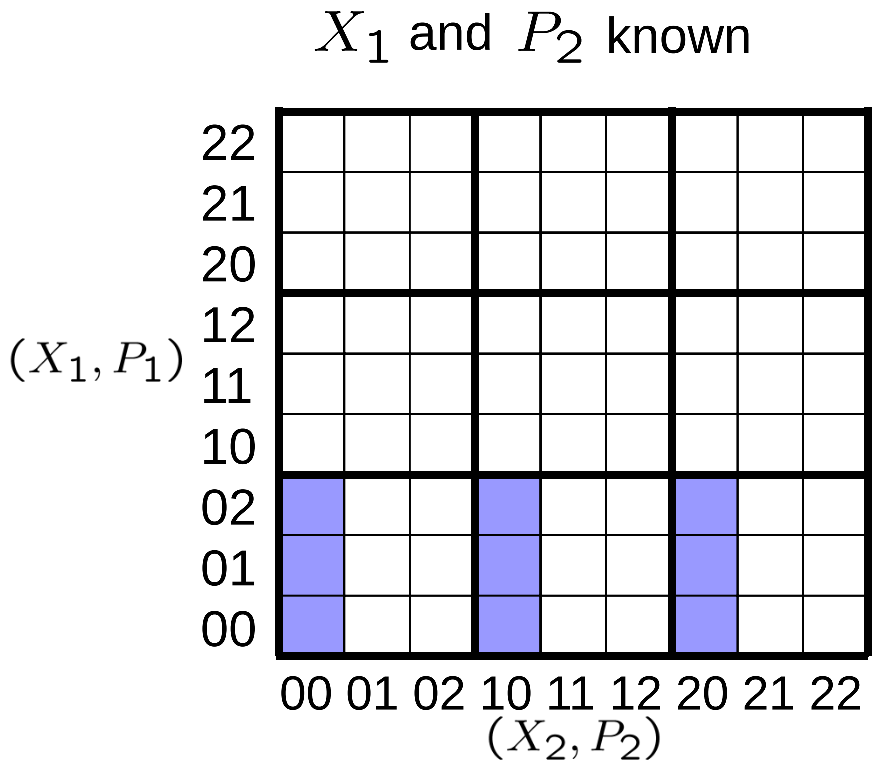


1 variable known



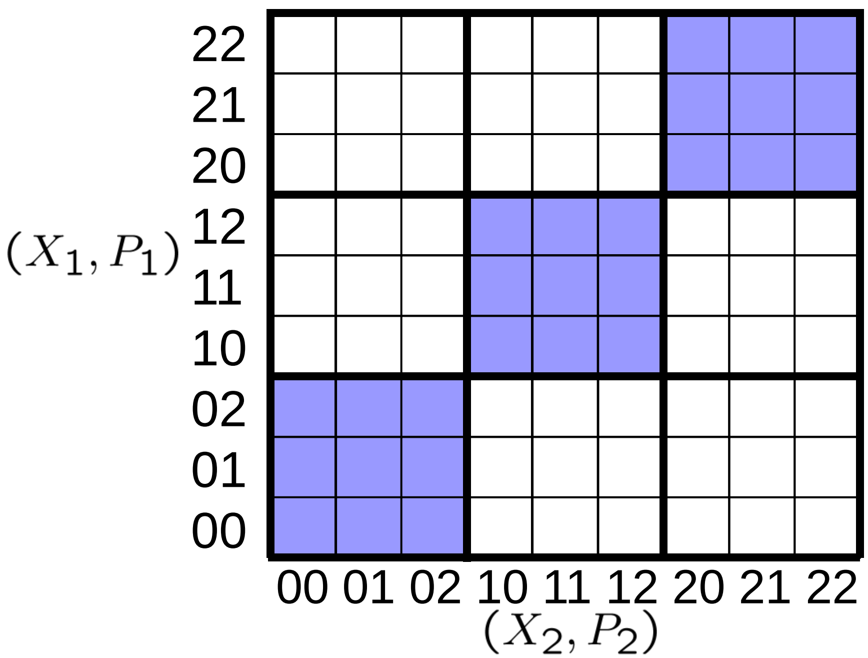


2 variables known

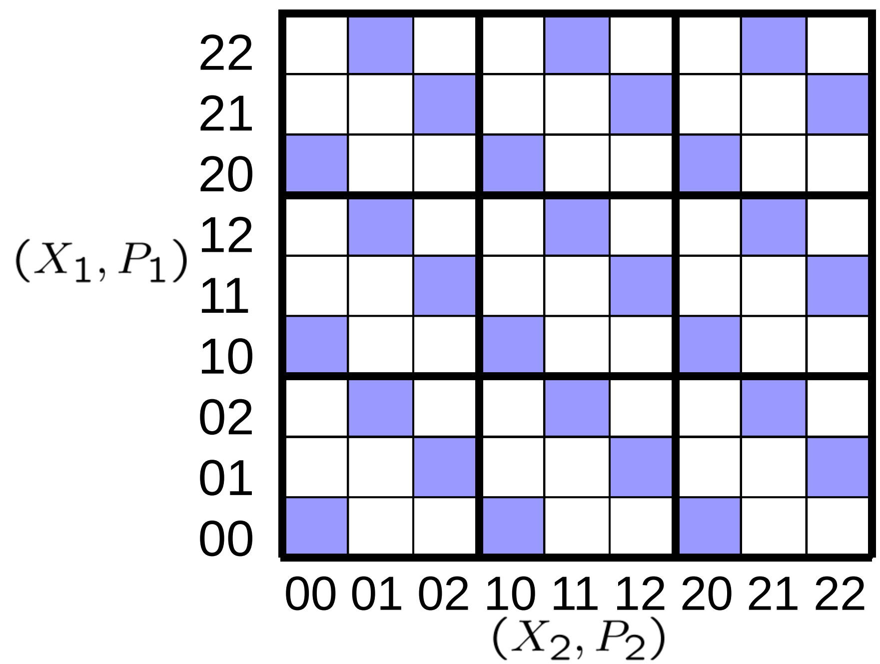


1 variable known

$X_1 - X_2$ known

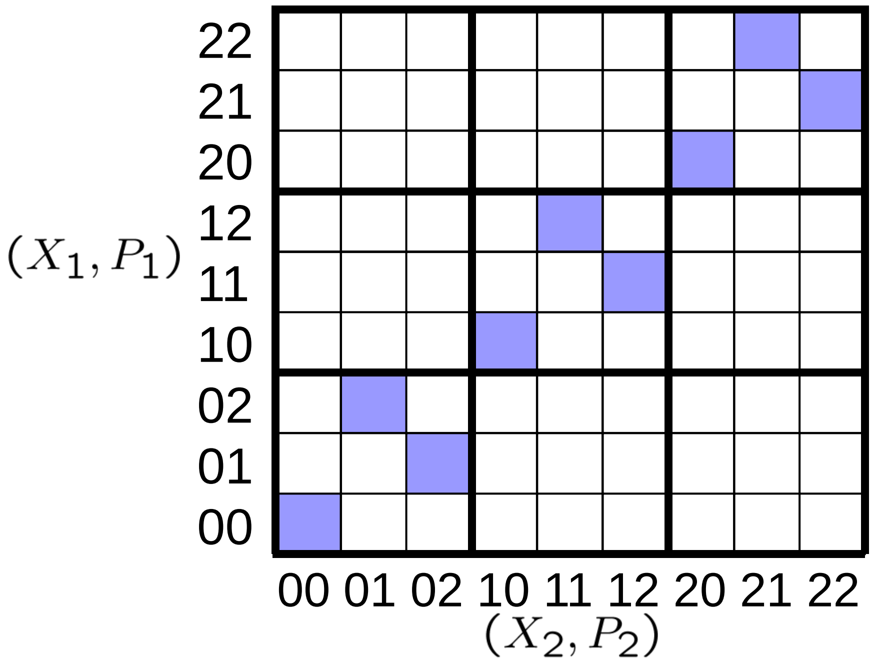


$P_1 + P_2$ known



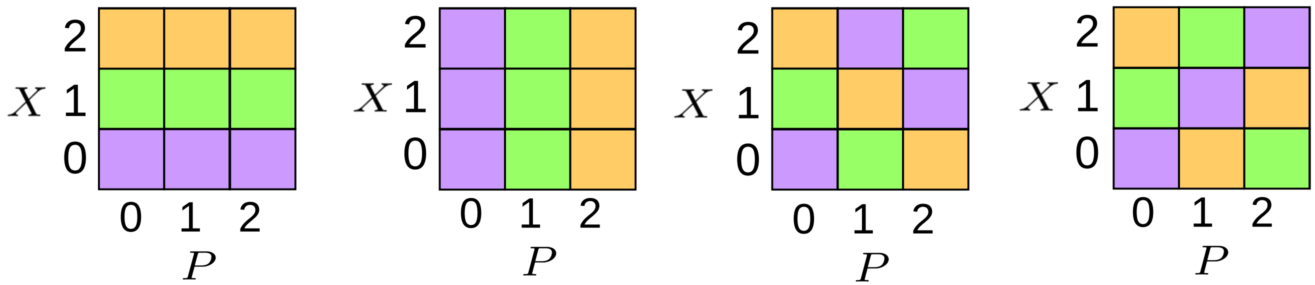
2 variables known

$X_1 - X_2$ and $P_1 + P_2$ known

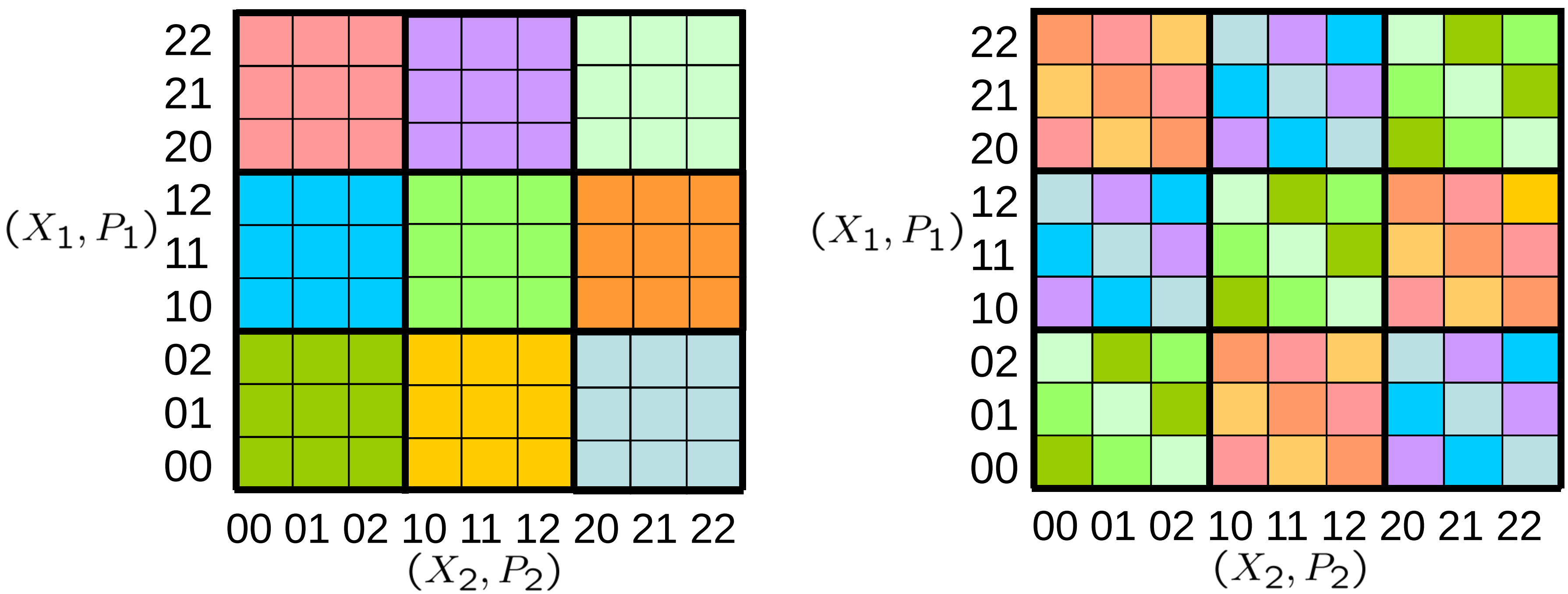


Valid reproducible measurements

On a single trit



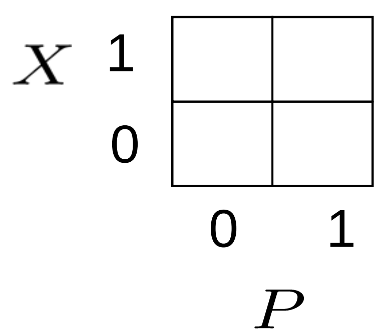
On a pair of trits



etc.

Restricted statistical theory of **bits**

$$- \;=\; (\mathbb{Z}_2)^{2n}$$



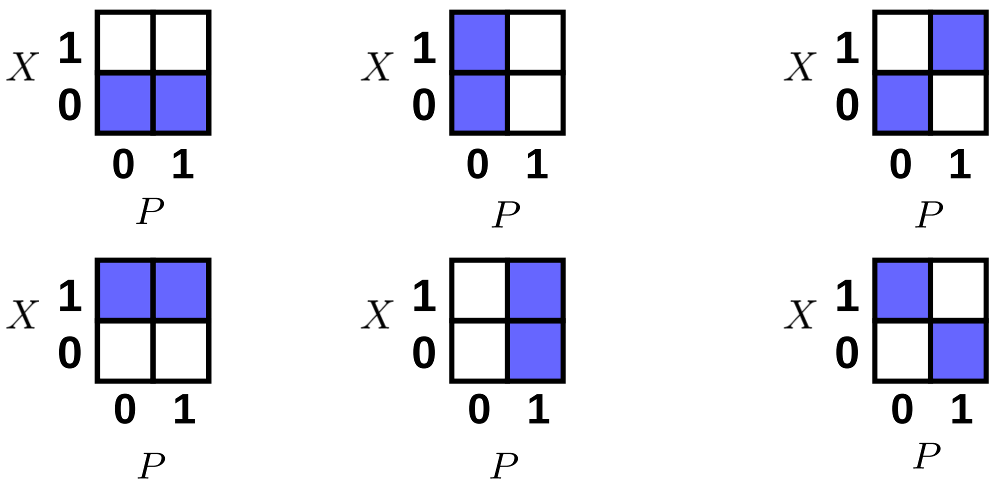
A single bit

Canonical variables $aX + bP$ $a, b \in \mathbb{Z}_2$

$$X, \quad P, \quad X + P (= X - P)$$

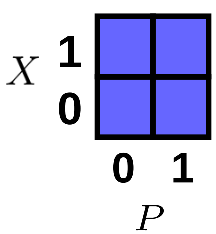
Epistemic states of maximal knowledge

X known P known $X + P$ known



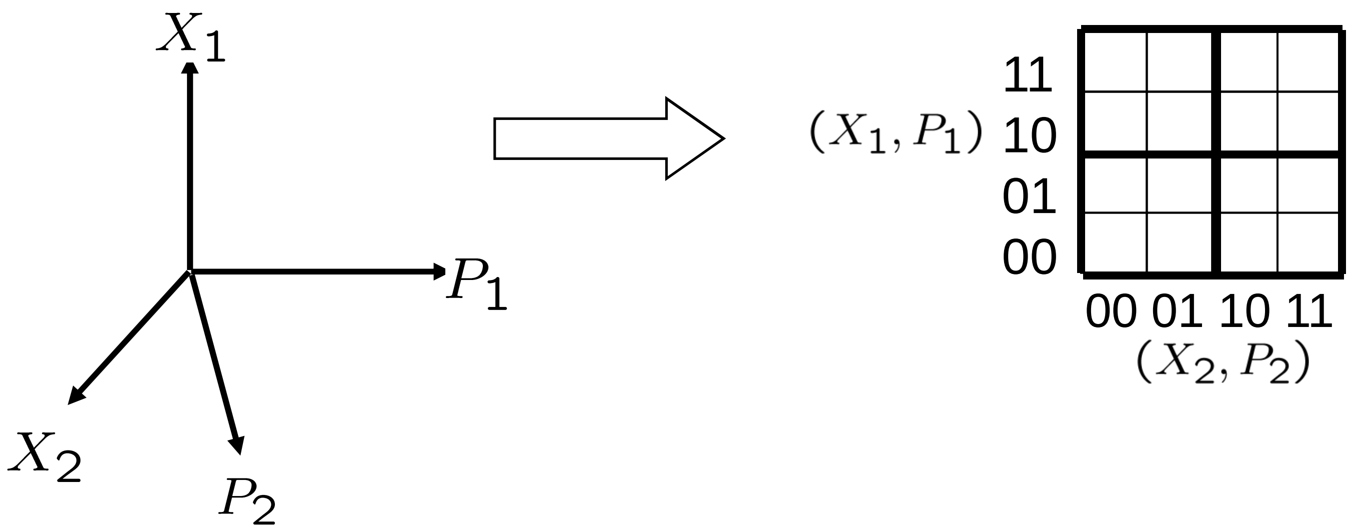
Epistemic states of non-maximal knowledge

Nothing known



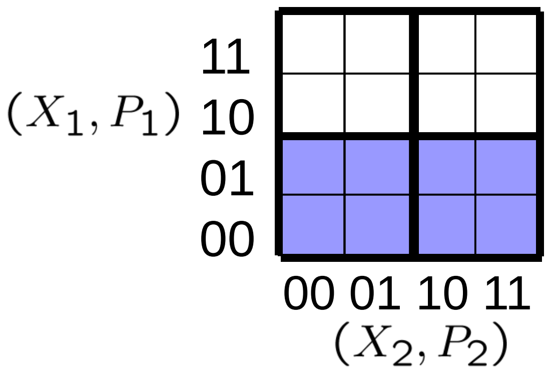
A pair of bits

Canonical variables $a_1 X_1 + b_1 P_1 + a_1 X_2 + b_2 P_2$ $a_1, b_1, a_2, b_2 \in \mathbb{Z}_2$

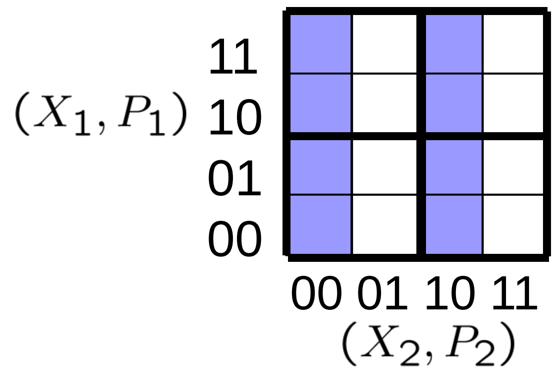


1 variable known

X_1 known

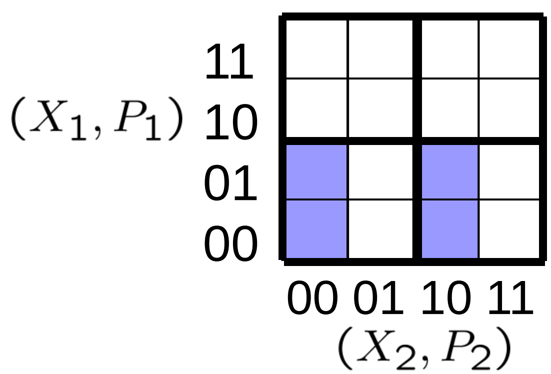


P_2 known



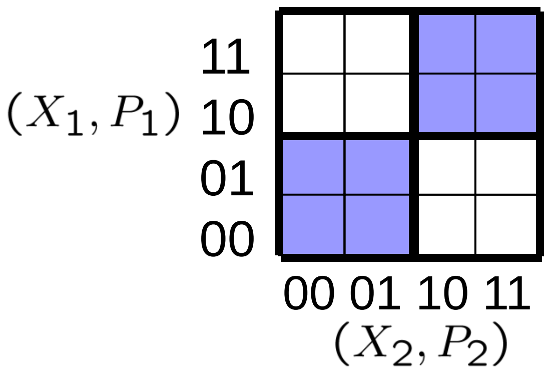
2 variables known

X_1 and P_2 known

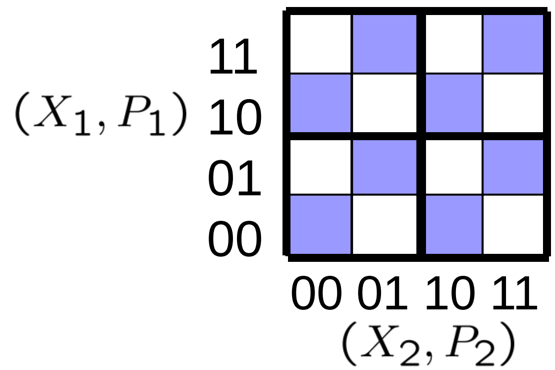


1 variable known

$X_1 - X_2$ known

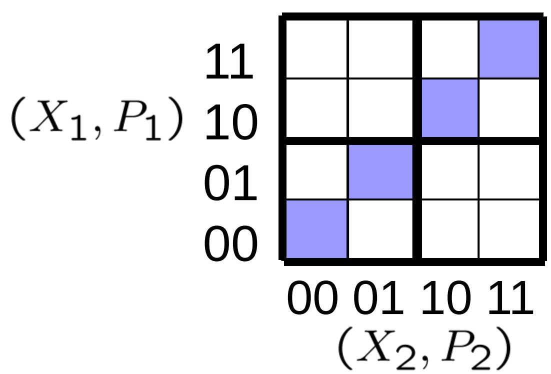


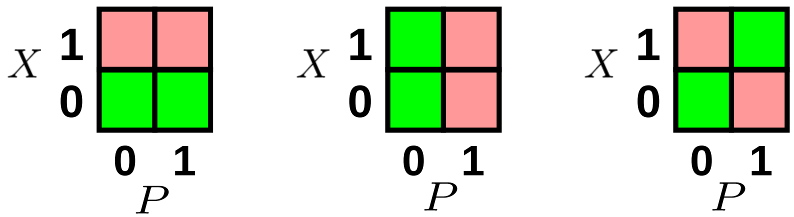
$P_1 + P_2$ known

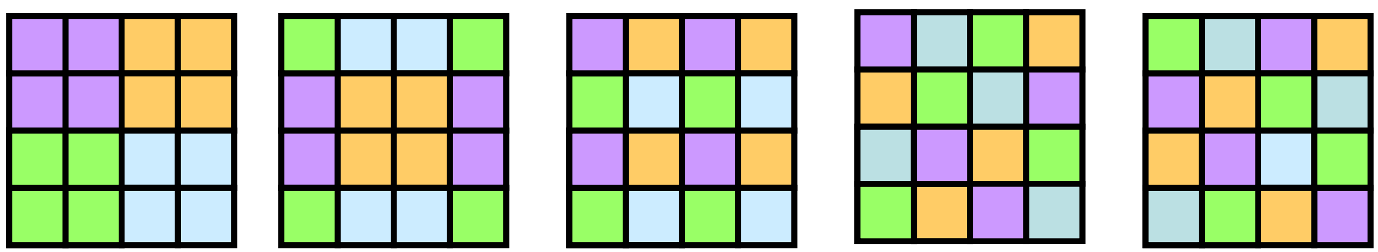


2 variables known

$X_1 - X_2$ and $P_1 + P_2$ known




X 1 0
0 1
P
X 1 0
0 1
P
X 1 0
0 1
P

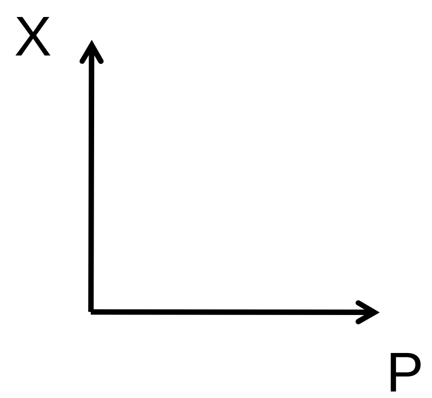


Equivalence of these restricted statistical theories to “subtheories” of quantum theory

Look to a representation of quantum theory on phase space
– the Wigner representation

Restricted Liouville mechanics
= Quadrature Quantum Mechanics

$$- \quad = \quad \mathbb{R}^{2n}$$



Quadrature quantum mechanics

Hermitian operators: $\hat{F} : \mathcal{L}^2(\mathbb{R}^n) \to \mathcal{L}^2(\mathbb{R}^n)$

Commutator:

$$[\widehat{F}, \widehat{G}] \equiv \widehat{F}\widehat{G} - \widehat{G}\widehat{F}$$

The quadrature operators are:

$$\widehat{F} = a_1\widehat{X}_1 + b_1\widehat{P}_1 + \cdots + a_n\widehat{X}_n + b_n\widehat{P}_n \qquad a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$$

Quadrature states are eigenstates of a commuting set of quadrature operators

Specified by an isotropic subspace V and a valuation vector $v \in V$

(Quadrature transformations and measurements take quadrature states to quadrature states)

Wigner representation of quantum mechanics

Weyl operator $\widehat{w}(m) = e^{-i \sum_i q_i \widehat{P}_i + p_i \widehat{X}_i}$

Quantum state ρ

Characteristic function $\chi_\rho(m) = \mathrm{Tr}(\rho\, \widehat{w}(m)^\dagger)$

Wigner function $W_\rho(m) = \sum_a e^{-i[m,a]} \chi_\rho(m)$

For quadrature state associated with V, v

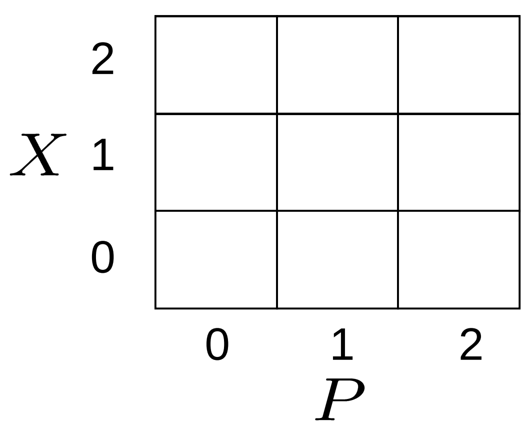
$W_{V,v}(m) = \frac{1}{\mathcal{N}} \delta_{V^\perp + v}(m)$

Equivalence of states implies equivalence of measurements and transformations
Therefore

Theorem: Restricted statistical Liouville mechanics is empirically equivalent to quadrature quantum mechanics

Restricted statistical theory of trits
= Stabilizer theory for qutrits

$$- \;=\; (\mathbb{Z}_3)^{2n}$$



C

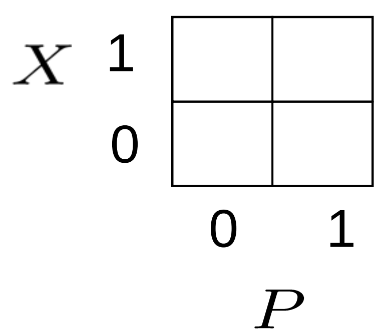
$\mathbb{C}_3$

Equivalence of states implies equivalence of measurements and transformations
Therefore

Theorem: The restricted statistical theory of trits is empirically equivalent to the Stabilizer theory for qutrits

Restricted statistical theory of bits
$\simeq$ Stabilizer theory for qubits

$$- = (\mathbb{Z}_2)^{2n}$$



Analogously to what we did for trits, one can:

Define stabilizer theory for qubits
Define Gross' discrete Wigner function for qubits

Find: Wigner function can be negative for qubit stabilizer states

The restricted statistical theory of bits is not equivalent but very close to the Stabilizer theory for qubits

Knowledge balance vs. classical complementarity

Contrast:

The principle of classical complementarity:

An observer can only have knowledge of the values of a commuting set of canonical variables and otherwise is maximally ignorant.

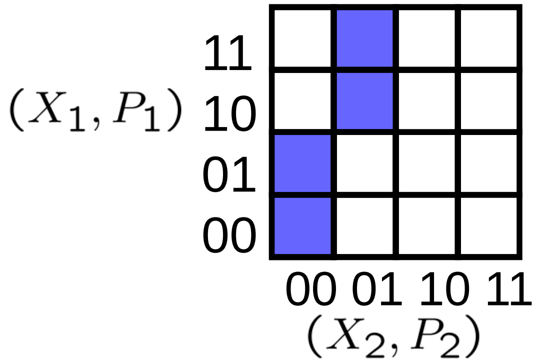
The knowledge-balance principle:

The only distributions that can be prepared are those that correspond to knowing at most half the information

From: RS, quant-ph/0401052 [Phys. Rev. A 75, 032110 (2007)]

The same epistemic states are found to be valid, but the logic is different…

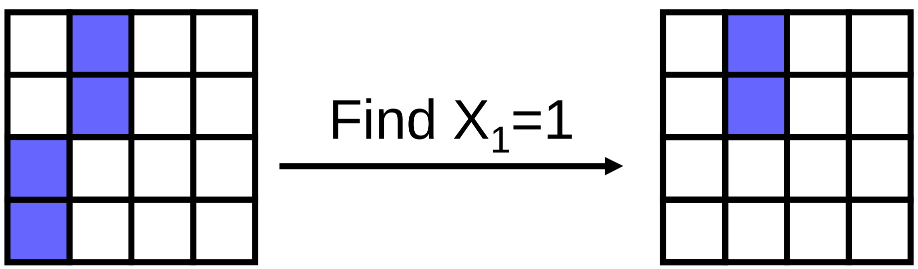
Example:



is forbidden

<u>Knowledge-balance principle:</u>

It is forbidden by an assumption of locality and the existence of nontrivial measurements:



<u>Principle of epistemic complementarity:</u>

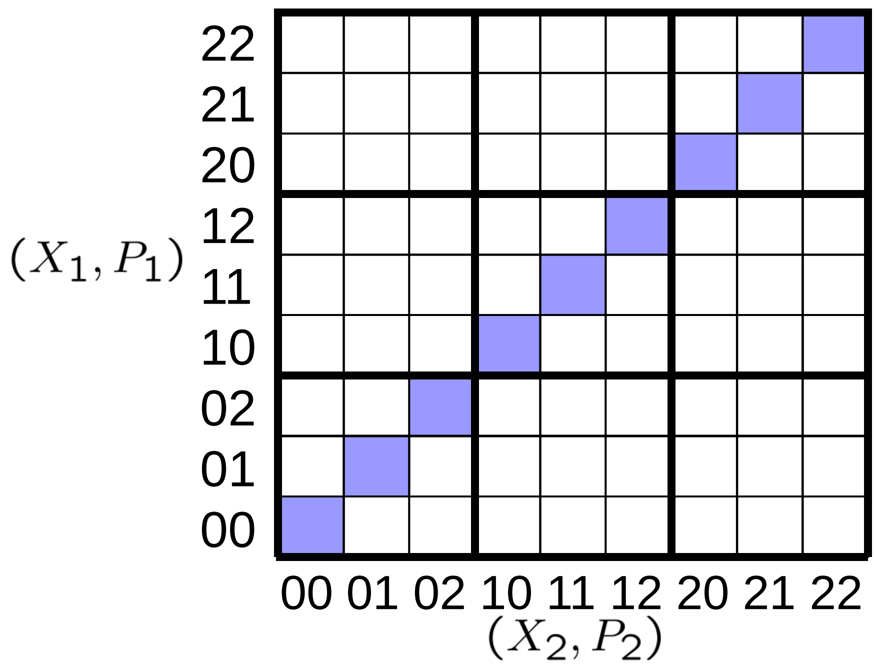
It is forbidden because it corresponds to

$X_2 = 0$ and $X_1 + P_2 = 0$ but $[X_2, X_1 + P_2] \neq 0$

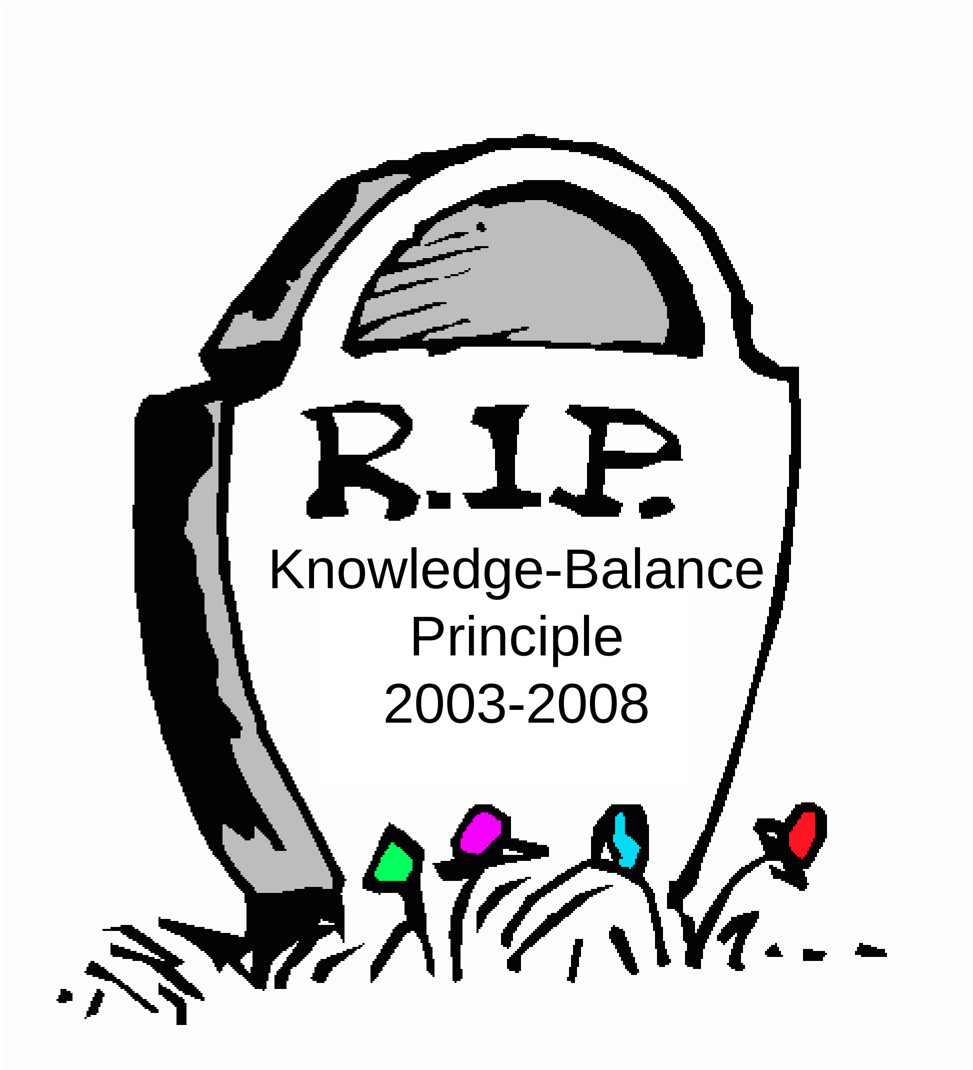
What about applying knowledge-balance to trits?
(See S. van Enk, arxiv:0705.2742)

Valid epistemic states for a pair of systems are different!

$X_1 - X_2$ and $P_1 - P_2$ known



Allowed by knowledge-balance, but corresponding to nothing in QM!



Long live Symplectic Structure!

Beyond classical complementarity: could a different statistical restriction get us closer to quantum theory?

NO for discrete degrees of freedom

Supplementing the unitary representation of the Clifford group with a single non-Clifford unitary yields all unitaries

YES for continuous degrees of freedom

In addition to rotations and displacements in phase space, one can add squeezing – one gets all the quadratic Hamiltonians

(Bartlett, Rudolph, Spekkens, unpublished)

The classical uncertainty principle:

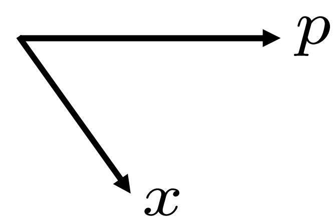
The only Liouville distributions that can be prepared are those satisfying

$$\gamma(\mu) + i\hbar J \geq 0$$

and that have maximal entropy for a given set of second-order moments.

$$\gamma(\mu) = 2\begin{pmatrix} \Delta^2 x_1 & C_{x_1,p_1} & C_{x_1,x_2} & C_{x_1,p_2} & \cdots \\ C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} & \\ C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} & \\ C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & -1 & & & \cdots \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$



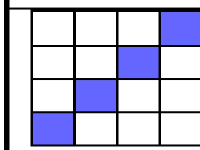
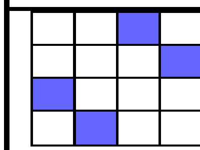
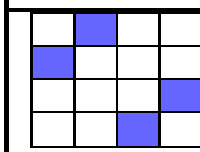
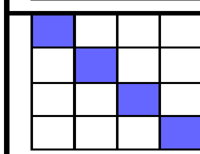
$$\mu(x_1, p_1, x_2, p_2, \ldots)$$

The theory is empirically equivalent to Gaussian quantum mechanics

Why the restricted statistical theory of bits Is **not** equivalent to qubit stabilizer theory

	$\{\lvert 0\rangle, \lvert 1\rangle\}$	$\{\lvert +\rangle, \lvert -\rangle\}$	$\{\lvert +i\rangle, \lvert -i\rangle\}$
$\lvert\Phi^+\rangle$	C	C	A
$\lvert\Phi^-\rangle$	C	A	C
$\lvert\Psi^+\rangle$	A	C	C
$\lvert\Psi^-\rangle$	A	A	A

Even number of correlations

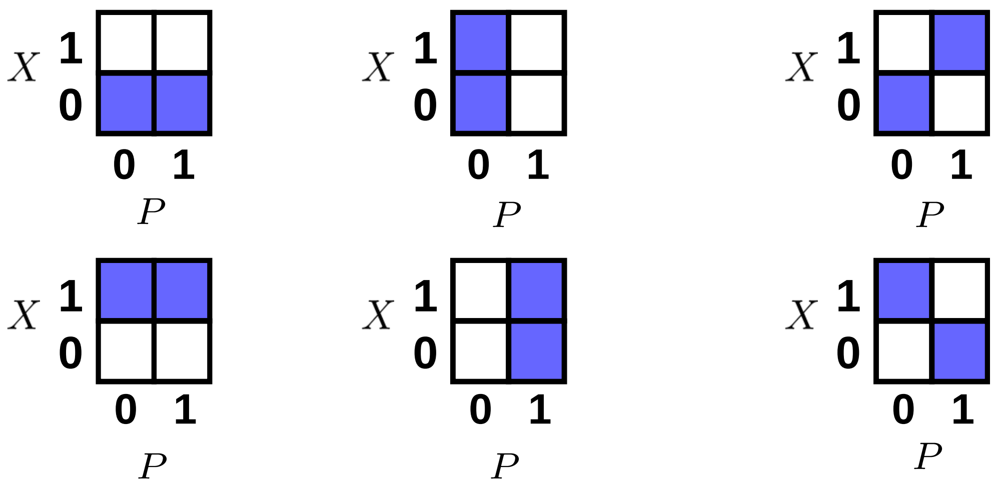
	I | I | II | II	I | II | I | II	I | II | II | I
	C	C	C
	C	A	A
	A	C	A
	A	A	C

Odd number of correlations

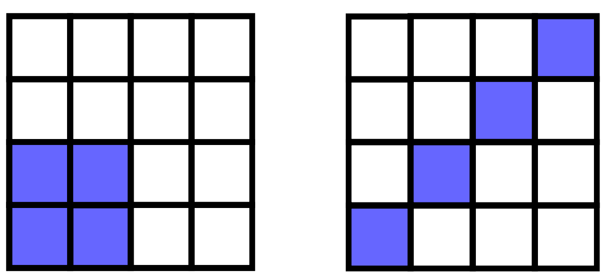
Qubit stabilizer theory is nonlocal and contextual (e.g. GHZ)
Restricted statistical theory of bits is local and noncontextual

According to Knowledge-Balance

Valid epistemic states for a single system



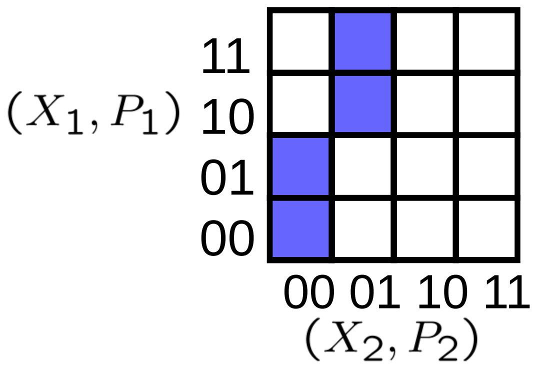
Valid epistemic states for a pair of systems



Plus permutations of rows and columns

The same epistemic states are found to be valid, but the logic is different…

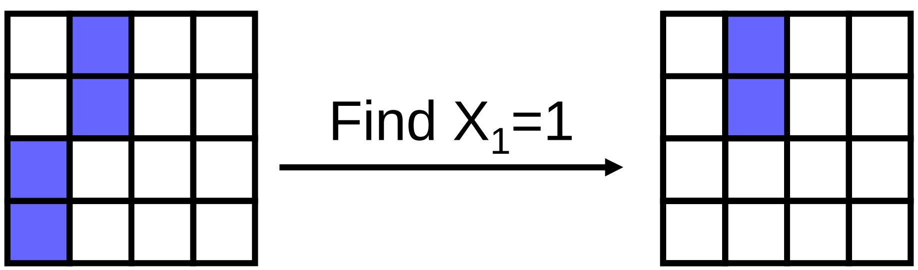
Example:



is forbidden

Knowledge-balance principle:

It is forbidden by an assumption of locality and the existence of nontrivial measurements:



Are the two theories always equivalent?

Principle of epistemic complementarity:

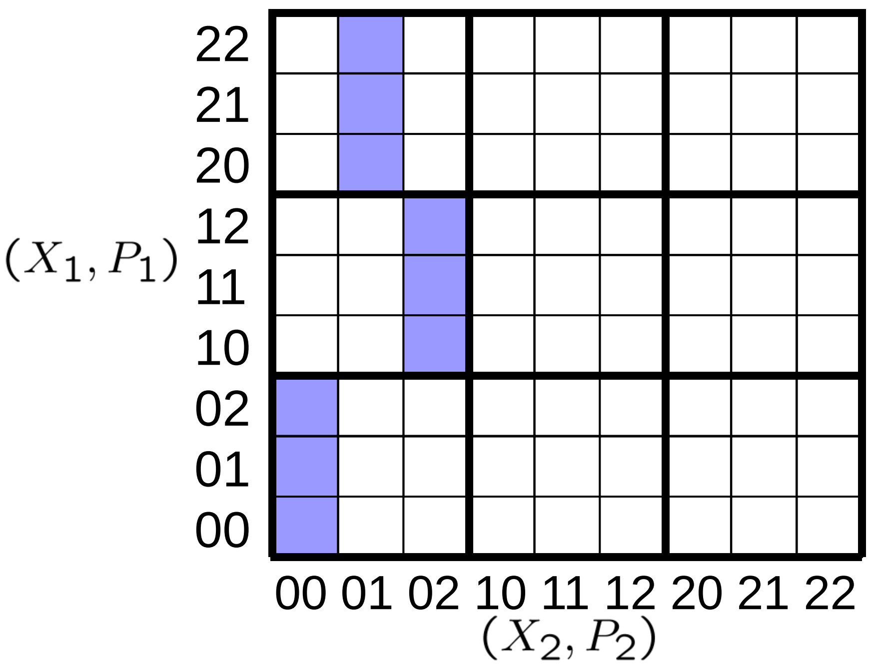
It is forbidden because it corresponds to

$X_2 = 0$ and $X_1 + P_2 = 0$ but $[X_2, X_1 + P_2] \neq 0$

What about applying knowledge-balance to trits?
(See S. van Enk, arxiv:0705.2742)

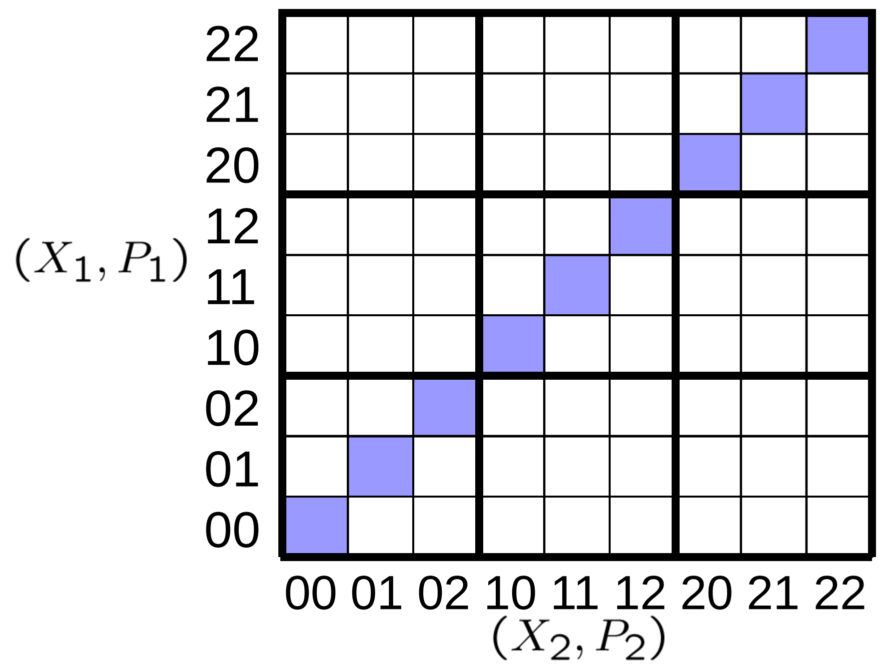
Valid epistemic states for a single system are the same
Valid epistemic states for a pair of systems are slightly different!

X_2 and $X_1 + P_2$ known



Ruled out by locality

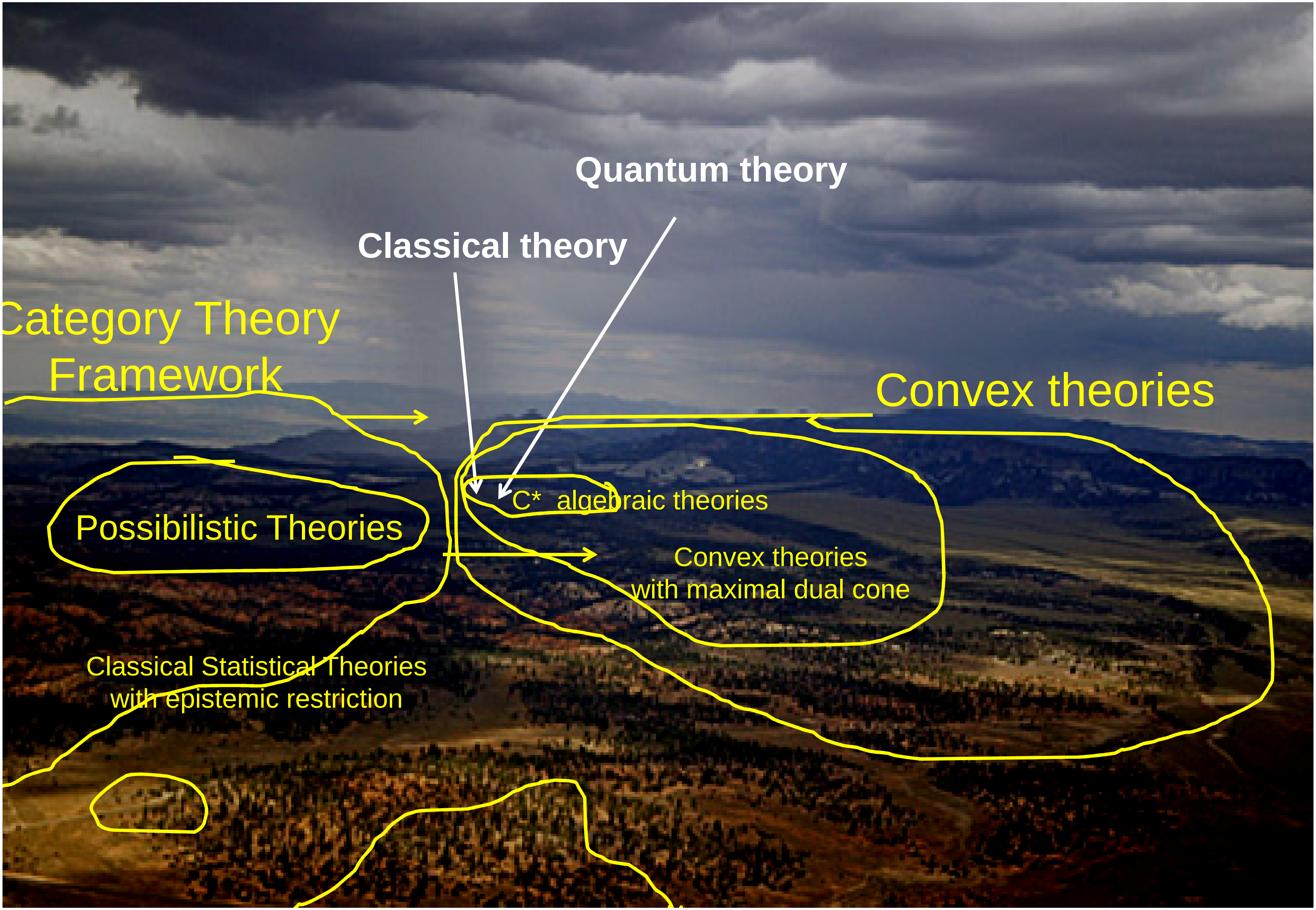
$X_1 - X_2$ and $P_1 - P_2$ known



Allowed by locality, but corresponding to nothing in QM!

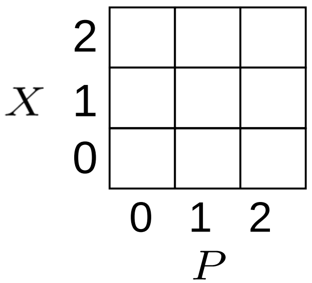
[redacted]

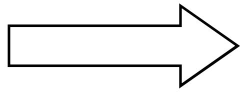
Valid epistemic states for a pair of degrees of freedom

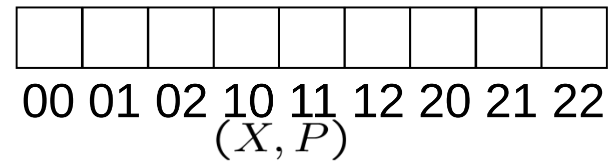


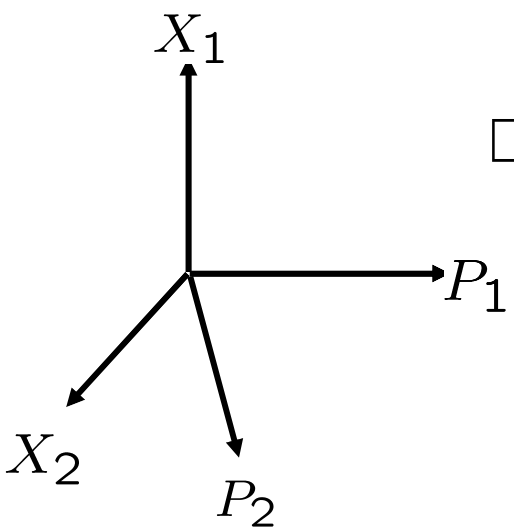
Quantum theory
Classical theory
Category Theory
Framework
Convex theories
Possibilistic Theories
C* algebraic theories
Convex theories
with maximal dual cone
Classical Statistical Theories
with epistemic restriction

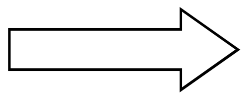
How to represent this graphically

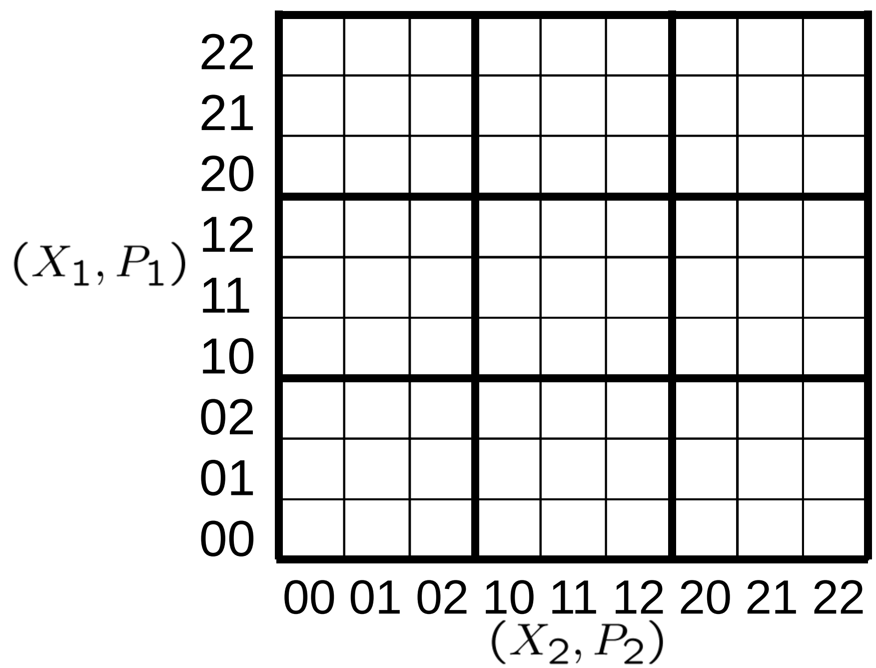




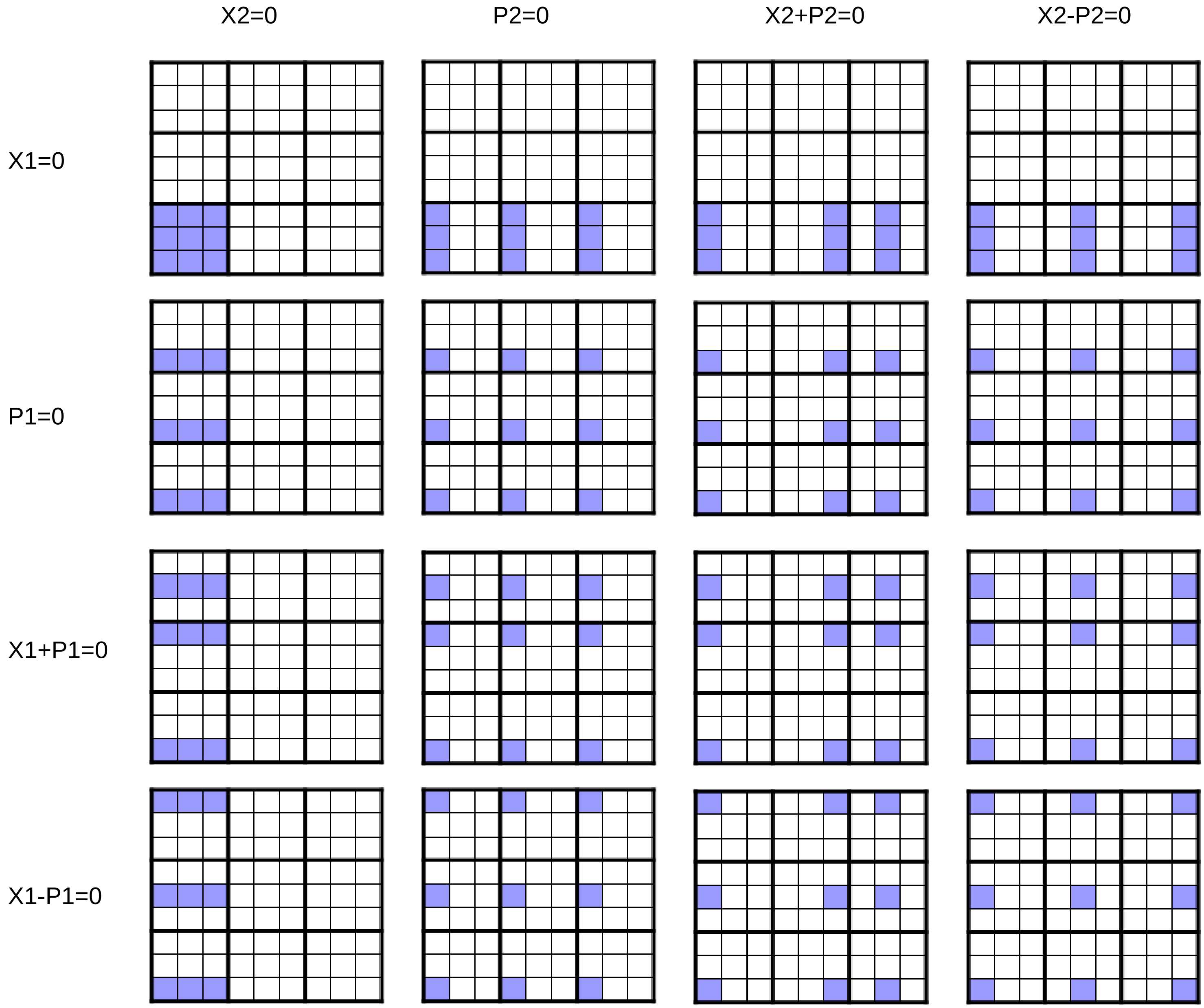




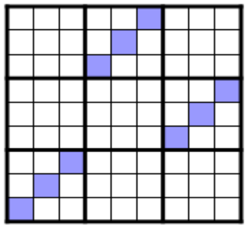


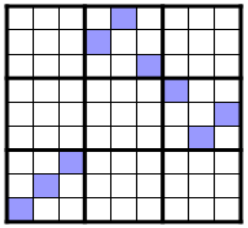


Uncorrelated pure epistemic states



Correlated pure epistemic states


X1+X2=0
P1-P2=0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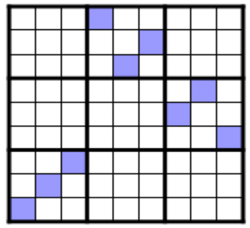
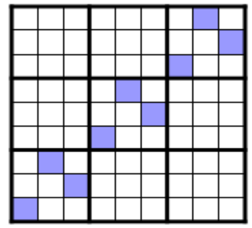
X1+P1-P2=0


X1+X2=0
X1-P1+P2=0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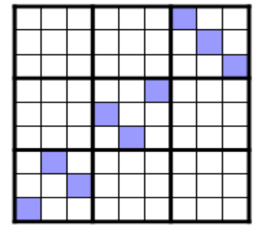
P1+P2=0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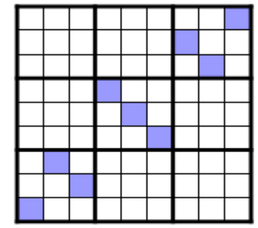
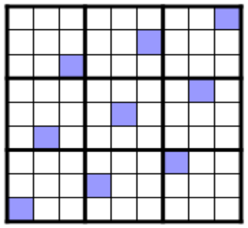
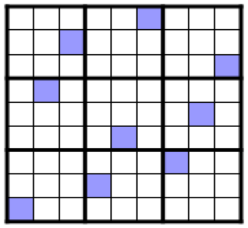
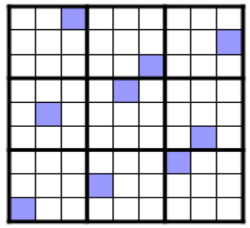
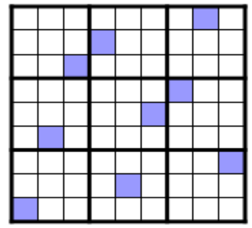
X1-P1-P2=0

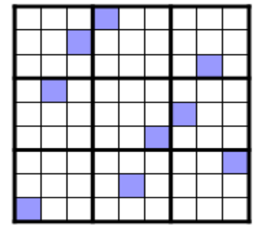

X1-X2=0
P1+X2+P2=0


X1-P2=0
P1-X2=0


X1-P2=0
P1-X2+P2=0


X1-P2=0
P1-X2-P2=0


P1-X2=0
X1+P1-P2=0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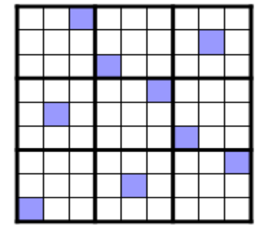
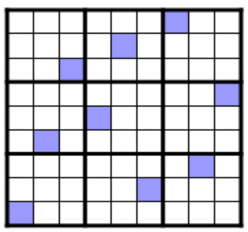
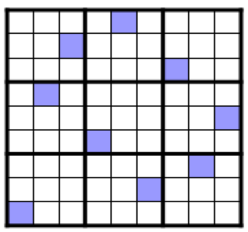
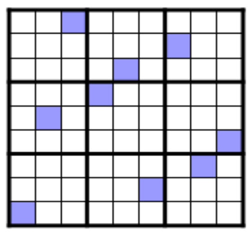
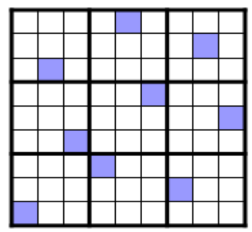
X1+X2-P2=0

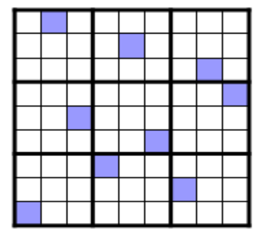

P1-P2=0
X1-P1+X2=0

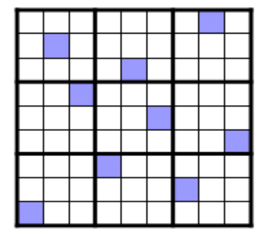

P1-X2=0
X1-P1-P2=0

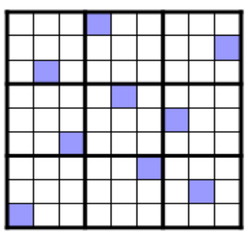

X1+P1-X2=0
X1-P1-X2-P2=0

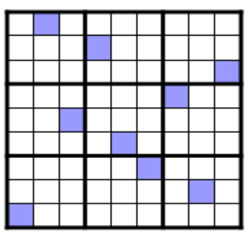

X1-X2-P2=0
X1-P1+X2=0

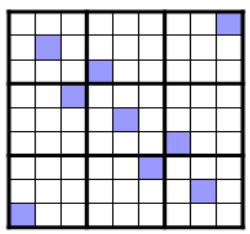

X1+P2=0
P1+X2=0


X1+P2=0
P1+X2+P2=0


X1+P2=0
P1+X2-P2=0


P1+X2=0
X1+X2+P2=0


X1+P1-P2=0
P1-X2+P2=0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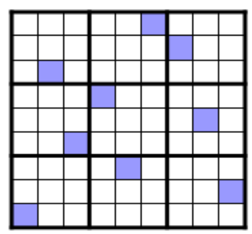
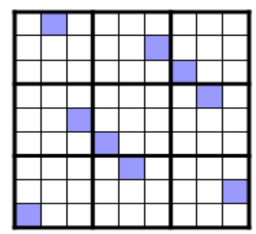
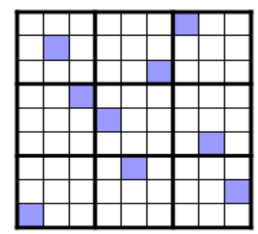
P1-P2=0


P1+X2=0
X1+P1+P2=0

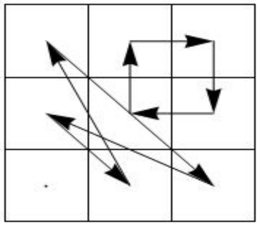

P1+P2=0
X1-X2+P2=0


X1-P1-P2=0
P1-X2-P2=0

Valid reversible transformations

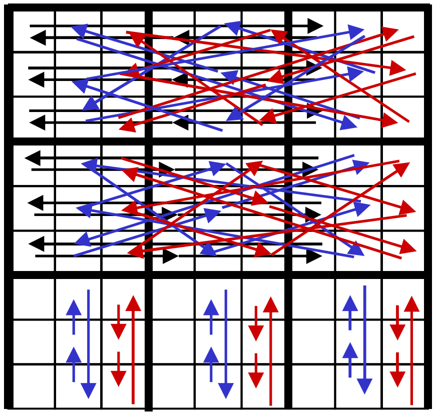
1 trit example:

$$X \mapsto P$$
$$P \mapsto -X$$



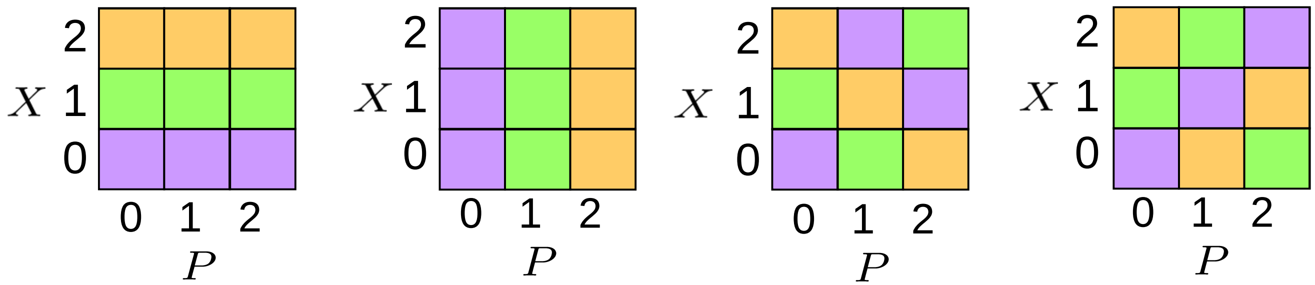
2 trit example:

$$X_1 \mapsto X_1$$
$$P_1 \mapsto P_1 - P_2$$
$$X_2 \mapsto X_1 + X_2$$
$$P_2 \mapsto P_2$$

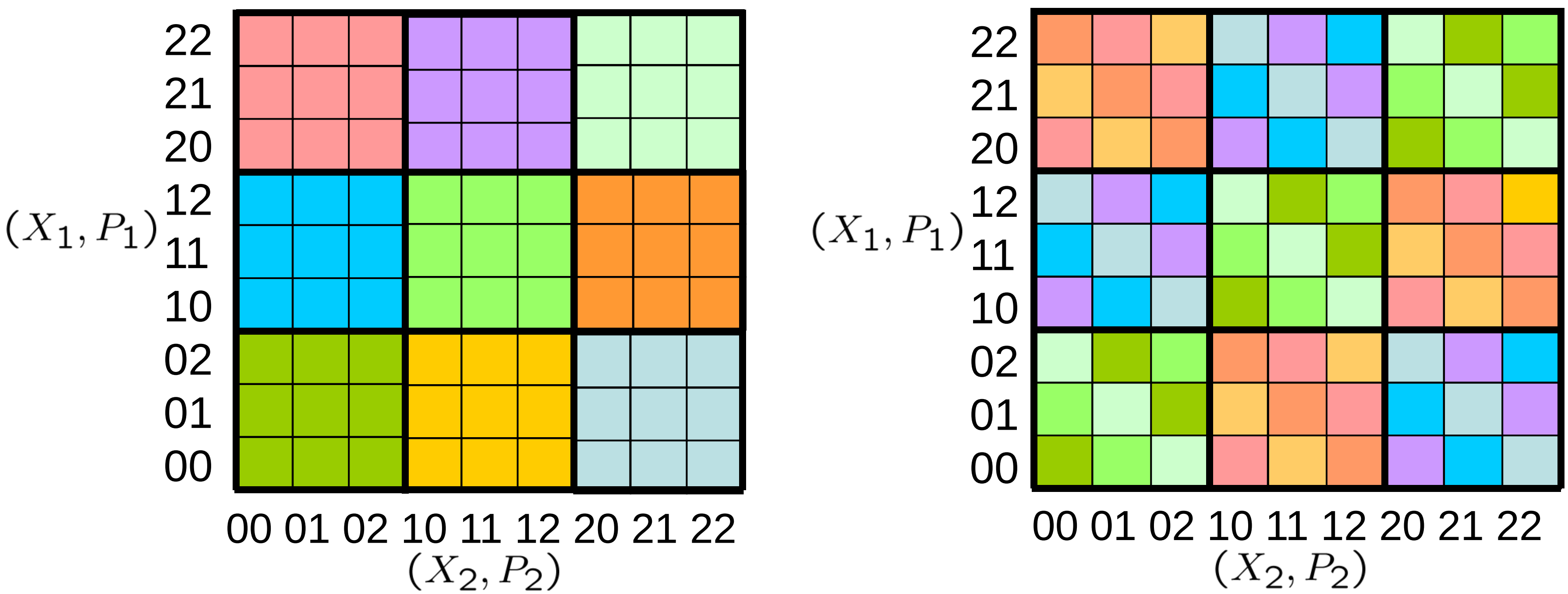


Valid reproducible measurements

On a single trit

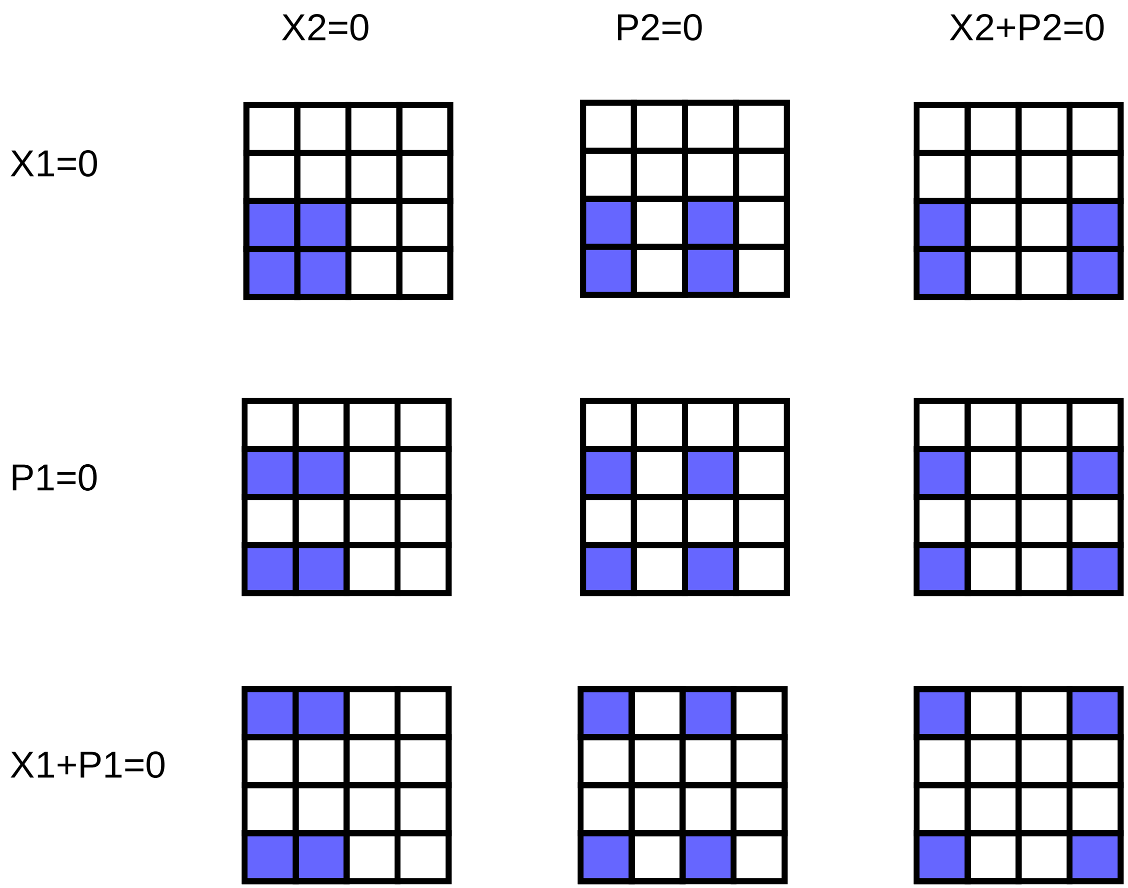


On a pair of trits

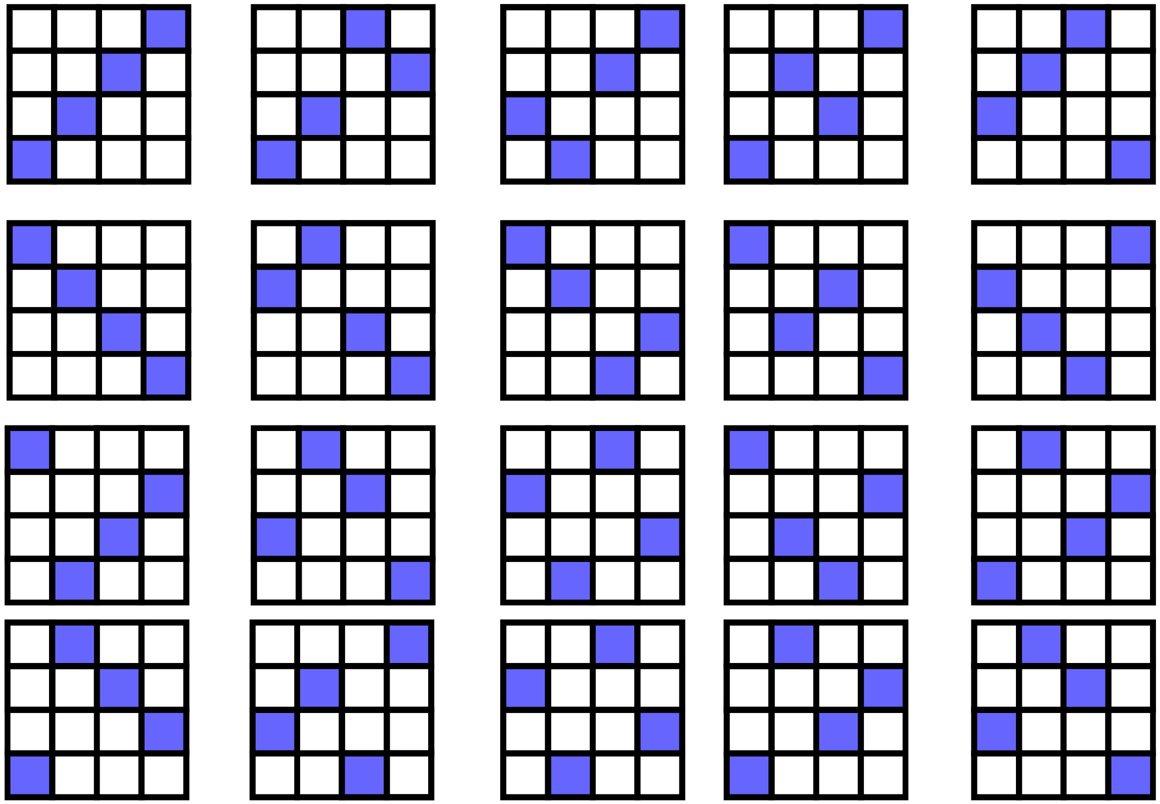


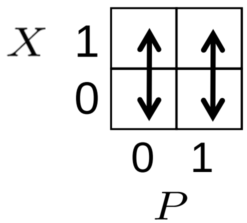
etc.

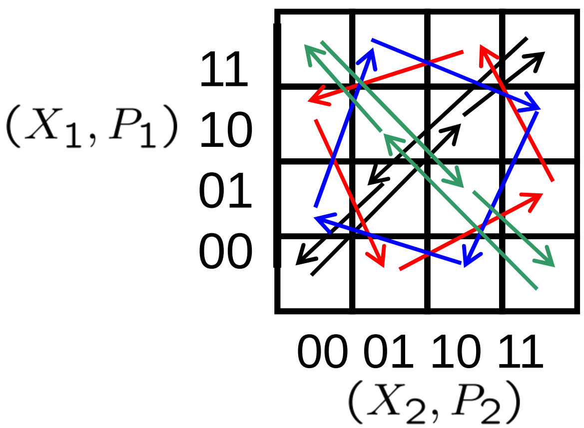
Uncorrelated pure epistemic states

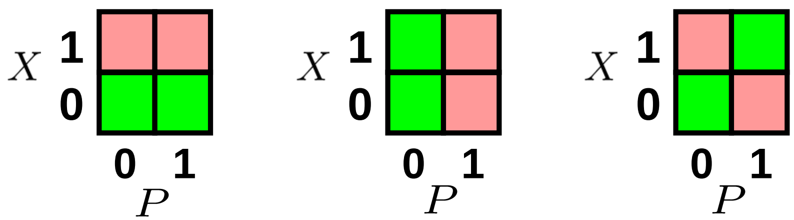


Correlated pure epistemic states




X
1
0
0 1
P


(X_1, P_1)
11
10
01
00
00 01 10 11
(X_2, P_2)


X
1
0
0
1
P
X
1
0
0
1
P
X
1
0
0
1
P