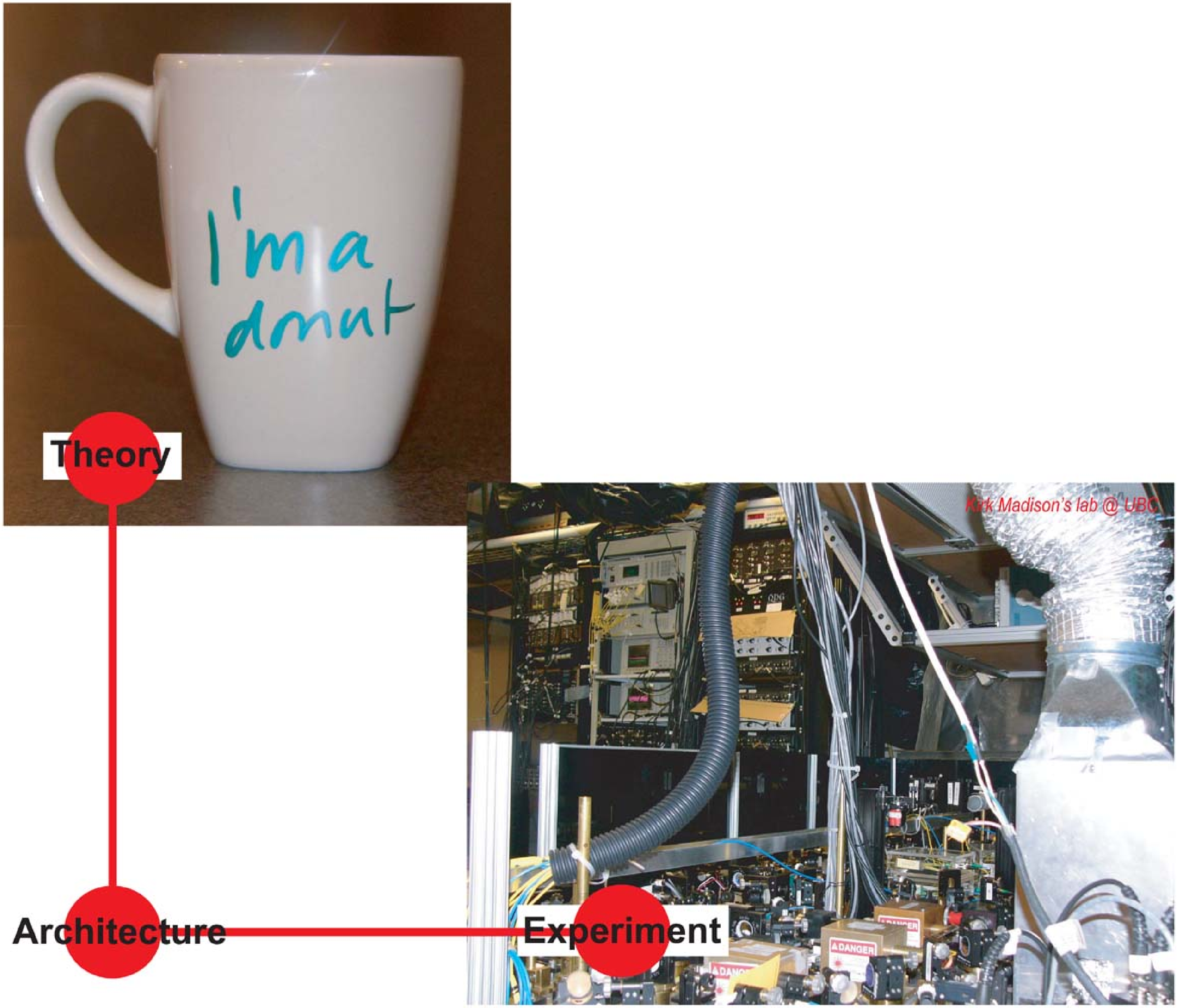
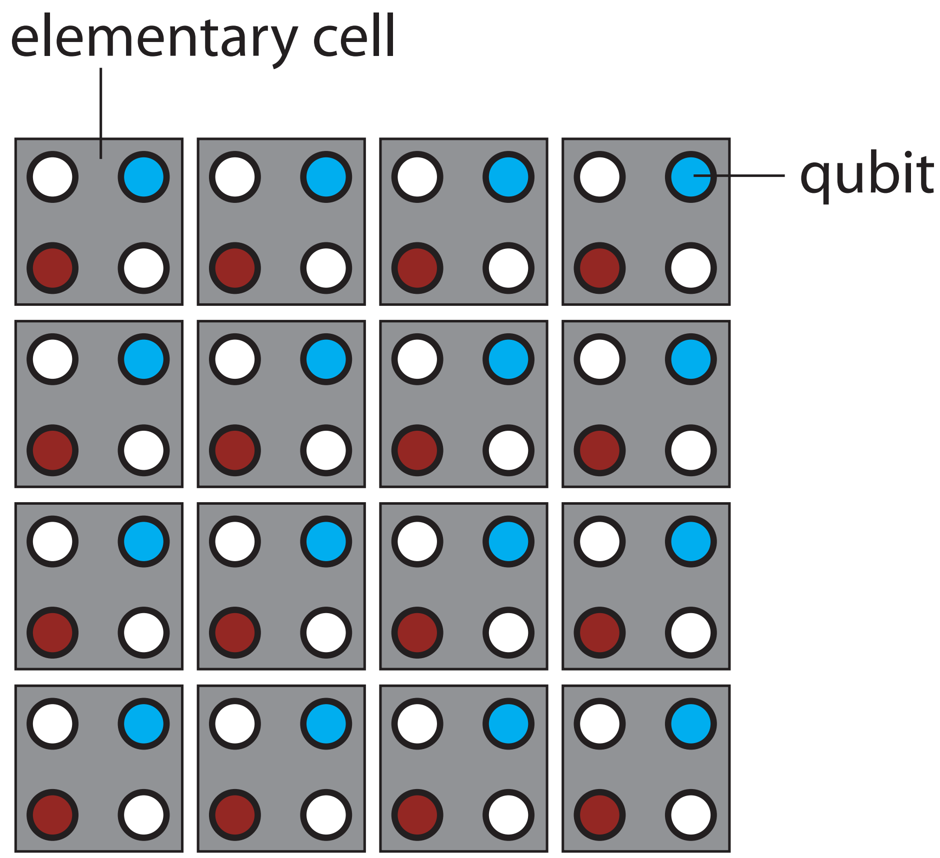
# Fault-tolerant quantum computation with cluster states

Robert Raussendorf, UBC

QI10 @ UBC, July 29, 2010

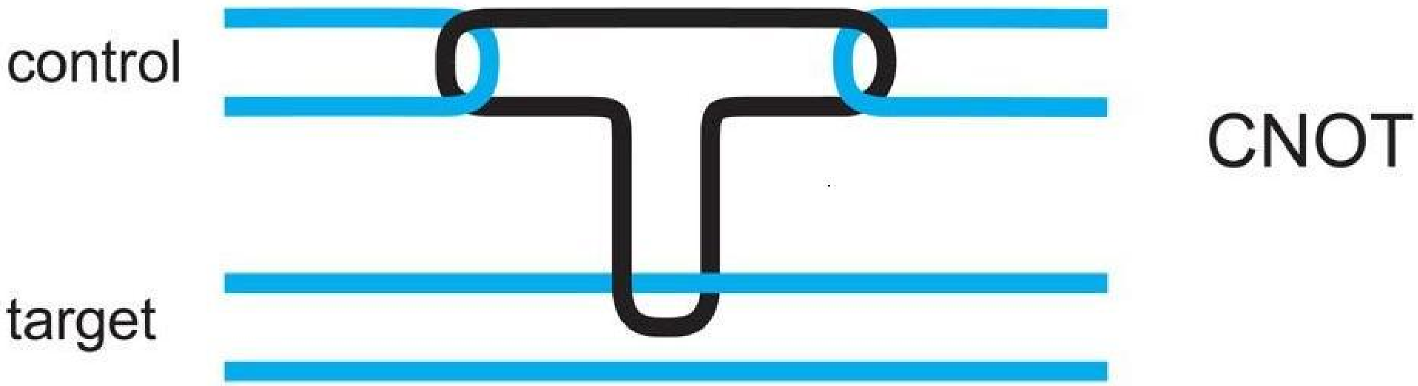

Theory
Architecture
Experiment
Kirk Madison's lab @ UBC

# Our setting




- 2D/3D: *Nearest-neighbor translation-invariant interaction*.
- High fault-tolerance threshold

# Our setting



# Fault-tolerant quantum computation

Task:

- *Maintain the quantum speedup in the presence of decoherence.*

Solution:

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*Fault-tolerance theorem**: If for a universal quantum computer the noise per elementary operation is below a constant non-zero error threshold $\epsilon$ then arbitrarily long quantum computations can be performed efficiently with arbitrary accuracy.

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*: Aharonov & Ben-Or (1996), Kitaev (1997), Knill & Laflamme & Zurek (1998), Aliferis & Gottesman & Preskill (2005)

# Talk outline

1. Universal cluster state computation.
   - The scheme: computation by local measurements
   - Cluster states: creation, definition, experiment

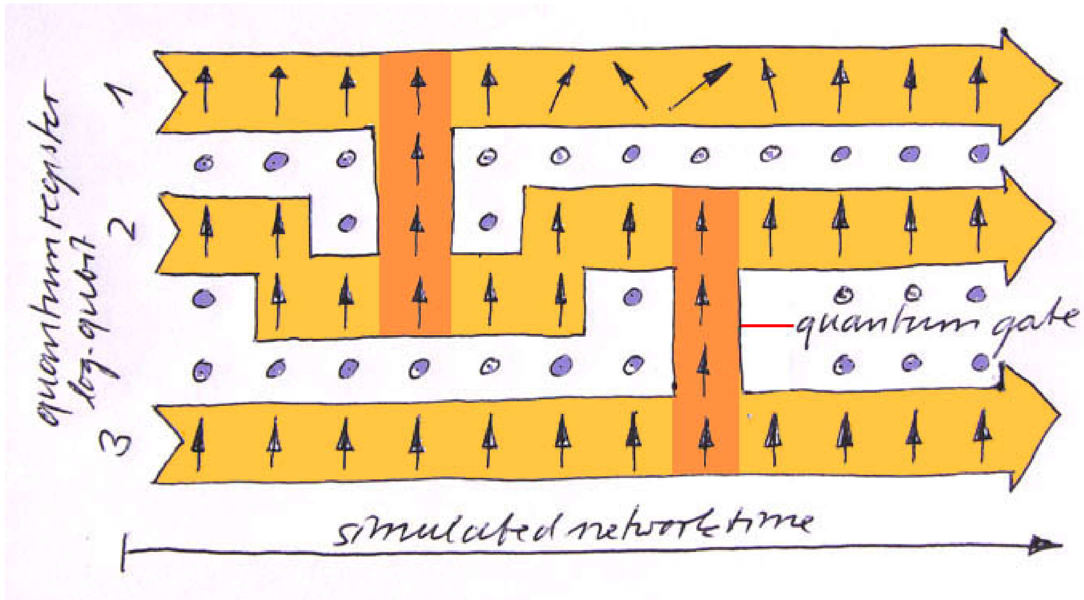
2. General introduction to quantum error-correction

3. Making cluster state computation fault-tolerant

# Part I:

## Cluster state quantum computation

# 1.1 Cluster state quantum computation




measurement of $Z$ ($\odot$), $X$ ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

R. Raussendorf and H.J Briegel, PRL **86**, 5188 (2001).

## 1.2 Cluster states - creation

1. Prepare product state $\bigotimes_{a\in\mathcal{C}} \frac{|0\rangle_a + |1\rangle_a}{\sqrt{2}}$ on $d$-dimensional qubit lattice $\mathcal{C}$.

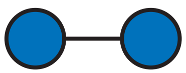
2. Apply the Ising interaction for a fixed time $T$:

$$U_{\text{Ising}} = e^{-i\frac{gT}{\hbar}\sum_{\langle i,j\rangle}\sigma_z^{(i)}\sigma_z^{(j)}}, \text{ with}$$

$$\frac{gT}{\hbar} = \frac{\pi}{4}.$$

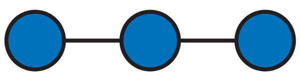
- Interaction time $T$ *independent* of cluster size.

## 1.2 Cluster states - simple examples



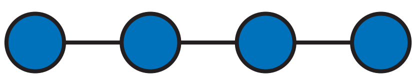
$$|\psi\rangle_2 = |0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2$$

Bell state



$$|\psi\rangle_3 = |+\rangle_1|0\rangle_2|+\rangle_3 + |-\rangle_1|1\rangle_2|-\rangle_3$$

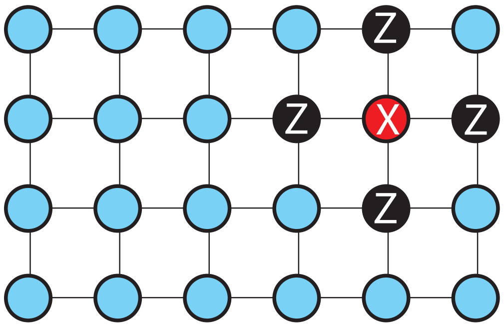
GHZ-state



$$|\psi\rangle_4 = |0\rangle_1|+\rangle_2|0\rangle_3|+\rangle_4 + |0\rangle_1|-\rangle_2|1\rangle_3|-\rangle_4 + \\ + |1\rangle_1|-\rangle_2|0\rangle_3|+\rangle_4 + |1\rangle_1|+\rangle_2|1\rangle_3|-\rangle_4$$

Number of terms exponential in number of qubits!

# 1.2 Cluster states - definition




A cluster state $|\phi\rangle_{\mathcal{C}}$ on a cluster $\mathcal{C}$ is the single common eigenstate of the stabilizer operators $\{K_a\}$,

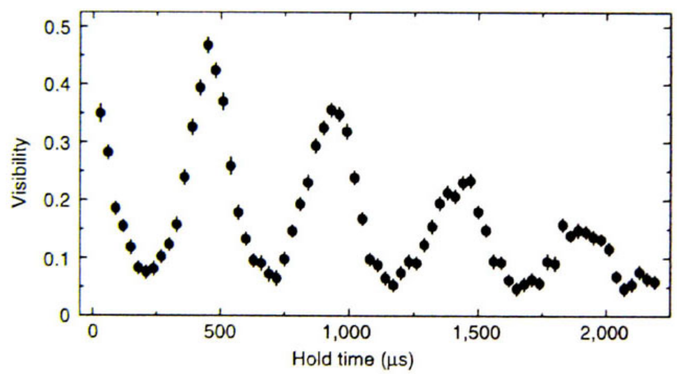
$$K_a|\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}},\ \forall a \in \mathcal{C},$$

with

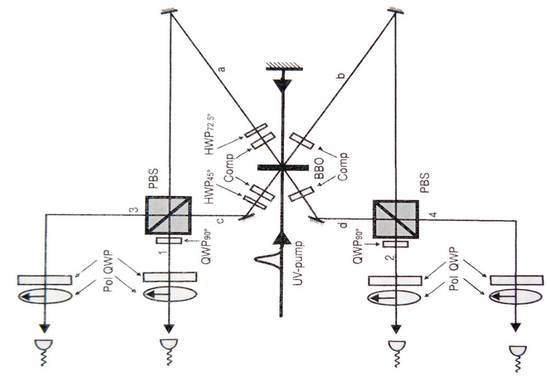
$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in \mathcal{C}. \tag{1}$$

Therein, $b \in N(a)$ if $a$,$b$ are spatial next neighbors in $\mathcal{C}$.

# 1.2 Cluster states - experiment




Cold atoms in optical lattices [1,2]




The $QC_{\mathcal{C}}$ with photons [3].


1: Greiner, Mandel, Esslinger, Hänsch, and Bloch, *Nature* 415, 39-44 (2002),

2: Greiner, Mandel, Hänsch and Bloch, *Nature*, 419, 51-54 (2002).

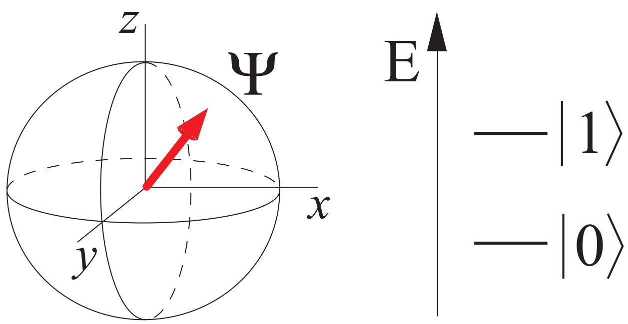
3: P. Walther *et al.*, *Nature* 434, 169 (2005).

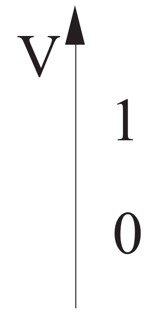
# Part II:

## Introduction to quantum error correction

*... take a break from cluster states*

# 2.1 Quantum vs. classical bits




$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$s = 1,0$$

**quantum bit**

– Measurement affects state

– Set of states continuous

**classical bit**

– Mmnt does not affect state

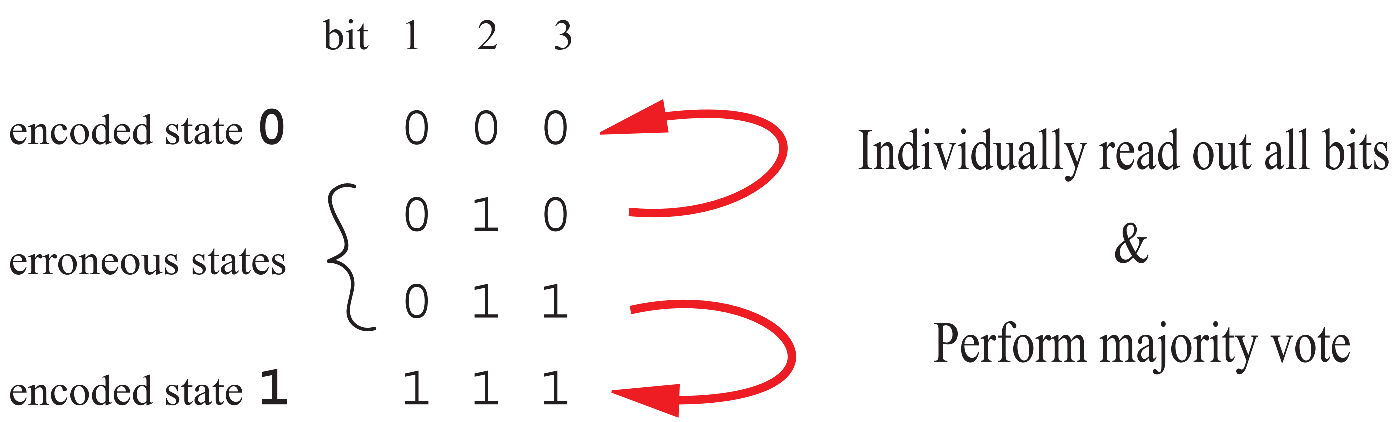
– Set of states discrete

Despite the differences:

- *Quantum error-correction (QEC) is possible.*
- *QEC is based on classical error correction.*

# 2.2 Starting point: Classical EC

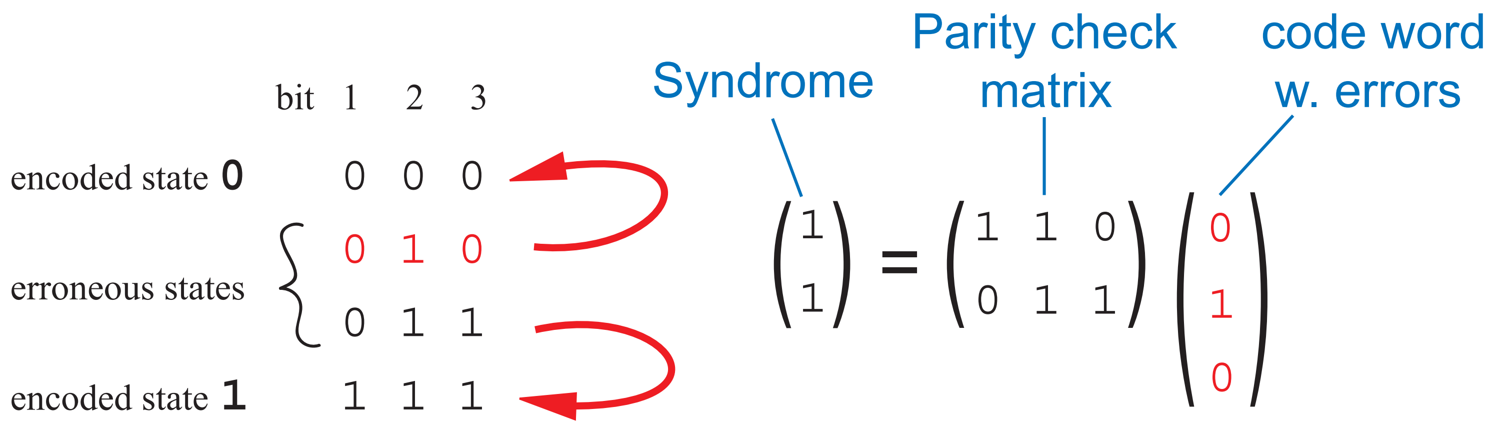
An example: the repitition code.




- Procedure on $n$-bit code corrects $\lfloor \frac{n-1}{2} \rfloor$ errors.
- Error-correction procedure learns encoded state.

# 2.2 Starting point: Classical EC

Same effect without state measurement: Read out parities only.




- Syndrome only reveals error, *not* encoded state: $Sy(c) = 0$, $\forall$ codewords $c$.

$$Sy(E \oplus c) \equiv Sy(E). \tag{2}$$

*Learning the state is not crucial for classical error-correction.*

## 2.3 How Quantum Error Correction works

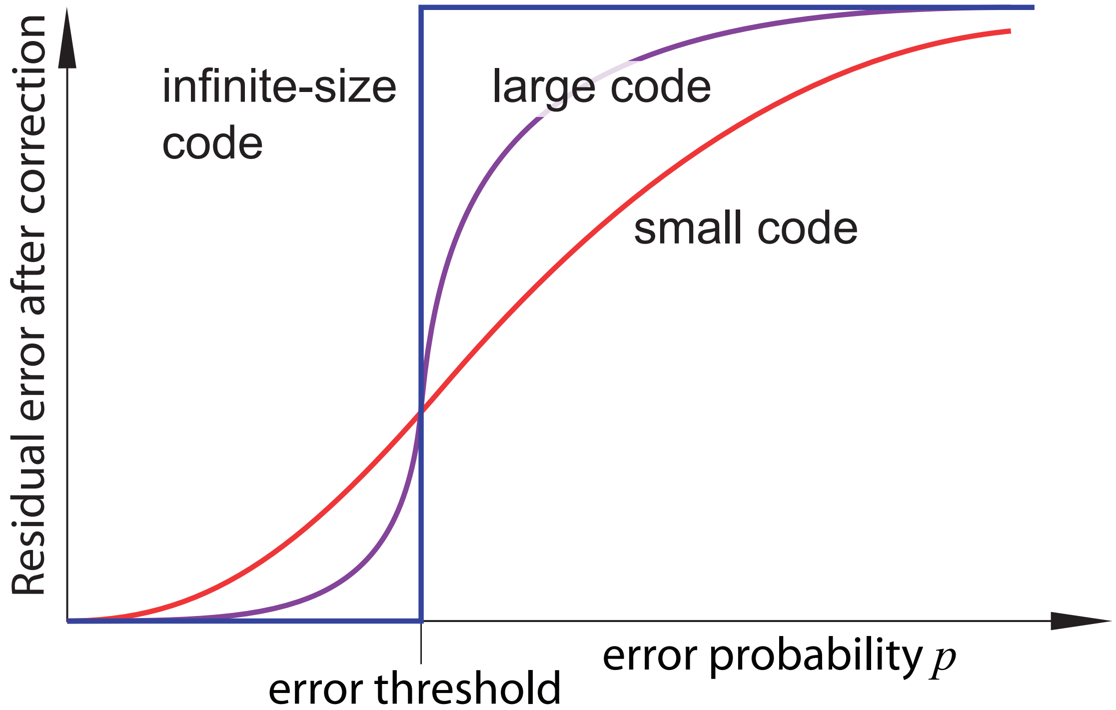
Classical-to-quantum dictionary:

| | | |
|---|---|---|
| $c \in \{000, 111\}$ | $\longrightarrow$ | $\overline{\|\Psi\rangle} = \alpha\|000\rangle + \beta\|111\rangle$ |
| Errors: bit flip | $\longrightarrow$ | spin & phase flips $\sigma_x$, $\sigma_y$, $\sigma_z$ |
| Parity check matrix $\overbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}$ | $\longrightarrow$ | stabilizer operators $Z_1 \otimes Z_2, Z_2 \otimes Z_3$ |
| Syndrome | $\longrightarrow$ | Measured eigenvalues of stabilizer operators. |

## 2.3 How Quantum Error Correction works

- Repeated measurement of the stabilizer operators, and conditional correction.
- Correctable errors *anti-commute* with at least one stabilizer operator $\rightarrow$ error-syndrome.
- Syndrome informs about an error, not the encoded state.

# Emergence of the error threshold

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*Fault-tolerance theorem*: For a universal quantum computer, an error per gate $< \epsilon$ is effectively as good as zero error.

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# So far...

- Have explained the basics of quantum error-correction.

- Have ignored:

  - Errors introduced by error-correction itself.

  - Computation.

*... but that can be fixed*

# Part III:

## Fault-tolerant quantum computation with 3D cluster states

# Part III outline

3.1 Topological quantum error-correction with 3D cluster states

3.2 Topologocal quantum gates

3.3 Fault-tolerance threshold, overhead scaling, mapping to 2D

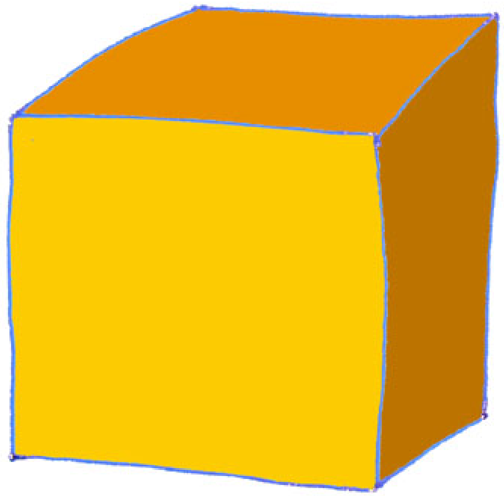
# Known threshold values

| no constraint | geometric constraint: 2D | geometric constraint: 1D |
| --- | --- | --- |
| [1] 0.03, est. | [5] $7 \cdot 10^{-3}$, est. | [7] $10^{-8}$, bd. |
| [2] $10^{-3}$, est. | [6] $2 \cdot 10^{-5}$, bd. | |
| [3] $10^{-4}$, est. | | |
| [4] $10^{-5}$, bd. | | |

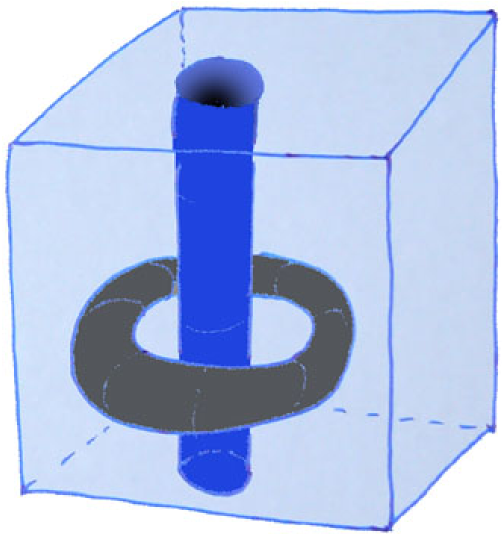
- Error sources:
  $|+\rangle$-Preparation, $\Lambda(Z)$-gates, Hadamard gates, measurement.

[1] Knill, (2005); [2] Zalka (1999); [3] Dawson & Nielsen (2005); [4] Aliferis & Gottesman & Preskill (2005), [5] Raussendorf & Harrington, quant-ph/0610062; [6] Svore & DiVincenzo & Terhal, quant-ph/0604090, [7] Aharonov & Ben-Or (1999)

# Main idea

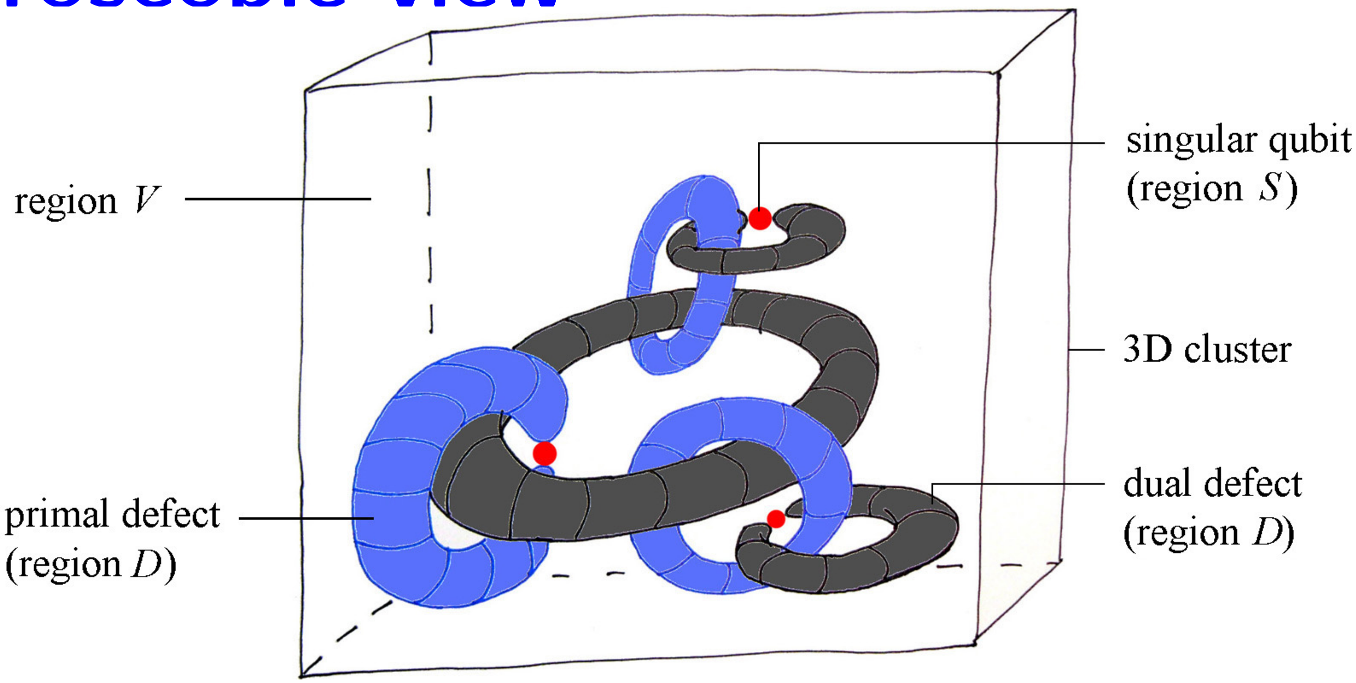


3D cluster state = fault-tolerant substrate



Gates from non-trivial cluster topology

# Macroscopic view



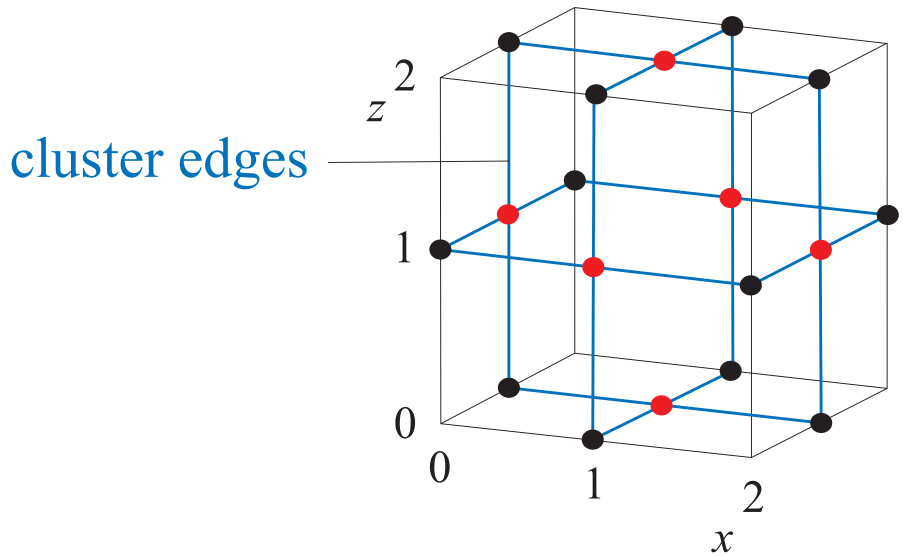

Three cluster regions:

$V$ (Vacuum), $D$ (Defect) and $S$ (Singular qubits).

| | |
|---|---|
| Qubits $q \in V$: | local $X$-measurements, |
| Qubits $q \in D$: | local $Z$-measurements, |
| Qubits $q \in S$: | local measurements of $X \pm Y$. |

R. Raussendorf, J. Harrington and K. Goyal, Ann. Phys. 321, 2242 (2006).

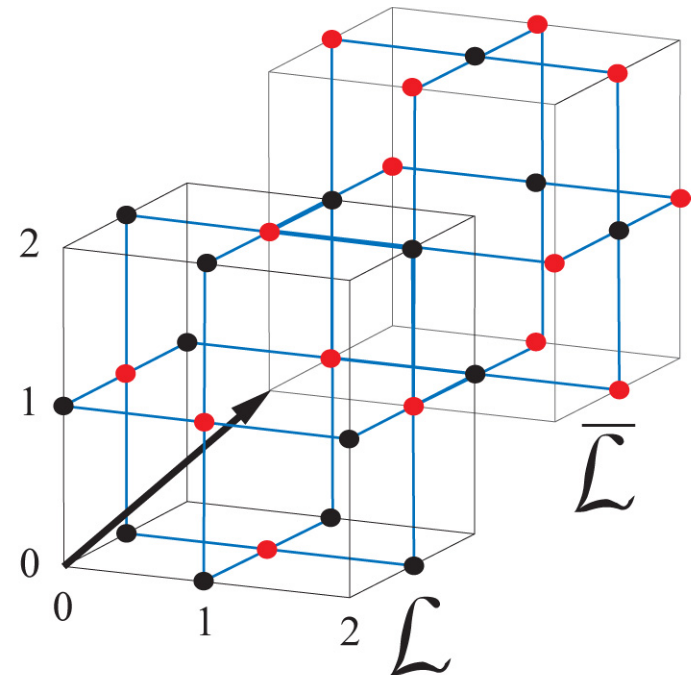
# Microscopic view




elementary cell of $\mathcal{L}$

| | | | |
|---|---|---|---|
| qubit location: | (even, odd, odd) | - | face of $\mathcal{L}$, |
| qubit location: | (odd, odd, even) | - | edge of $\mathcal{L}$, |
| syndrome location: | (odd, odd, odd) | - | cube of $\mathcal{L}$, |
| syndrome location: | (even, even, even) | - | site of $\mathcal{L}$. |

# Lattice duality $\mathcal{L} \longleftrightarrow \overline{\mathcal{L}}$




Translation by vector $(1,1,1)^T$:

- Cluster $\mathcal{C}$ invariant,
- $\mathcal{L}$ (primal) $\longrightarrow \overline{\mathcal{L}}$ (dual).

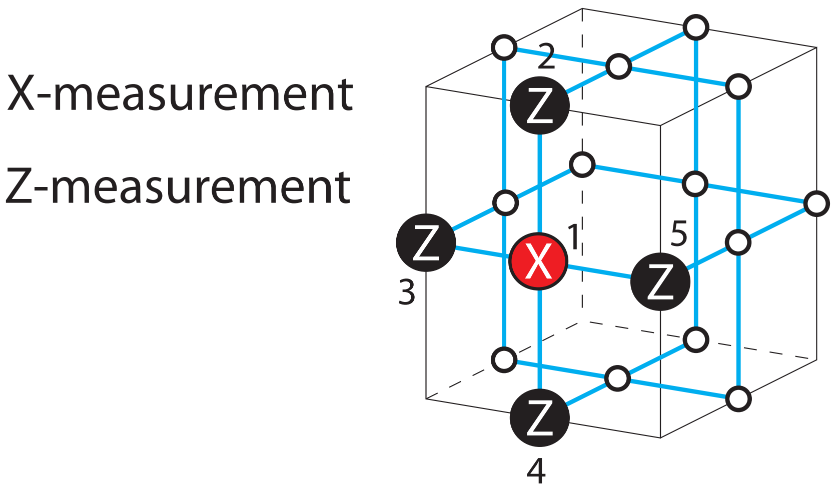
$$\begin{aligned} \text{face of } \mathcal{L} &\;-\; \text{edge of } \overline{\mathcal{L}}, \\ \text{edge of } \mathcal{L} &\;-\; \text{face of } \overline{\mathcal{L}}, \\ \text{site of } \mathcal{L} &\;-\; \text{cube of } \overline{\mathcal{L}}, \\ \text{cube of } \mathcal{L} &\;-\; \text{site of } \overline{\mathcal{L}}, \end{aligned} \tag{3}$$

- Many objects in this scheme exist as 'primal' and 'dual'.

# Part III.1:

## Quantum Error-correction in 3D cluster states

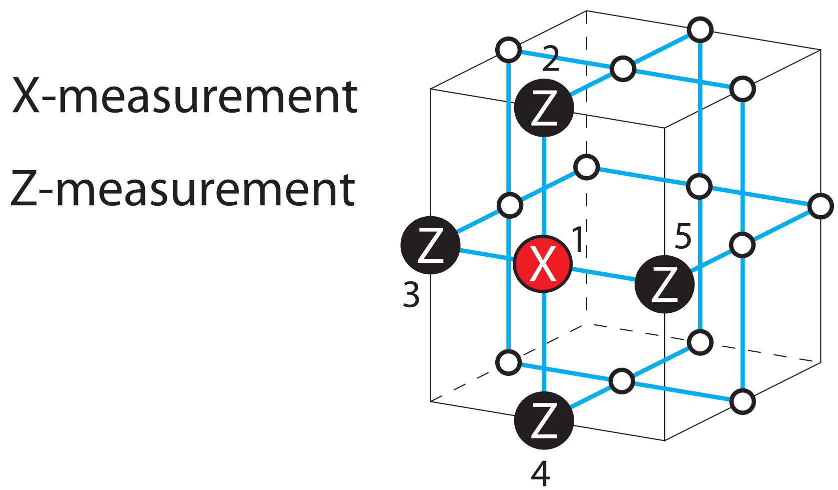
# 3.1 Measuring the cluster state stabilizer




$$K_1 = X_1 Z_2 Z_3 Z_4 Z_5$$

But ...

# 3.1 Measuring the cluster state stabilizer



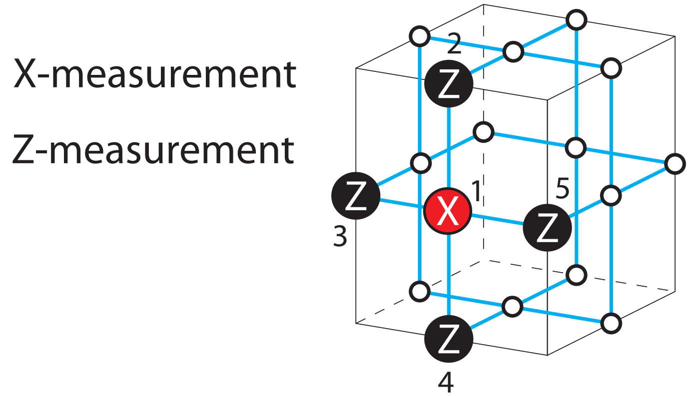

$$K_1 = X_1 Z_2 Z_3 Z_4 Z_5$$

$$\begin{matrix} \lambda_{X,1} & \lambda_{Z,2} & \lambda_{Z,3} & \lambda_{Z,4} & \lambda_{Z,5} & = +1. \\ \pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \end{matrix}$$

Measure eigenvalue of $K_1$ by local measurements on qubits 1 - 5.

But ...

# 3.1 Measuring the cluster state stabilizer




But ... all measurements in cluster region $V$ are in the $X$-basis.

*Are there stabilizer elements that we can measure by local $X$-measurements only?*

Criterion:

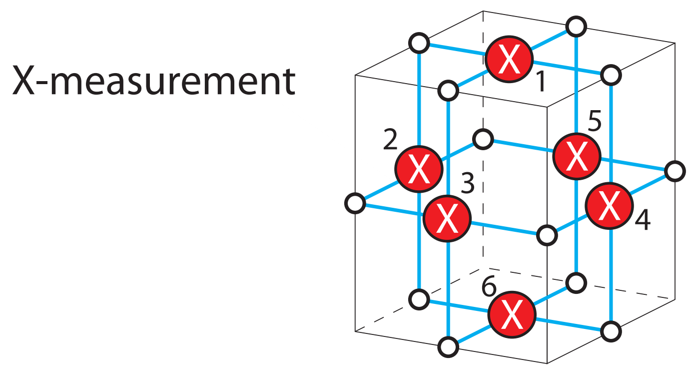
$$K_J = \bigotimes_{a \in J} X_a.$$

## 3.1 Measuring the cluster state stabilizer

Criterion: $$K_J = \bigotimes_{a\in J} X_a.$$

Such stabilizer elements exist!

Example:

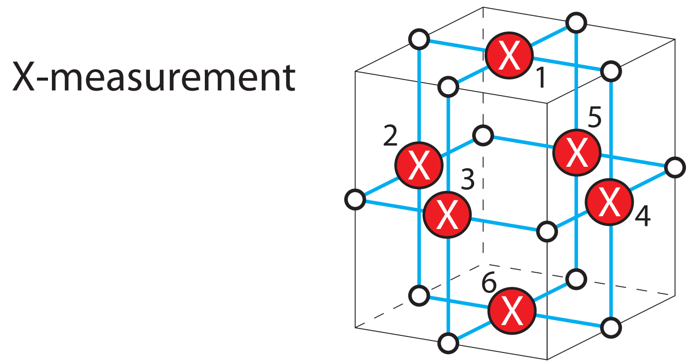



$$X_1 X_2 X_3 X_4 X_5 X_6 = K_1 K_2 K_3 K_4 K_5 K_6$$

Correlation of measured eigenvalues:

$$\begin{array}{cccccc} \lambda_{X,1} & \lambda_{X,2} & \lambda_{X,3} & \lambda_{X,4} & \lambda_{X,5} & \lambda_{X,6} \\ \pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 \end{array} = +1, \text{ if no error.}$$

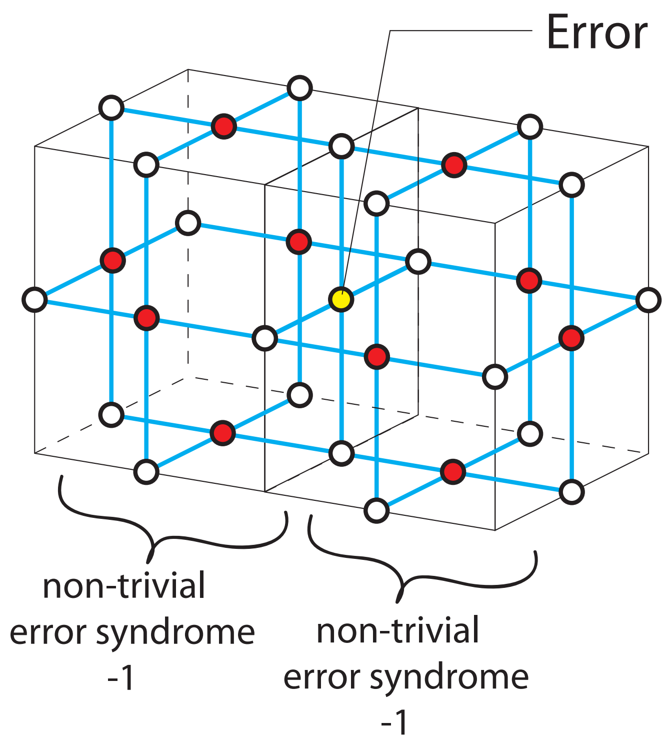
## 3.1 Measuring the cluster state stabilizer




$$\underbrace{\lambda_{X,1}\lambda_{X,2}\lambda_{X,3}\lambda_{X,4}\lambda_{X,5}\lambda_{X,6}}_{\text{Error syndrome}} = \text{-1}$$ indicates an error.

- One bit of error syndrome per lattice cell.

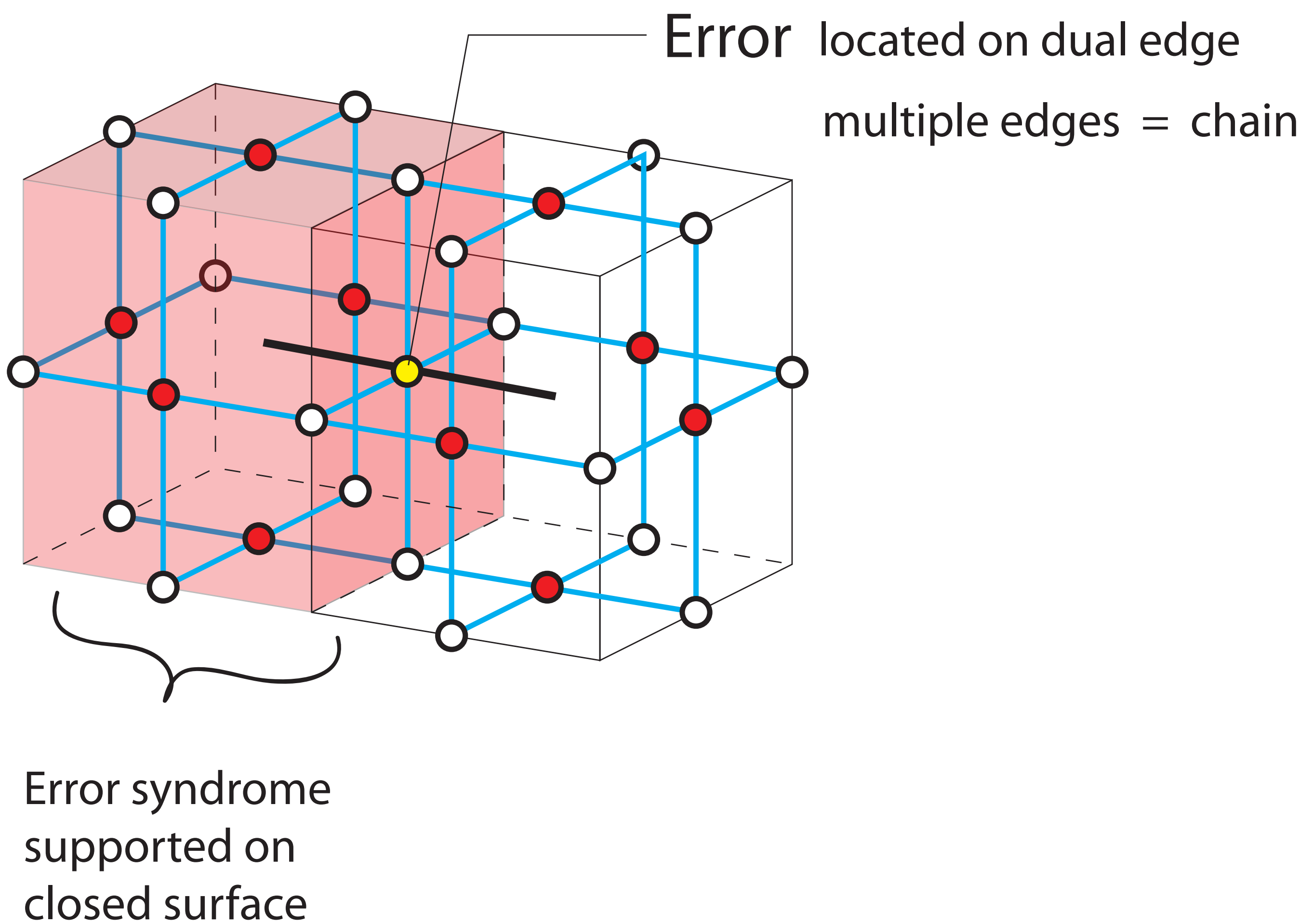
## 3.1 Measuring the cluster state stabilizer



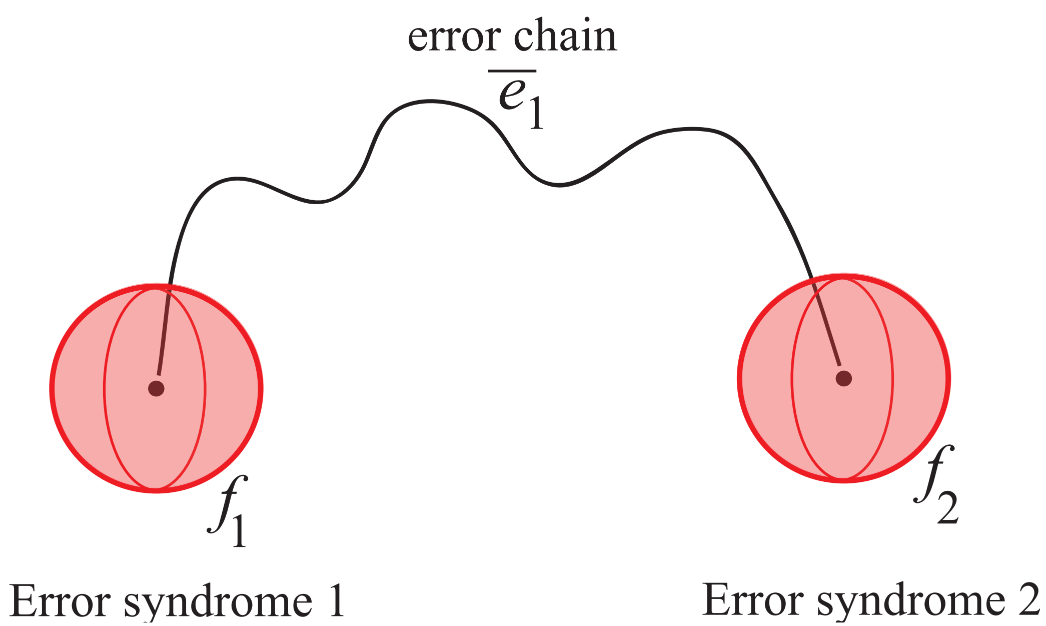

$Z$-error on face qubit yields non-trivial syndrome on adjacent cells.

- Each error leaves characteristic signature in the syndrome.
- Identify error by that syndrome.

# 3.1 Geometry and topology

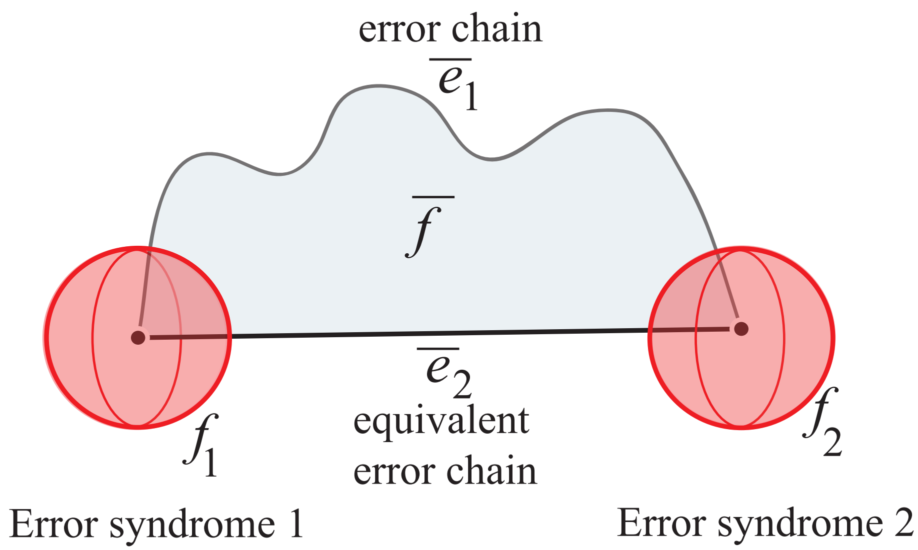


# 3.1 Geometry and topology




- An error chain $Z(\overline{e})$ is detected by a syndrome Sy($f$) if $e$ and $f$ *interesect* an even number of times.
- Intersection number is a topological invariant.

# 3.1 Geometry and topology

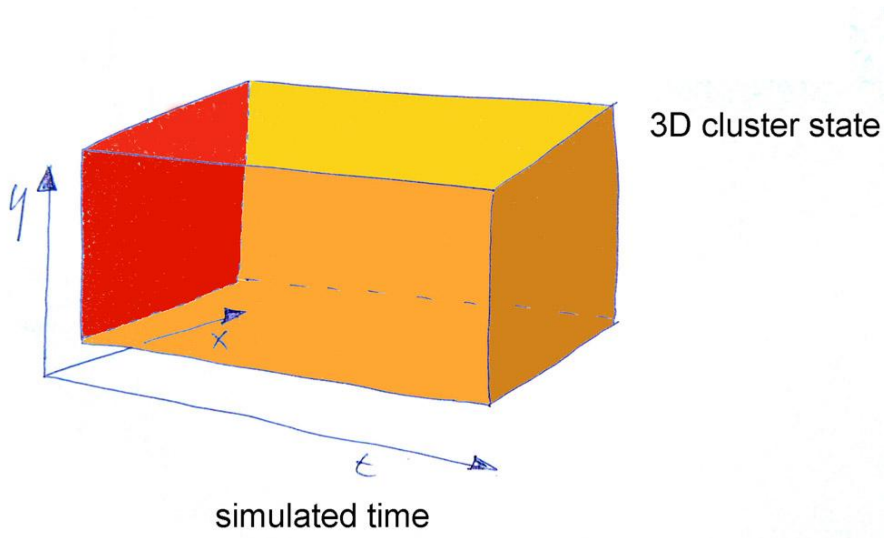



- Homologically equivalent error chains have same effect on the computation:

$$\overline{e}_2 = \overline{e}_1 + \partial \overline{f} \longrightarrow Z(\overline{e}_2) \equiv Z(\overline{e}_1).$$

- Only need to identify the *homology class* of the error.

# 3.1 Topological error-correction



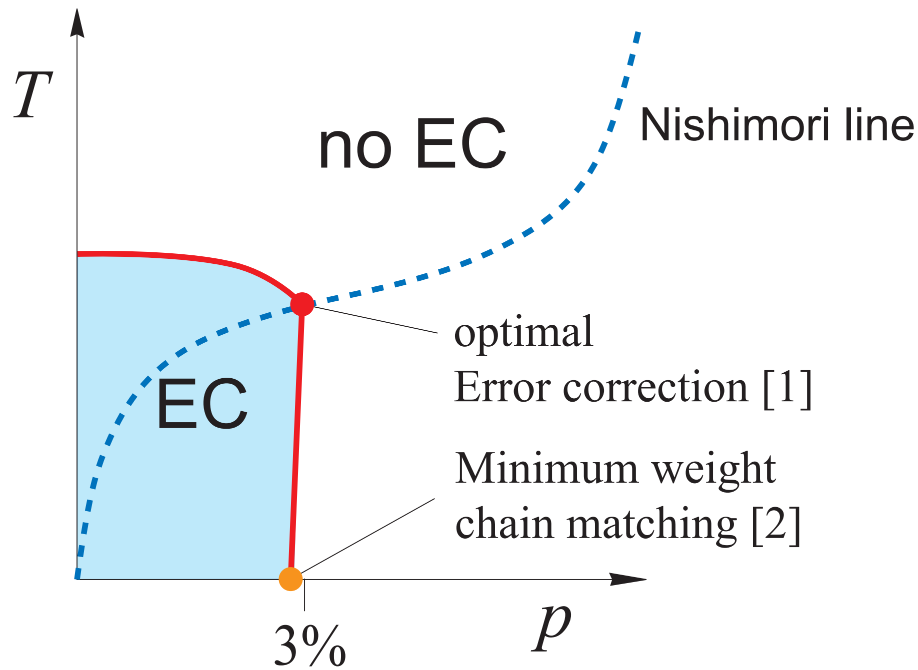

- Topological error-correction in 3D cluster states described by *Random plaquette* $Z_2$*-gauge model* (RPGM) [1].

- FT quantum memory with toric code described by RPGM as well [1].

[1] Dennis et *al.*, quant-ph/0110143 (2001).

# 3.1 Phase diagram of the RPGM

Map error correction to statistical mechanics:



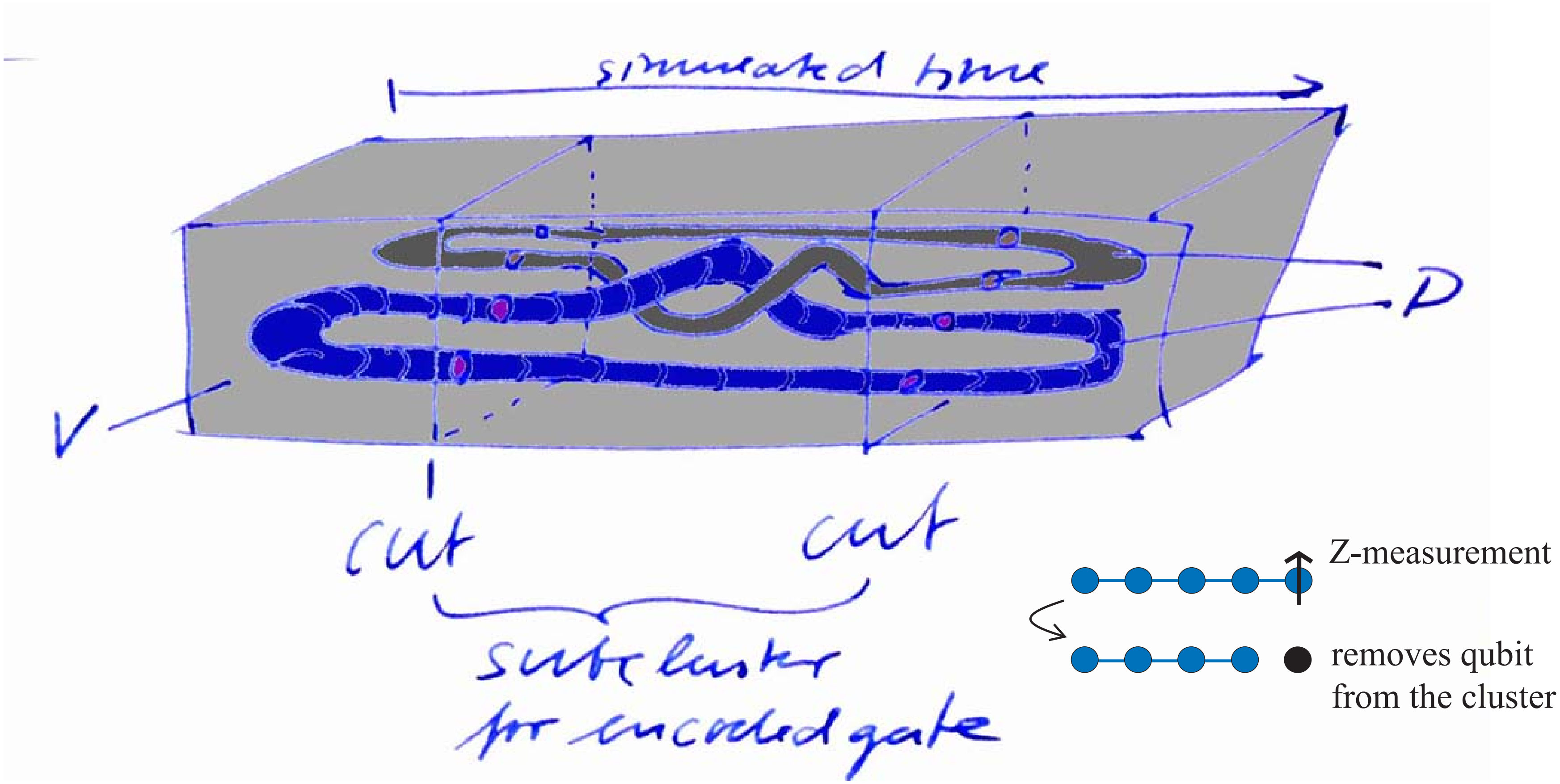

- Have an error budget of 3%.

[1] T. Ohno et *al.*, quant-ph/0401101 (2004). [2] E. Dennis et *al.*, quant-ph/0110143 (2001); J. Edmonds, Canadian J. Math. **17**, 449 (1965).

# Part III.2:

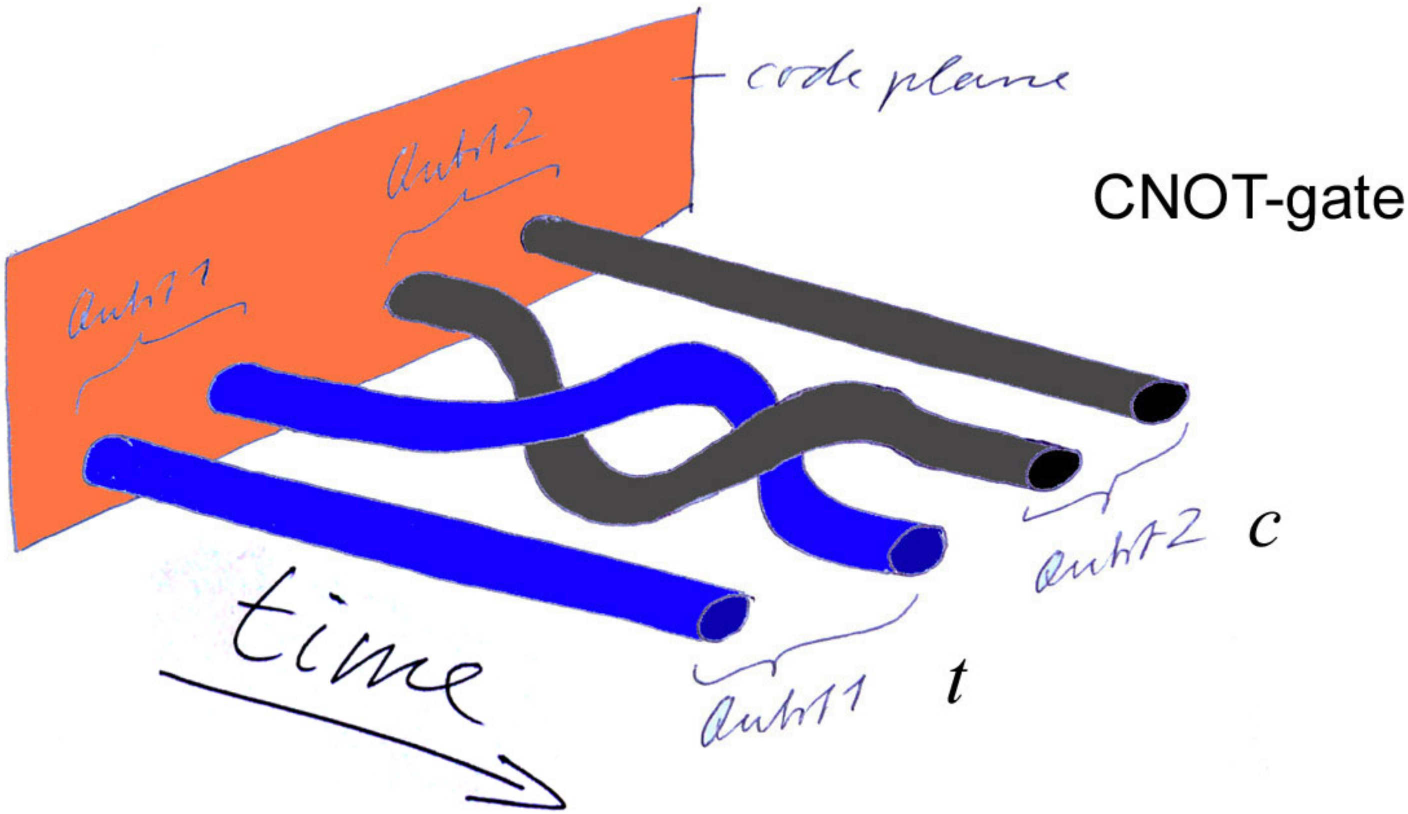
## Topological quantum gates

# 3.2 Encoded quantum gates



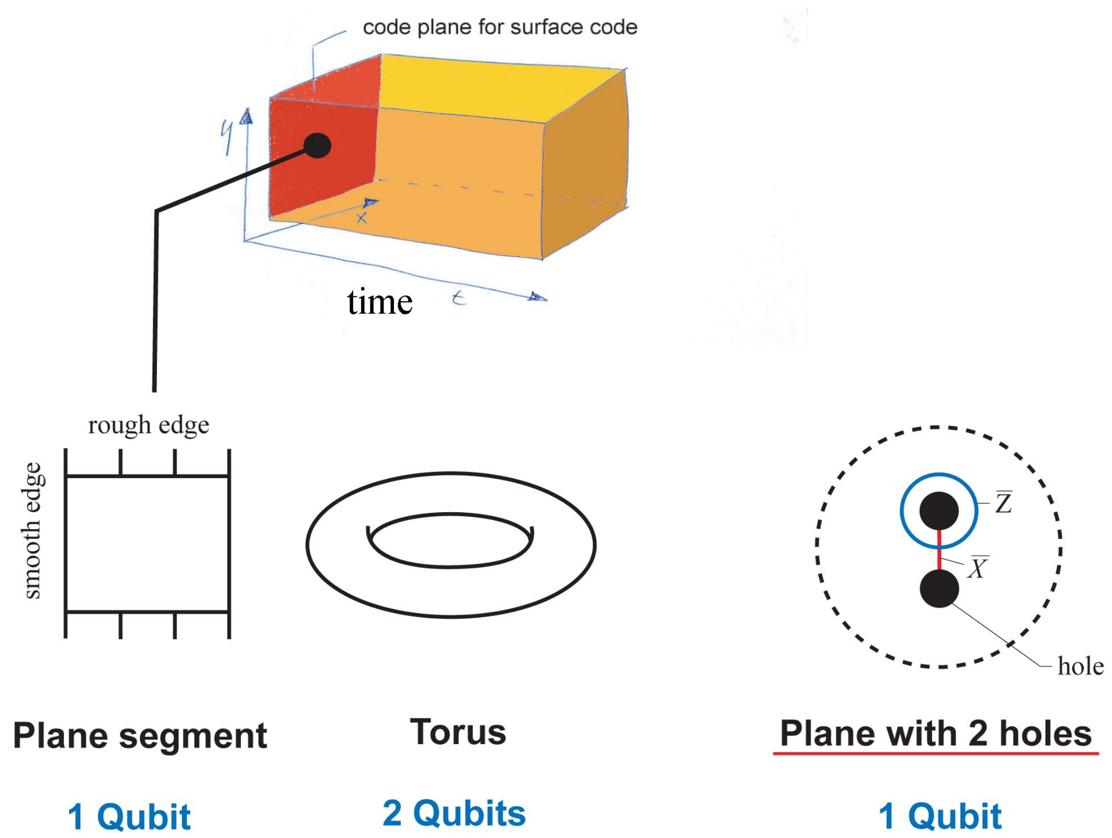

- Local $Z$-measurements remove the qubits in region $D$ from the cluster.
- Remaining cluster has non-trivial topology.

# 3.2 Encoded quantum gates



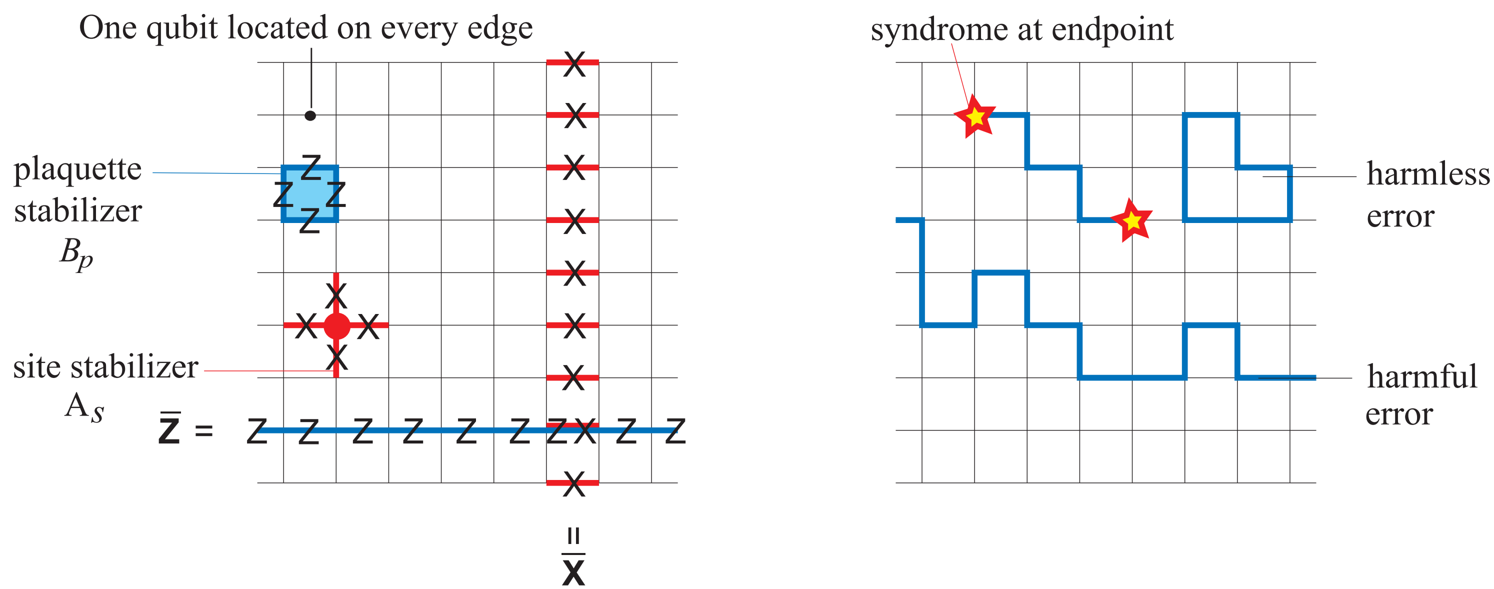

*Surface perpendicular to "time" supports a quantum code*

# 3.2 Surface codes




- Storage capacity of the code depends upon the topology of the code surface.

# 3.2 The surface code

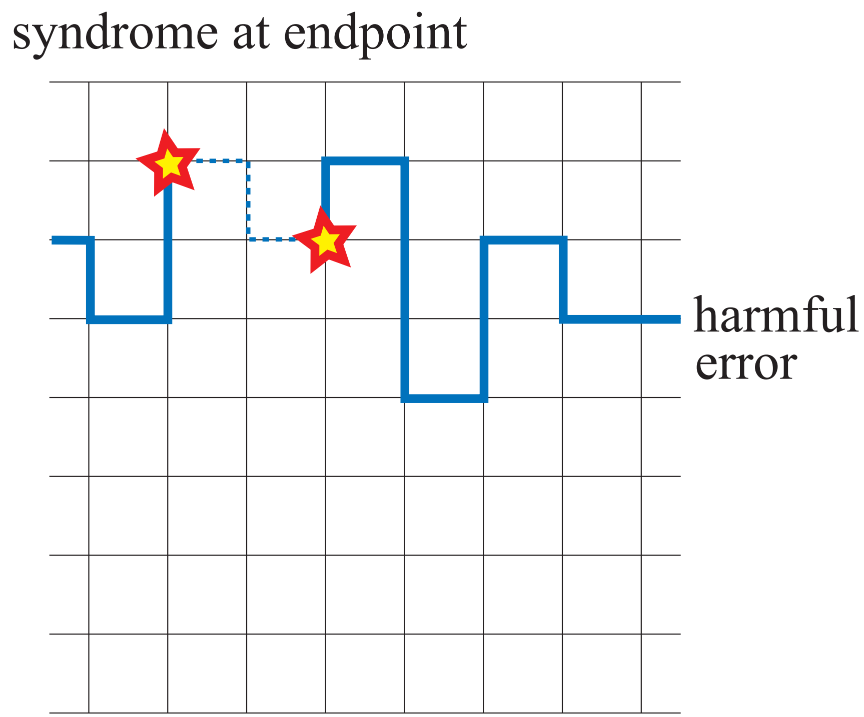



$$|\psi\rangle = A_s|\psi\rangle = B_p|\psi\rangle,\ \forall|\psi\rangle \in \mathcal{H}_C, \forall s, p. \qquad (4)$$

- Surface codes are stabilizer codes associated with 2D lattices.
- Only the *homology class* of an error chain matters.

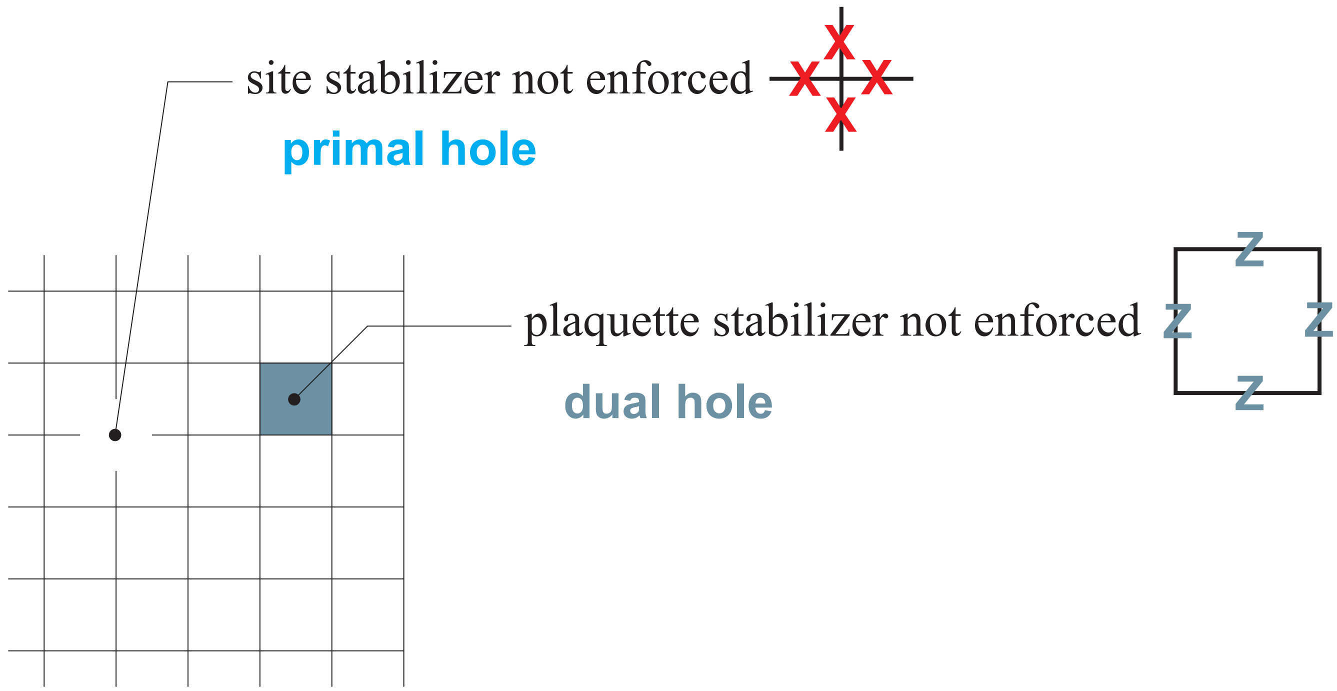
A. Kitaev,quant-ph/9707021 (1997).

# 3.2 The surface code




Non-correctable error: small weight-distance away
from non-trivial cycle.

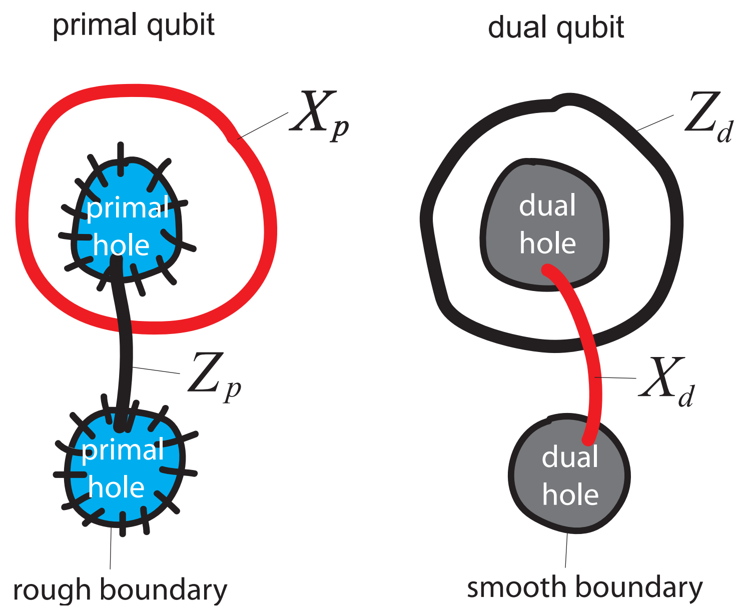
## 3.2 Surface code on plane with holes




- There are two types of holes: primal and dual.
- A pair of same-type holes constitutes a qubit.

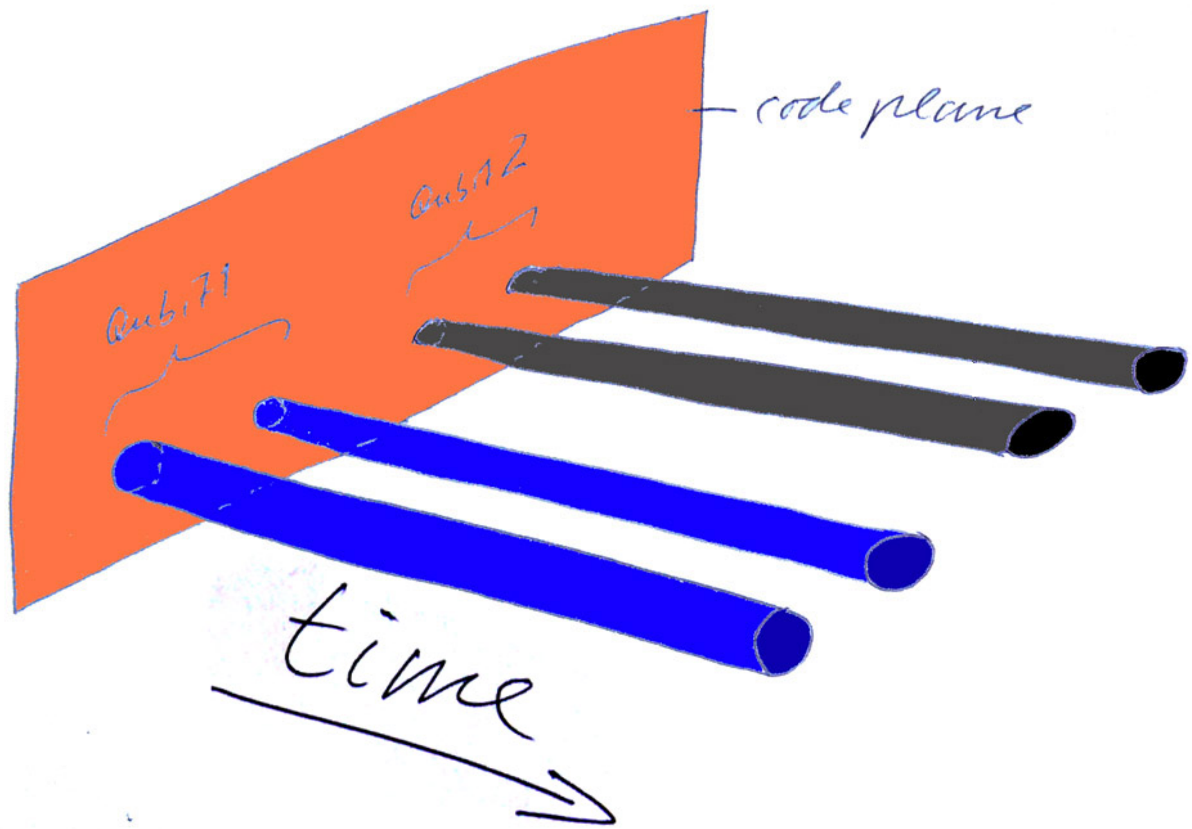
# 3.2 Surface code on plane with holes

Surface code with boundary:



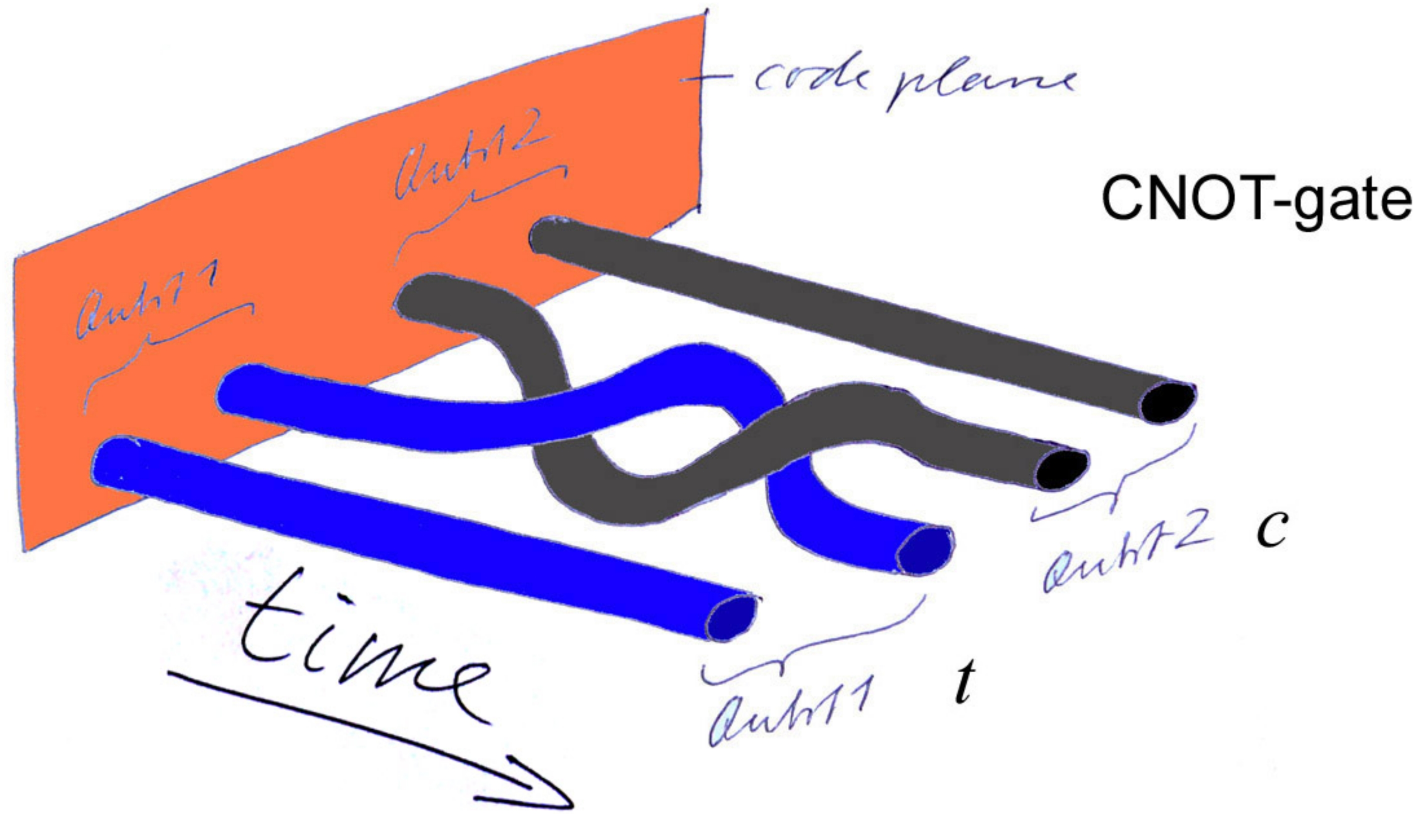

- $X$-chain cannot end in primal hole, can end in dual hole.
- $Z$-chain can end in primal hole, cannot end in dual hole.

# 3.2 Encoded quantum gates



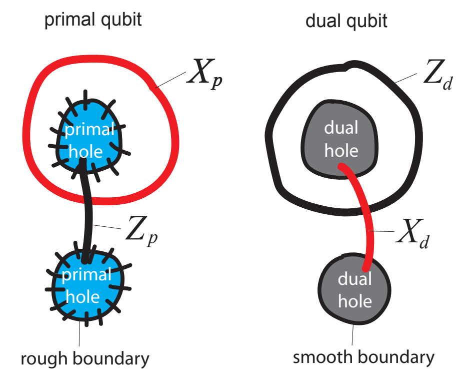

*Defect $D$ = worldline of hole.*

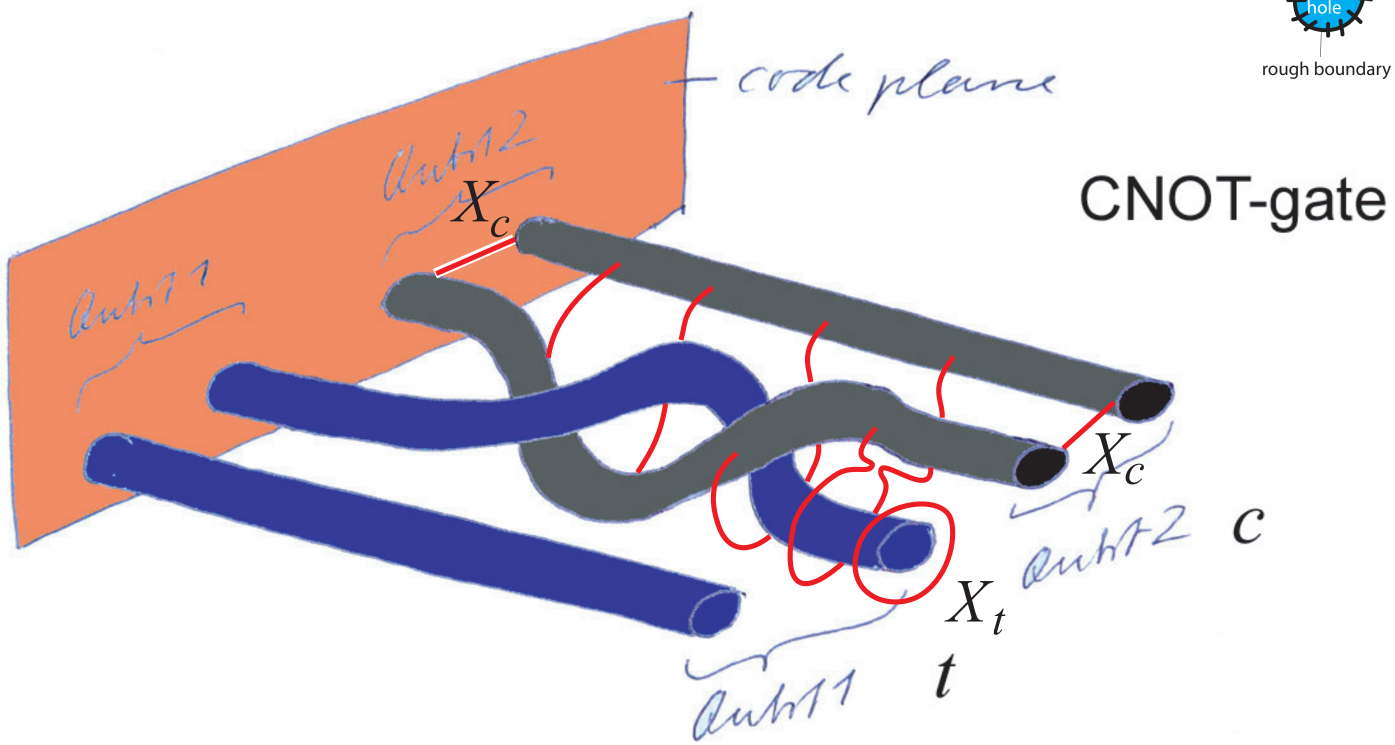
# 3.2 Encoded quantum gates




*Topological quantum gates are encoded in the way worldlines of primal and dual holes are braided.*

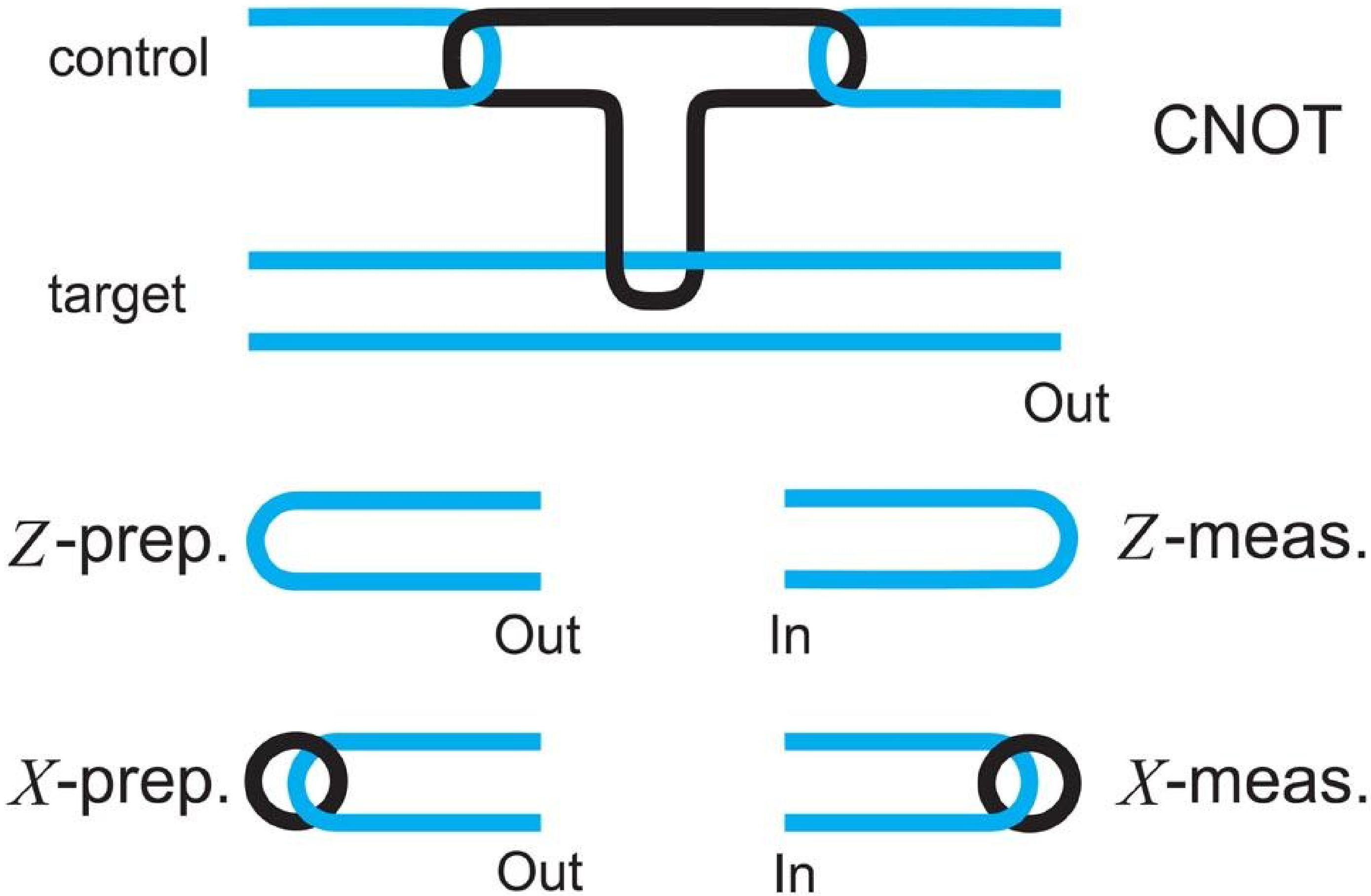
# 3.2 A CNOT-gate



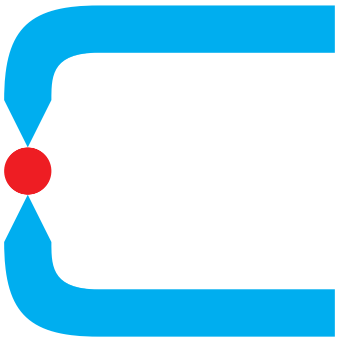

- Propagation relation: $X_c \longrightarrow X_c \otimes X_t$.
- Remaining prop rel $Z_c \rightarrow Z_c$, $X_t \rightarrow X_t$, $Z_t \rightarrow Z_c \otimes Z_t$ for CNOT derived analogously.

# 3.2 Topological quantum gates



# 3.2 Universal gate set

- Need one non-Clifford element:
  fault-tolerant preparation of $|A\rangle := \frac{X+Y}{\sqrt{2}}|A\rangle$.



Encoding of $|A\rangle$.

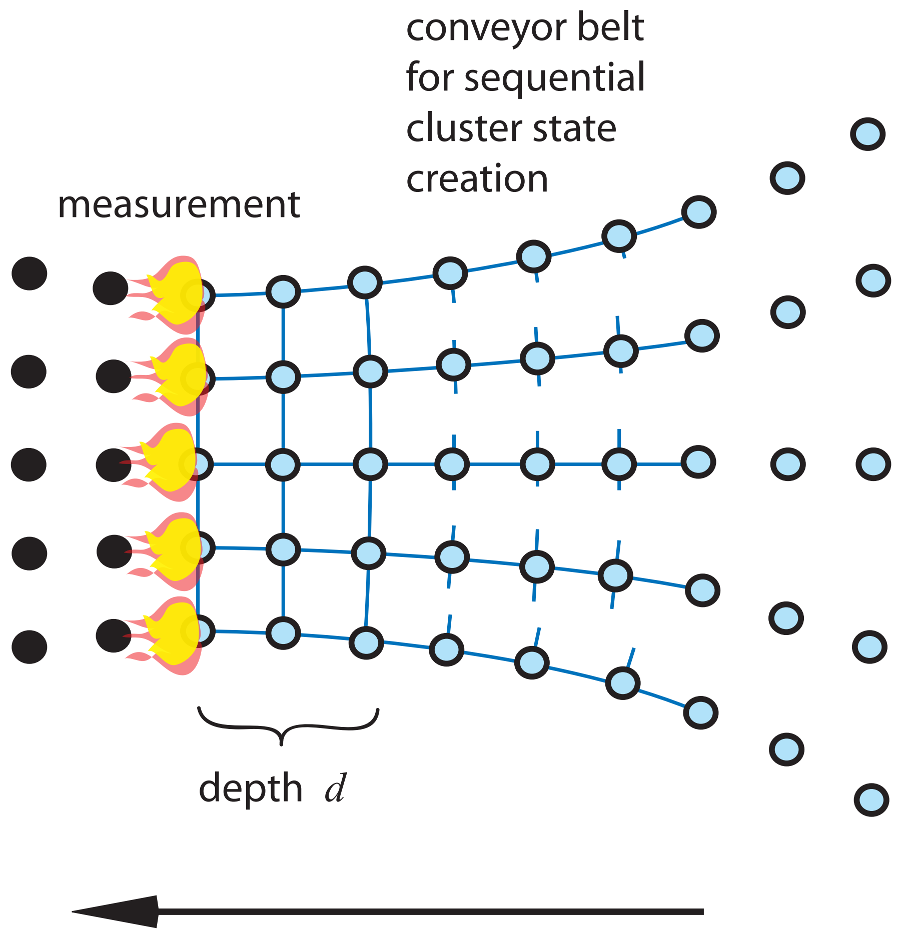
- FT prep. of $|A\rangle$ provided through realization of *magic state distillation**.

*: S. Bravyi and A. Kitaev, Phys. Rev. A 71, 022316 (2005).

# Part III.3:

## Threshold and Overhead Scaling

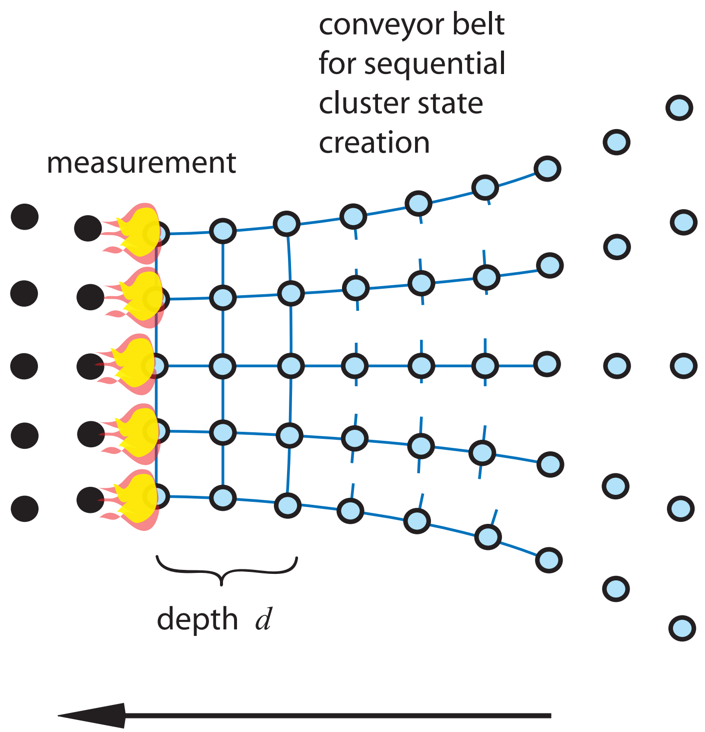
# 3.3 Sequential cluster state creation




*How large or small can the depth $d$ be?*

# 3.3 Sequential cluster state creation

*How large?*

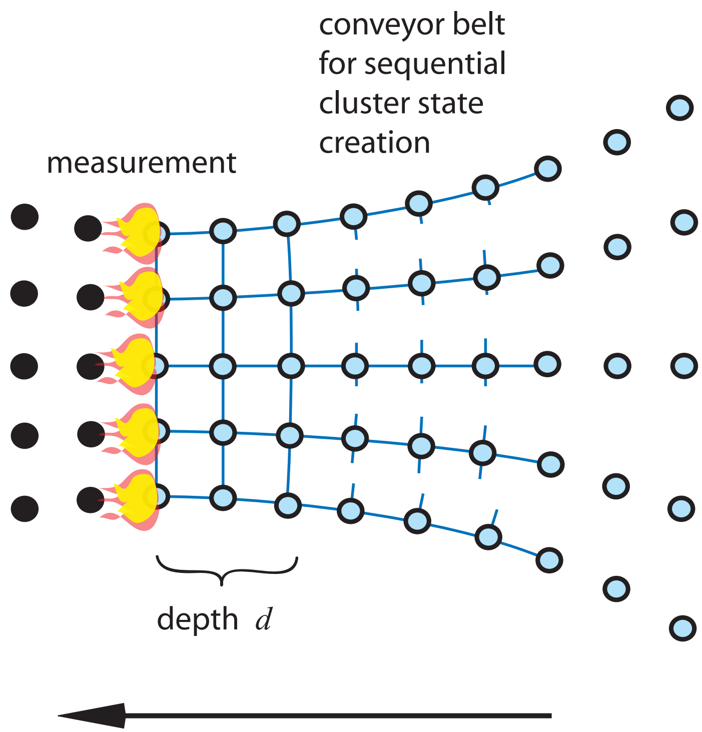



Limitation:

- Temporal order of measurements $\longrightarrow$ accumulating memory error.

# 3.3 Sequential cluster state creation

*How small?*




- $d \geq 1$.

- $d = 1$: mapping to circuit model, $d \geq 2$: cluster state.

## 3.3 Error model

- **Locality:** Errors are assocoated with the elementary gates of a quantum computer. Errors act where the gates act.

- **Independence:** Errors associated with different gates are stochastically independent.

- **Probabilistic error-model:** The elementary errors are probabilistic Pauli-flips $\sigma_x$, $\sigma_y$, $\sigma_z$ on all qubits.

All these assumptions can be relaxed.

Example: "Fault-tolerance with long-range correlated noise" (Dorit Aharonov, Alexei Kitaev, John Preskill, Phys. Rev. Lett. 96 (2006) 050504)

## 3.3 Error Model

- Error sources:

| 2D ($d = 1$) | 3D ($d \geq 2$) |
| --- | --- |
| 1. $\vert+\rangle$-preparation | 1. $\vert+\rangle$-preparation |
| 2. cPhase-gates | 2. cPhase-gates |
| 3. Hadamard-gates | 3. Memory error |
| 4. Local measurement | 4. Local measurement |

- Every quantum operation has same error $p$.

- Instant classical processing.

## 3.3 Fault-tolerance threshold

Topological threshold in cluster region $V$:

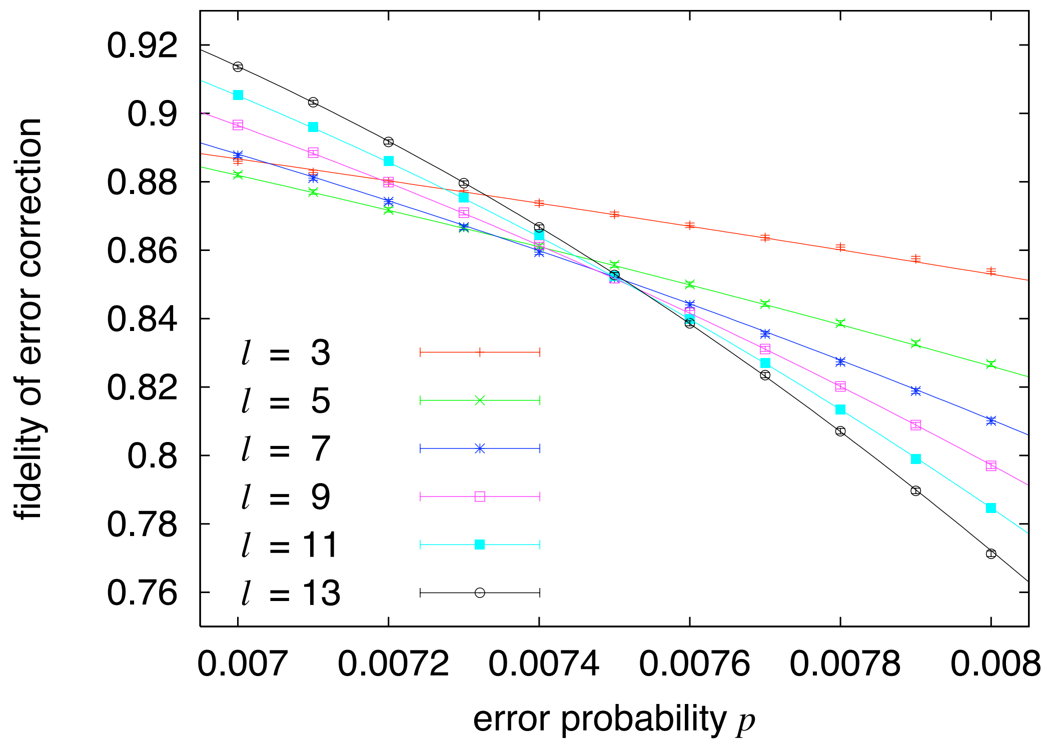
$$\begin{aligned} p_c &= 7.5 \times 10^{-3} \text{ (2D)}, \\ p_c &= 6.8 \times 10^{-3} \text{ (3D)}. \end{aligned} \quad (5)$$

Purification threshold for fault-tolerant $|A\rangle$-preparation:

$$p_c = 3.7 \times 10^{-2}. \quad (6)$$

*Topological EC sets the overall threshold.*

# 3.3 Fault-tolerance threshold




Numerical estimate of the fault-tolerance threshold in 2D.

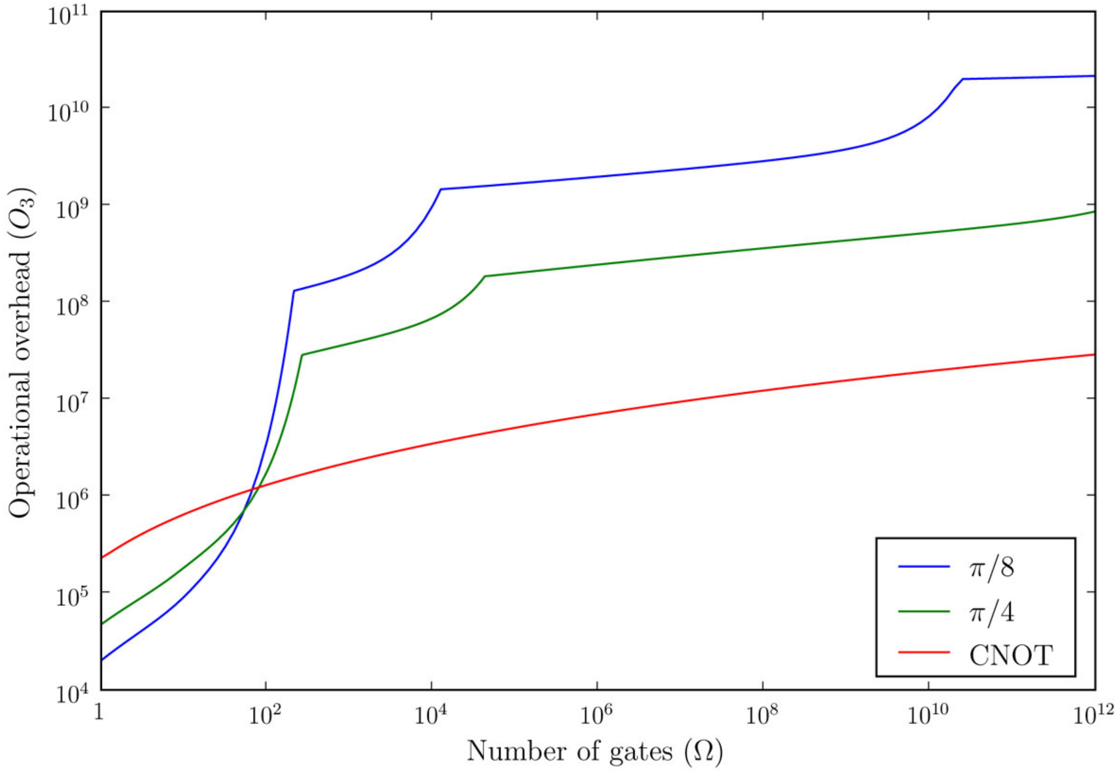
## 3.3 Overhead and Robustness

- Denote by $S$ ($S'$) the bare (encoded) size of a quantum circuit. Then, for the described method:

$$S' \sim S \log^3 S. \tag{7}$$

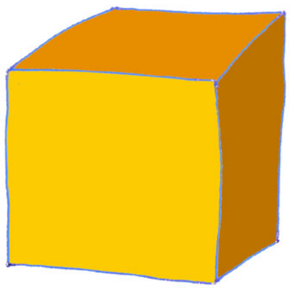
- The threshold is robust against variations in the error model such as higher weight elemetary errors, long-distance errors.

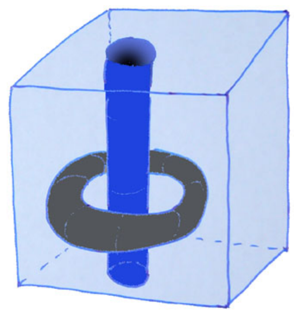
# 3.3 Overhead in absolute terms

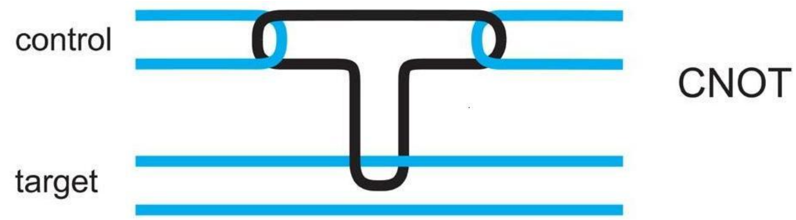



Operational cost of a fault-tolerant gate, at 1/3 threshold.

# Summary








- The 3D cluster state has error-correction built-in
- Encoded gates by topology
- High error threshold of 0.7%

Reading:

Raussendorf and Harrington, Phys. Rev. Lett. 98, 190504 (2007).

Raussendorf, Harrington and Goyal, New J. Phys. 9, 199 (2007).