CLASSICAL SIMULATIONS OF QUANTUM COMPUTATION

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Motivation

- The quantum computational speed-up is much more subtle than originally believed
- Turns out to be very difficult to build quantum algorithms for natural problems which exponentially outperform classical computers
- There exist non-trivial quantum computations that provably can be simulated classically in poly-time

Motivation

- So:

  What are the essential ingredients responsible for quantum computational power?

  What is the relationship between P and BQP?

  Is there any problem where quantum computers provably outperform classical ones?

- Understanding such questions may:
  - give insight into fundamental difference between classical and quantum physics
  - expose where we should look for new quantum algorithmic primitives

Outline

- Classical simulation
- Gottesman - Knill i.e. Clifford circuits
- Valiant i.e. Matchgate circuits
- Tensor contraction methods

CLASSICAL SIMULATION

OF Q.C.

What is classical simulation (ctd)?

- weak classical simulation : sample once from $\{p(x)\}$ in poly(n) time on classical computer

- Remarks : – weak simulation (obviously) more natural notion
  – $\exists$ q. circuits that can be weakly simulated but <u>not</u> strongly [cf. workshop talk]
  – strong simulation implies weak simulation

What is classical simulation of QC ?

- quantum computation : $|0\rangle^n \longrightarrow \underbrace{\mathcal{C}}_{\text{poly-size q. circuit}} |0\rangle^n \longrightarrow$ measure k qubits in $\{|0\rangle, |1\rangle\}$ basis

  (k: say, first k qubits $\{1, \dots, k\}$)

  output = k-bit string $x = (x_1 \dots x_k)$ $\quad x_i \in \{0,1\}$

  with probability $p(x) = \langle 0|^n \mathcal{C}^\dagger \, |x\rangle\langle x| \otimes \mathbb{1} \, \mathcal{C} |0\rangle^n$

- strong classical simulation :
  - evaluate $x \to p(x)$ up to m bits in poly(m,n) time on classical computer
  - evaluate all marginal distributions of $\{p(x)\}$ up to m bits in poly(m,n) time on class. comp!

    i.e. $S \subseteq \{1, \dots, k\}$ arbitrary subset ; bit string $y = (y_i : i \in S)$

    then $p_S(y) \equiv$ prob. of measuring y when performing measurement on qubits in S

    then $y \to p_S(y)$ up to m bits in poly(m,n) time $\forall S$

Strong simulation implies weak simulation

- $\{ p(x_1, \dots, x_k) \}$ prob. distribution on k bits

  Marginals : $S \subseteq \{1, \dots, k\}$ $y = (y_i : i \in S)$ $\quad p_S(y) \equiv p(y) = \sum_{z} p(y;z)$

  z ↳ all bits outside S

Suppose $\forall S, \forall y$: $p(y)$ can be computed eff. $\Rightarrow$ possible to sample from $\{p(x_1, \dots, x_k)\}$

- proof: take $k = 4$ i.e. goal is to sample from $p(x_1 x_2 x_3 x_4)$
  - central ingredient : conditioned probabilities e.g. $p(x_2 | x_3 x_4) = \dfrac{p(x_2 x_3 x_4)}{p(x_3 x_4)}$ can be computed eff. also !

  - $$p(x_1 x_2 x_3 x_4) = p(x_4) \cdot \frac{p(x_3 x_4)}{p(x_4)} \cdot \frac{p(x_2 x_3 x_4)}{p(x_3 x_4)} \cdot \frac{p(x_1 x_2 x_3 x_4)}{p(x_2 x_3 x_4)}$$
    $$= p(x_4) \cdot p(x_3 | x_4) \cdot p(x_2 | x_3 x_4) \cdot p(x_1 | x_2 x_3 x_4) \quad \otimes$$

  - algorithm : sample from $\{ p(x_4 = 0), p(x_4 = 1) \}$ → outcome x_4 → sample from $\{ p(x_3 = 0 | \underline{x_4}), p(x_3 = 1 | \underline{x_4}) \}$ → outcome x_3 → sample from $\{ p(x_2 = 0 | \underline{x_3 x_4}), p(x_2 = 1 | \underline{x_3 x_4}) \}$ → outcome x_2 → ...

Then total prob of obtaining $x_1 x_2 x_3 x_4$ is precisely $\otimes$ + procedure is efficient !

THE GOTTESMAN-KNILL THEOREM

Gottesman - Knill theorem

- (non-precise version:) Every quantum circuit composed of CNOT, H and PHASE can be simulated classically efficiently
  $$\mathrm{PHASE} = \mathrm{diag}(1, i)$$

[henceforth: Clifford circuit ≡ composed of CNOT, H, PHASE]

Gottesman - Knill thm

- 1st key example of nontrivial class of simulatable q. circuits
- Clifford operations : central in QIT
- conceptual importance of GK : provides insight in power of QC
- practical importance : Clifford operations alone do not yield good quantum algorithms

Outline of this chapter

- 1st variant of GK theorem + proof ("Heisenberg picture")
- 2nd variant of GK theorem + proof ("Schrödinger picture")

- Remarks

GK theorem, 1st variant

- **GK theorem ①**: Consider poly-size Clifford circuit acting on arbitrary n-qubit product input, followed by standard basis measurement of first qubit; denote output probabilities p_0 & $p_1 = 1 - p_0$. Then p_0 and p_1 can be computed classically up to m bits in poly(m,n) time.

- Note: arbitrary product state as input + single-qubit measurement
- Note: strong simulation

Proof of 1st variant

- Main ingredient : CNOT, H and PHASE are <u>Clifford</u> operations
  - Pauli operation $P = P_1 \otimes \ldots \otimes P_n$ $\quad P_i \in \{ \mathbb{1}, X, Y, Z \}$
  - Clifford operation C : for every Pauli P there exists Pauli P' such that $C^\dagger P C = \pm P'$

    [important class of operations, cf. vast literature]

  - Easy to check : CNOT, H and PHASE are Clifford (Exercise!)
  - if C and C' are Clifford then so is $C'C$

Proof of 1st variant, ctd

- Now consider poly-size Clifford circuit $\mathcal{C}$ + product input $|\alpha\rangle = |\alpha_1\rangle \otimes \ldots \otimes |\alpha_n\rangle$
- Then $p_0 = \langle\alpha| \mathcal{C}^\dagger\, |0\rangle\langle 0| \otimes \mathbb{1}\; \mathcal{C} |\alpha\rangle$
  $p_1 = \langle\alpha| \mathcal{C}^\dagger\, |1\rangle\langle 1| \otimes \mathbb{1}\; \mathcal{C} |\alpha\rangle$
  so $p_0 - p_1 = \langle\alpha| \mathcal{C}^\dagger \Big[\underbrace{(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes \mathbb{1}} \Big] \mathcal{C} |\alpha\rangle$
  $= \langle\alpha| \mathcal{C}^\dagger\, Z \otimes \mathbb{1}\; \mathcal{C} |\alpha\rangle$
- $Z \otimes \mathbb{1}$ = Pauli operation so $\mathcal{C}^\dagger (Z \otimes \mathbb{1}) \mathcal{C} = \pm P$ for some Pauli P
- Moreover, $\pm P$ can be computed efficiently classically!
- So $p_0 - p_1 = \pm \langle\alpha| P |\alpha\rangle$ where
  - P = product operator
  - $|\alpha\rangle$ = product state
- This yields efficient classical computation of $p_0 - p_1$; hence also of p_0 and p_1

∎

GK theorem, 2nd variant

- GK theorem 2 | Consider poly-size Clifford circuit $\mathcal{C}$ acting on $|0\rangle^n$, followed by standard basis measurement off all qubits. Then it is possible to sample classically in poly-time from distribution $\text{Prob}(x) = |\langle x|\mathcal{C}|0\rangle^n|^2 \quad x \in \{0,1\}^n$

- Note: input $= |0\rangle^n$; Thm still true for any standard basis input but not for any product input !

- Note: Thm still true for measurement of subset of qubits

- Note: here weak simulation ; but strong simulation also possible

Proof of 2nd variant

- Main ingredient: consider states of the following form:

$$|\psi\rangle = \sum_{x \in A} i^{\ell(x)} (-1)^{q(x)} |x\rangle \equiv |\psi(A, q, \ell)\rangle$$

where:
- $A = \{ Ru + t : u \in \mathbb{Z}_2^k \}$ affine subspace of $\mathbb{Z}_2^n$
- $q(x) = x^T B x + b^T x$ quadratic form over $\mathbb{Z}_2$
- $\ell(x) = a^T x$ linear function over $\mathbb{Z}_2$

- if $U \in \{ \text{CNOT}, H, \text{PHASE} \}$ then $U |\psi(A, q, l)\rangle = |\psi(A', q', l')\rangle$

for some A', q', ℓ'

AND update $(A, q, \ell) \to (A', q', \ell')$ is efficient!

[Proof: straightforward] [Exercise!]

Proof of 2nd variant, ctd.

- Now consider Clifford circuit $\mathcal{C}$ and input $|0\rangle^n$
- Note: $|0\rangle^n$ corresponds to $|\psi(A, g, l)\rangle$ with $A \equiv \{0\}$, $g \equiv 0$, $l \equiv 0$

  i.e. trivial instance

- Therefore: state of quantum register has the form $|\psi(A, g, l)\rangle$ throughout entire computation and each update is efficient
- Thus we may efficiently compute final state $|\psi(A, g, l)\rangle_{final}$

Proof of 2nd variant, ctd.

- Final state: $|\psi_{final}\rangle = \sum_{x \in A} (-1)^{q(x)} i^{l(x)} |x\rangle$
- now consider $\{|0\rangle, |1\rangle\}$ measurement of qubits
  - outcome is independent of q and l !
  - measurement yields uniformly random x in A
  - classical simulation = trivial :

    $A = \{Ru + t : u \in \mathbb{Z}_2^k\}$

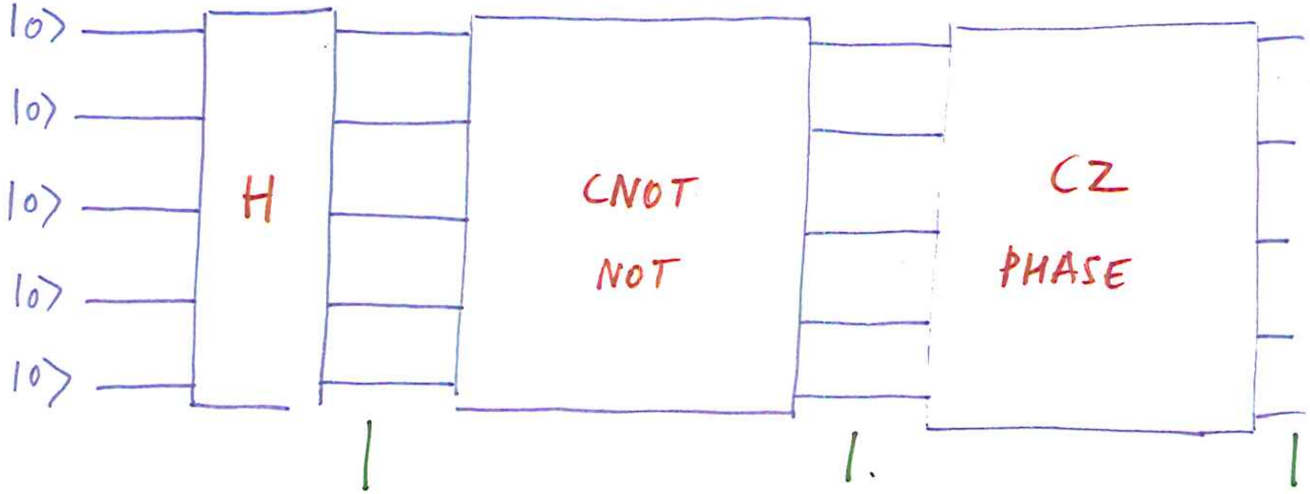
    so just generate random bit string u

    and apply R; then $x := Ru$ is random element in A

[Exercise: show that strong simulation is also possible] ∎

• Final remark:

$$C|0\rangle^n = |\psi\rangle = \sum_{x\in A} (-1)^{q(x)}\, i^{\ell(x)}\, |x\rangle$$ suggests alterative way to prepare $|\psi\rangle$



$\equiv |\psi\rangle$

$$\sum_u |u\rangle|0\rangle \qquad \sum_u |Ru+t\rangle = \sum_{x\in A} |x\rangle \qquad \sum_{x\in A} (-1)^{q(x)}\, i^{l(x)}\, |x\rangle$$

= Highly simplified circuit that prepares same output state

IMPORTANT: no "destructive" interference

GK variant ① versus ②

- ① focus on observables $\Longleftrightarrow$ ② focus on states
  "Heisenberg picture" "Schrödinger picture"
- crucial ingredient in both proofs: Clifford operations preserve certain closed family of states / observables
- Once this family is identified, proof is simple
  but this may be non-trivial

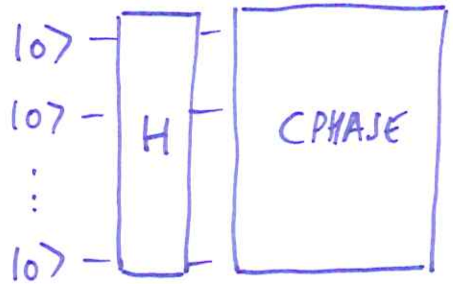
GK and entanglement

- Clifford circuits may generate complex entangled states

E.g. Cluster / graph states

$G = (V, E)$

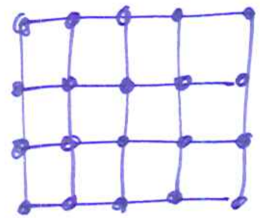
V: vertices, E: edges



$$\mathrm{CPHASE} \equiv CZ \equiv \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

$$|G\rangle = \prod_{ab \in E} CZ_{ab} \, |+\rangle^{n}$$

Graph states may be highly entangled: e.g. $G_{2D} \equiv$ 2D grid



$|G_{2D}\rangle \equiv$ 2D cluster state

- Thus, high degrees of entanglement are not sufficient for quantum speed-up!

Characterizing the power of Clifford circuits

- Clifford operations are not universal for q. computation; but what is their power?
- power of Clifford circuits is equivalent to power of classical circuits composed of CNOT and NOT
  - i.e. presence of H gate is essentially irrelevant (cf also ②)
  - associated complexity class: $\oplus L$ ("parity - L")

    [$\oplus L \subseteq P$; inclusion probably strict but no proof]
- Thus, Clifford circuits are (probably) not even universal for classical computation

VALIANT'S THEOREM

Valiant's theorem

- Matchgate G : 2-qubit unitary gate of the form

$$G = \begin{bmatrix} a & \cdot & \cdot & b \\ \cdot & x & y & \cdot \\ \cdot & z & t & \cdot \\ c & \cdot & \cdot & d \end{bmatrix} \quad \text{with } \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in SU(2)$$

- (Valiant's thm; non-precise version:) Every quantum circuit composed of matchgates acting on nearest-neighbor qubits can be simulated classically efficiently

[henceforth: Matchgate circuit $\equiv$ composed of n.n. matchgates]

Valiant's thm

- 2nd key example of simulatable quantum circuits
- conceptual importance: Matchgates highlight fragile gap between classical & quantum computation
- practical importance: Matchgates describe time evolution of important class of physical systems i.e. non-interacting fermions
  spin chains such as Ising, XY model
- Matchgates are interdisciplinary:
  - Computer science ("holographic algorithms")
  - Math (graph theory & planar matchings)

Outline of this chapter

- Matchgates, Clifford algebra's, non-interacting fermions
- Proof of Valiant's thm
- Achieving universal QC by adding SWAP
- Remarks

Clifford algebra's and quadratic Hamiltonians

- Jordan-Wigner operators on n qubits

$$c_1 = X \otimes 1 \otimes \ldots \otimes 1 \qquad c_3 = Z \otimes X \otimes \ldots \otimes 1 \qquad c_{2k-1} = Z \otimes \ldots \otimes Z \otimes \overset{\overset{k}{\downarrow}}{X} \otimes 1 \otimes \ldots \otimes 1$$

$$c_2 = Y \otimes 1 \otimes \ldots \otimes 1 \qquad c_4 = Z \otimes Y \otimes \ldots \otimes 1 \qquad c_{2k} = Z \otimes \ldots \otimes Z \otimes Y \otimes 1 \otimes \ldots \otimes 1$$

- anti-commutation relations $\{c_\mu, c_\nu\} = 2\delta_{\mu\nu} 1$ $\quad \mu, \nu = 1, \ldots, 2n$ [exercise: check!]

- quadratic Hamiltonian $H = i \sum_{\mu,\nu} h_{\mu\nu} c_\mu c_\nu$ $\quad (h_{\mu\nu})$ real + antisymmetric $2n \times 2n$ matrix

  - important class of Hamiltonians
  - describes all non-interacting fermionic systems $\quad a_k := \frac{1}{2}(c_{2k-1} + i c_{2k})$

    [check!] $\quad \{a_i, a_j\} = 0$ and $\{a_i, a_j^+\} = \delta_{ij} 1$

  - describes some 1D spin systems e.g. Ising model: $H = -J \sum X_k X_{k+1} - h \sum Z_k$

    $X_k X_{k+1} \propto c_{2k} c_{2(k+1)-1} \qquad Z_k \propto c_{2k-1} c_{2k}$

Matchgates and quadratic Hamiltonians

- Recall: Matchgate

$$G(A,B) := \begin{bmatrix} a & \cdot & \cdot & b \\ \cdot & x & y & \cdot \\ \cdot & z & t & \cdot \\ c & \cdot & \cdot & d \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$A, B \in SU(2)$$

- Consider n-qubit system;

Then every nearest-neighbor matchgate $G(A,B)_{k,k+1}$ is exponential of quadratic Hamiltonian !

E.g. Matchgate acting on qubit 1 and 2

$$G(A,B) = e^{iH} \quad \text{with} \quad H = \begin{bmatrix} k & \cdot & \cdot & l \\ \cdot & p & q & \cdot \\ \cdot & \bar{q} & -p & \cdot \\ \bar{l} & \cdot & \cdot & -k \end{bmatrix} \in \operatorname{span}\{\underset{-ic_1c_2}{\underset{\shortparallel}{Z\otimes 1}}, \underset{ic_1c_3}{\underset{\shortparallel}{Y\otimes X}}, \underset{ic_1c_4}{\underset{\shortparallel}{Y\otimes Y}}, \underset{-ic_2c_3}{\underset{\shortparallel}{X\otimes X}}, \underset{-ic_2c_4}{\underset{\shortparallel}{X\otimes Y}}, \underset{-ic_3c_4}{\underset{\shortparallel}{1\otimes Z}}\}$$

Matchgates and quadratic Hamiltonians

- Important :
  - nearest-neighbor
  - $A, B \in SU(2)$

  } crucial conditions !

Time evolutions of quadratic Hamiltonians

- **Theorem:** consider quadratic Hamiltonian H and $U := e^{iH}$.

  Then $U^\dagger c_\mu U = \sum_\nu R_{\mu\nu} c_\nu$ with $R = (R_{\mu\nu}) \in SO(2n)$

Proof: write $c_\mu(t) := e^{-itH} c_\mu e^{+itH}$

then $\frac{d c_\mu(t)}{dt} = \underbrace{i\,[H, c_\mu(t)]}$

- since H commutes with e^{itH} one has

  $i[H, c_\mu(t)] = i\, e^{-itH} [H, c_\mu]\, e^{+itH}$

- using commutation relations of c_ν's, one has

  $i[H, c_\mu] = \sum_\nu 4 h_{\mu\nu} c_\nu$ [Exercise: check]

$= \sum_\nu 4 h_{\mu\nu} c_\nu(t)$

Proof (ctd): thus $\frac{d c_\mu(t)}{dt} = \sum_\nu 4\, h_{\mu\nu}\, c_\nu(t)$

+ initial condition $c_\mu(0) = c_\mu$

unique solution: $c_\mu(t) = \sum_\nu R(t)_{\mu\nu}\, c_\nu$ with $R(t) = e^{4ht}$

[exercise: check!]

Remark: – $t = 1$ special case: $c_\mu(t=1) = U^\dagger c_\mu U$

thus $U^\dagger c_\mu U = \sum_\nu R(1)_{\mu\nu}\, c_\nu$

– $R \equiv R(1) = e^{4h} \in SO(2n)$ since h antisymmetric + real

$\blacksquare$

Proof of Valiant's thm

- **Valiant:** Consider poly-size nearest-neighbor matchgate circuit $\mathcal{C}$ acting on arbitrary n-qubit product input $|\alpha\rangle$, followed by standard basis measurement of first qubit; denote p_0 and p_1 as before.
  Then p_0 and p_1 can be computed classically up to m bits in $\mathrm{poly}(m,n)$ time.

- Note: arbitrary product input + single-qubit measurement
- Note: strong simulation

Proof of Valiant's thm

- As before: $p_0 - p_1 = \langle \alpha | \mathcal{C}^\dagger Z \otimes 1 \otimes \ldots \otimes 1 \, \mathcal{C} | \alpha \rangle$

- $Z \otimes 1 \otimes \ldots \otimes 1 = -i\, c_1 c_2$ [recall $c_1 = X \otimes 1 \ldots \otimes 1$ and $c_2 = Y \otimes 1 \ldots \otimes 1$]

- $\mathcal{C} = U_N \ldots U_1$ where each U_i is n.n. matchgate

  so $U_i \leftrightarrow R_i \in SO(2n)$ such that $U_i^\dagger c_\mu U_i = \sum_\nu [R_i]_{\mu\nu} c_\nu$

  let $R := R_N \ldots R_1$; then $\mathcal{C}^\dagger c_\mu \mathcal{C} = \sum_\nu R_{\mu\nu} c_\mu$ + computing R is efficient!

- $p_0 - p_1 = -i \langle \alpha | \mathcal{C}^\dagger c_1 c_2 \mathcal{C} | \alpha \rangle = -i \langle \alpha | [\mathcal{C}^\dagger c_1 \mathcal{C}][\mathcal{C}^\dagger c_2 \mathcal{C}] | \alpha \rangle$

  $$= -i \sum_{\mu,\nu} R_{1\mu} R_{2\nu} \langle \alpha | c_\mu c_\nu | \alpha \rangle \qquad (\mu, \nu = 1, \ldots, 2n)$$

- sum contains $(2n)^2 = \text{poly}(n)$ terms; $R_{1\mu}$ and $R_{2\nu}$ are eff computable;

  also $\langle \alpha | c_\mu c_\nu | \alpha \rangle$ eff computable as $c_\mu c_\nu$ = product operator.

  $|\alpha\rangle$ = product state

  $\boxtimes$

Similarities GK & Valiant

- compare Valiant to variant ① of GK
- As in case of GK: Matchgate circuits preserve closed (algebraic) framework

  viz. quadratic Hamiltonians

Importance of nearest-neighbor + $SU(2)$ conditions

- It is known that $G(A,B)$ gates, with $A, B \in SU(2)$, acting on arbitrary qubit pairs are universal for QC !
- Equivalently, nearest-neighbor $G(A,B)$ + SWAP gate are universal
- Stronger: n.n $G(A,B)$ + next-n.n. $G(A,B)$ are also universal !
- Alternatively: $G(A,B)$ gates acting on n.n. with $A, B \in U(2)$ are also universal:

$$\mathrm{SWAP} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot \\ \cdot & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} = G(\mathbb{1}, X) \quad \text{where } X \in U(2)$$

Matchgates and log-space quantum computation

- $\exists$ equivalence between n-qubit matchgate circuits and arbitrary quantum circuits acting on log(n) qubits

- $\mathcal{C} \xrightarrow{\text{log-space classical comp}} \mathcal{C}'$

  Mg circuit acting on n qubits, composed of N gates

  standard q. circuit acting on $O(\log n)$ qubits composed of $O(N \log n)$ gates

  $\mathcal{C}$ and $\mathcal{C}'$ are equivalent i.e. have same output distribution p_0, p_1

- reverse arrow $\mathcal{C}' \to \mathcal{C}$ also holds
- IDEA OF PROOF: regard $R \in SO(2n)$ as q. computation on $O(\log n)$ qubits !

TENSOR CONTRACTION METHODS

Tensor contraction methods

- Classical simulation based on structural properties of certain q. circuits
- conceptual importance: highlights role of entanglement as necessary ingredient for quantum speed-up
- practical importance: used in major investigations of strongly correlated systems (viz MPS, PEPS, MERA, ...)

Outline of this chapter

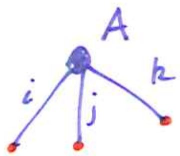
- Tensor networks
- Example: nearest-neighbor constant-depth circuits
- Role of entanglement

Tensor networks

- Tensor A_{ijk} $\quad i,j,k = 1,\dots,d$ $\qquad d^3$ complex numbers

  [rank = 3 ; dimension = d]

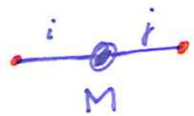
- graphically :



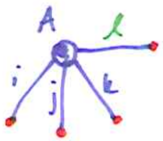
[often without labels]

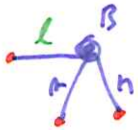
Examples : – vector $v = (v_i : i = 1,..,d)$

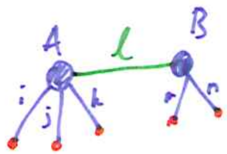
– Matrix $M = (M_{ij} : i,j = 1,..,d)$



- Contraction : $\displaystyle\sum_{l=1}^{d} A_{ijkl} B_{mln} = C_{ijkmn}$







rank(A) = 4 $\qquad$ rank(B) = 3 $\qquad$ rank(C) = 4 - 1 + 3 - 1 = 5

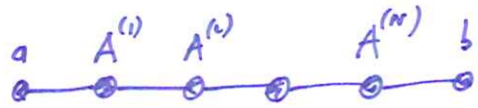
Tensor networks

- Tensor network = collection of tensors, contracted w. each other at certain indices, according to some graphical pattern [edge ≡ contraction]

- Examples : – [matrix element of] product of N $d \times d$ matrices

  $A^{(\alpha)}_{ij}$: $i,j = 1, \dots, d$ $a = (a_i : i = 1, \dots, d)$ $b = (b_i : i = 1, \dots, d)$

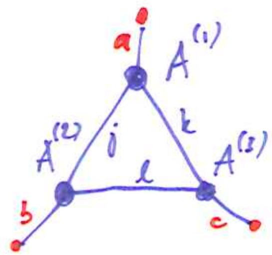
  Matrices ; $\alpha = 1, \dots, N$ vector   vector

  $a^T A^{(1)} \dots A^{(N)} b \equiv$ 

  – contraction of rank 3 tensors w.r.t. cycle graph

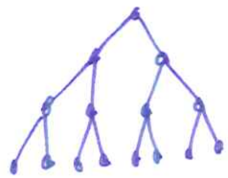
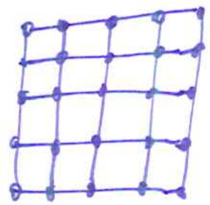
  $A^{(\alpha)}_{ijk}$ $i,j,k = 1, \dots, d$

  $\alpha = 1, 2, 3$

  $$\sum_{jkl} A^{(1)}_{ajk} A^{(2)}_{bjl} A^{(3)}_{ckl} \equiv$$ 

Tensor networks

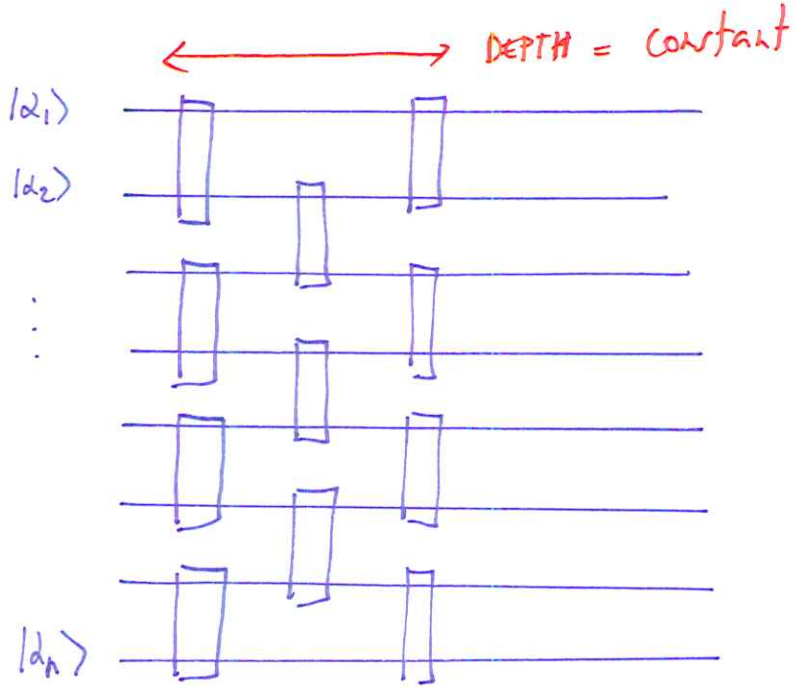
- Central problem: given network of N tensors $A^{(\alpha)}$ of $\begin{cases} \text{rank} & r \equiv \text{const} \\ \text{dimension} & d \equiv \text{const}, \end{cases}$
  how hard is it to contract this network?
  → efficient if contraction possible in poly (N) time

- Examples:
  -  $\equiv$ matrix product = efficient
  -  $\in$ Tree tensor network = efficient
    [exercise! TIP: contract from bottom to top]
  -  2D grid $N \times N$ not efficient (#P hard)

- General result: Every tensor network with is sufficiently "tree-like" can be contracted efficiently (→ notion of TREE-WIDTH)

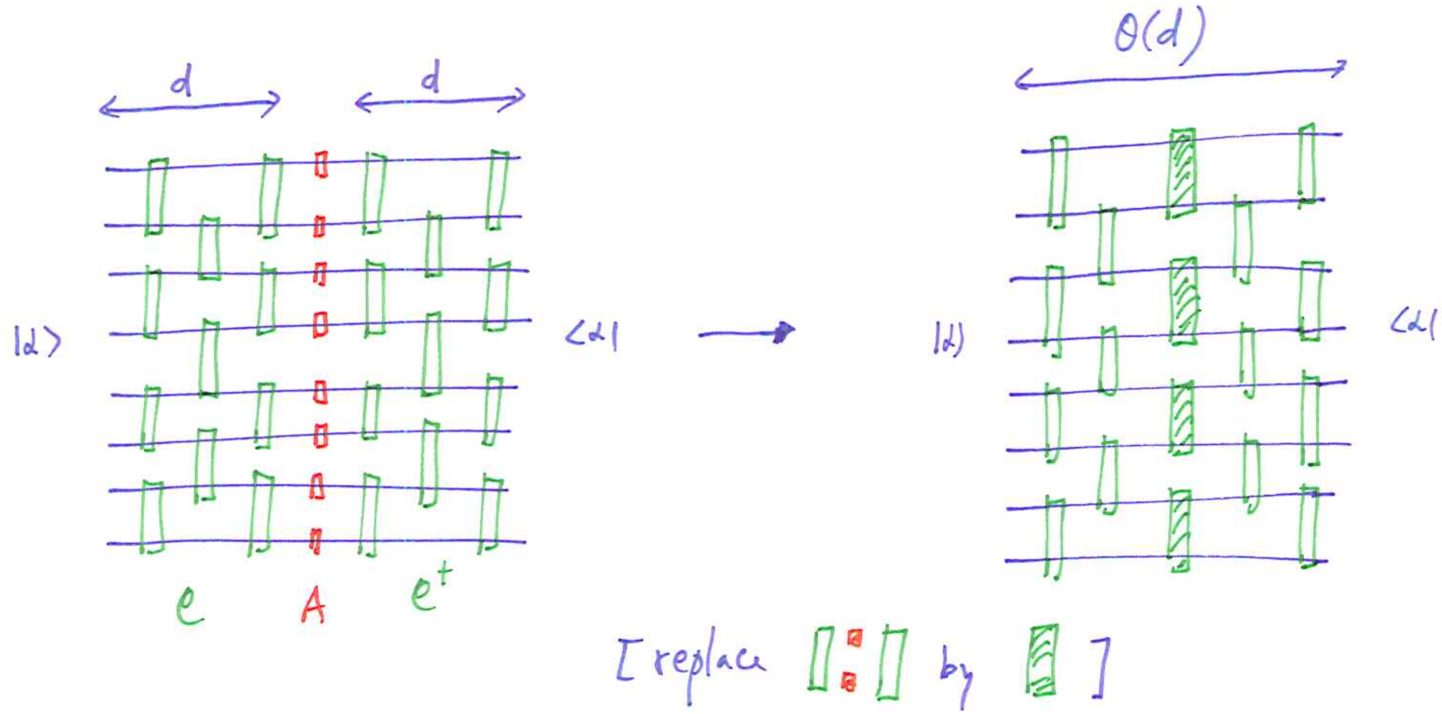
Example : n.n. constant-depth circuits

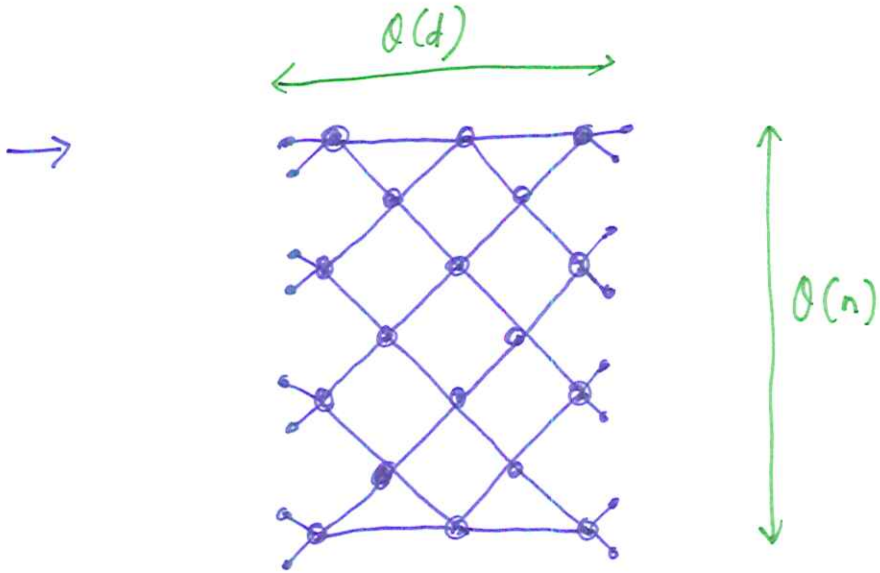
- THEOREM : consider n-qubit depth-d quantum circuit $\mathcal{C}$ composed of [potentially non-unitary] 2-qubit gates. The input state is any product state $|\alpha\rangle$. Let $A = A_1 \otimes \ldots \otimes A_n$ be any product operator. Then $\langle \alpha | \mathcal{C}^{+} A \, \mathcal{C} | \alpha \rangle$ can be computed classically up to m bits in poly(m, n) time.



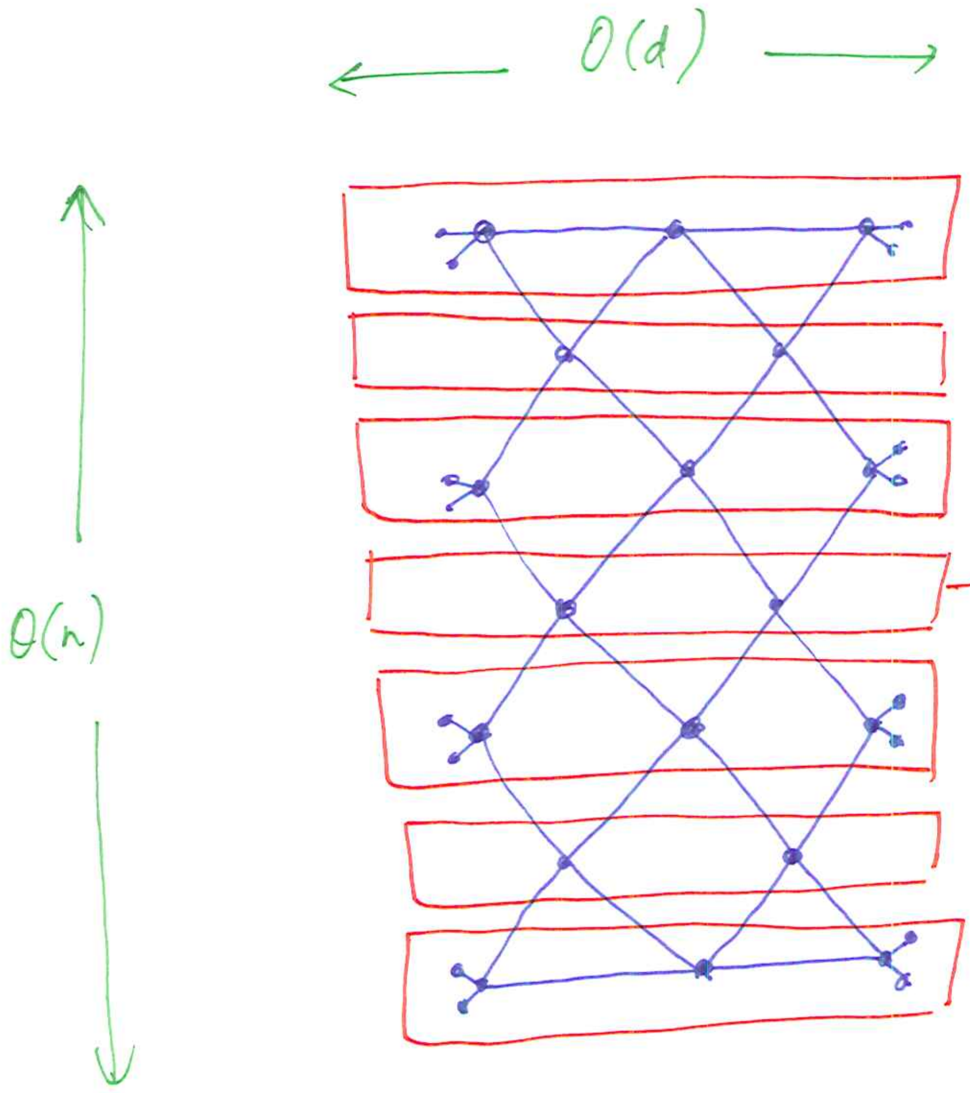
Proof: [sketch]

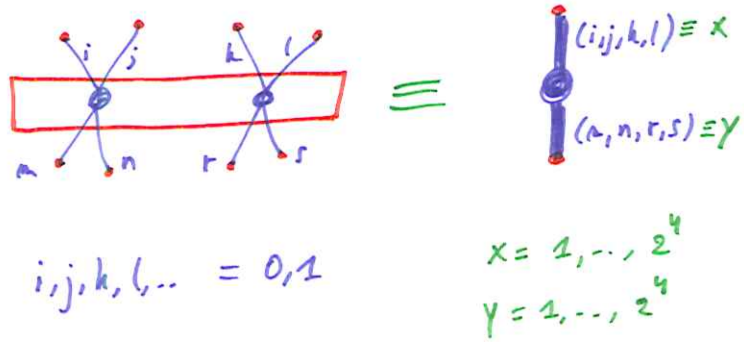
$\langle\alpha|\, e^{\dagger} A\, e\, |\alpha\rangle =$





Proof (ctd)





$i,j,k,l,\ldots = 0,1$

$x = 1,\ldots,2^4$

$y = 1,\ldots,2^4$

Thus  has rank 2

and dimension = constant

(i.e. $O(e^d)$)

→ contraction is now similar to matrix product of $O(n)$ $D \times D$ matrices with D constant !

Role of entanglement

- n.n. constant-depth circuits cannot produce much entanglement, in the following sense:
  - $|\psi\rangle$ multiqubit state; bipartion (A,B)

    $$\chi_{A,B}(|\psi\rangle) = \min\left\{ r : |\psi\rangle = \sum_{k=1}^{r} |\psi_k\rangle_A \otimes |\psi_k\rangle_B \right\}$$ schmidt rank

  - take $|\psi\rangle = \mathcal{C}|\alpha\rangle$ with $\mathcal{C}$ n.n. constant-depth

    take bipartition $\underbrace{1 \ldots m}_{A} \,|\, \underbrace{m+1 \ldots n}_{B}$ of n-qubit system

    then $\chi_{A,B}(|\psi\rangle) \leq$ Constant [Exercise!]

- This is general feature of tree-like tensor networks / circuits: they cannot generate much "schmidt-rank-type" entanglement

- Reverse statements are also true: "Bounded schmidt-rank-type entanglement implies classical simulation"

  i.e. entanglement as necessary ingredient for Q. speed-up

THANK YOU !