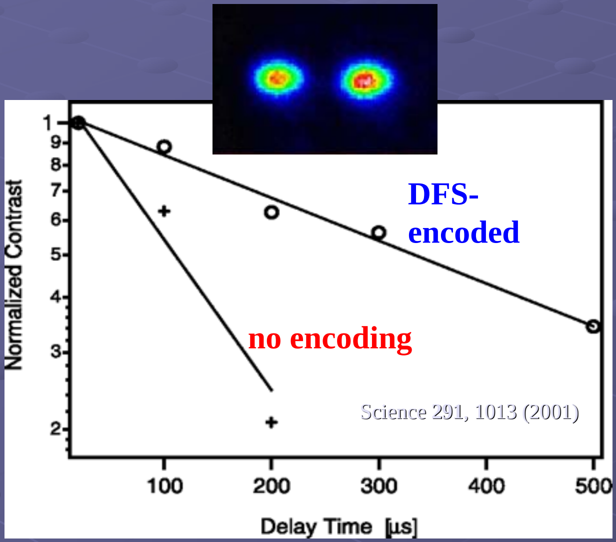
Hybrid quantum error prevention, reduction, and correction methods

10th Canadian Summer School on Quantum Information

UBC, July 21, 2010



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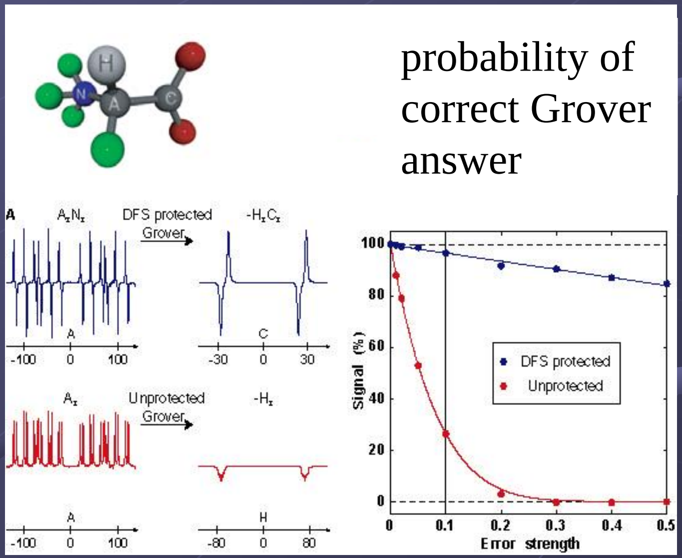

Outline

- Symmetry and preserved quantum information
- System-bath, decoherence and all that
- Unified view of decoherence-free subspaces, noiseless subsystems, quantum error correcting codes, operator quantum error correction
- DFS/NS examples: theory and experiment

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- Symmetry and preserved quantum information
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- DFS/NS examples: theory and experiment

Feel free to interrupt and ask lots of questions!

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Symmetry ➔ conserved quantity = quantum info.

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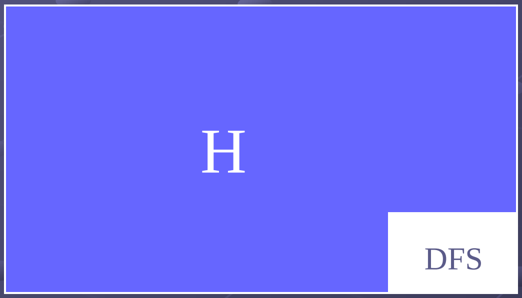
1) DFS/NS: Error Prevention
use an *existing* **exact** symmetry to **perfectly** hide q. info. from bath



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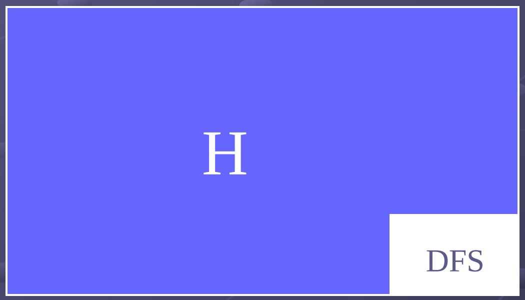
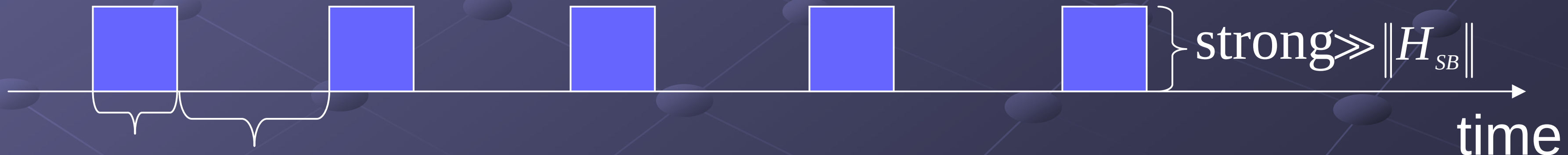


2) Dynamical Decoupling: Open-loop control
dynamically *generate* a symmetry



BANG + free evolution $\ll$ bath correlation time

Whence the Errors?
Decoherence from System-Bath Interaction

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Every real quantum system is coupled to an environment (“bath”).

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system

bath

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system bath

System dynamics alone:

$$\rho(t) = \mathrm{Tr}_B\left[e^{-iHt}\rho_{SB}(0)e^{iHt}\right]$$
$$= \sum_k c_k E_k(t)\rho(0)E_k(t)^\dagger$$

Decoherence

Non-unitary evolution of system

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$\Rightarrow$ Decoherence

Non-unitary evolution of system

Markovian master (Lindblad) equation:

$$\frac{\partial \rho}{\partial t} = -i[H_s, \rho] + \frac{1}{2}\sum_{\alpha,\alpha'} a_{\alpha,\alpha'}\left(2S_\alpha \rho S_{\alpha'}^\dagger - \rho S_{\alpha'}^\dagger S_\alpha - S_{\alpha'}^\dagger S_\alpha \rho\right)$$

What is there is a symmetry? Symmetric coin flipping noise

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How to reliably store a single bit?

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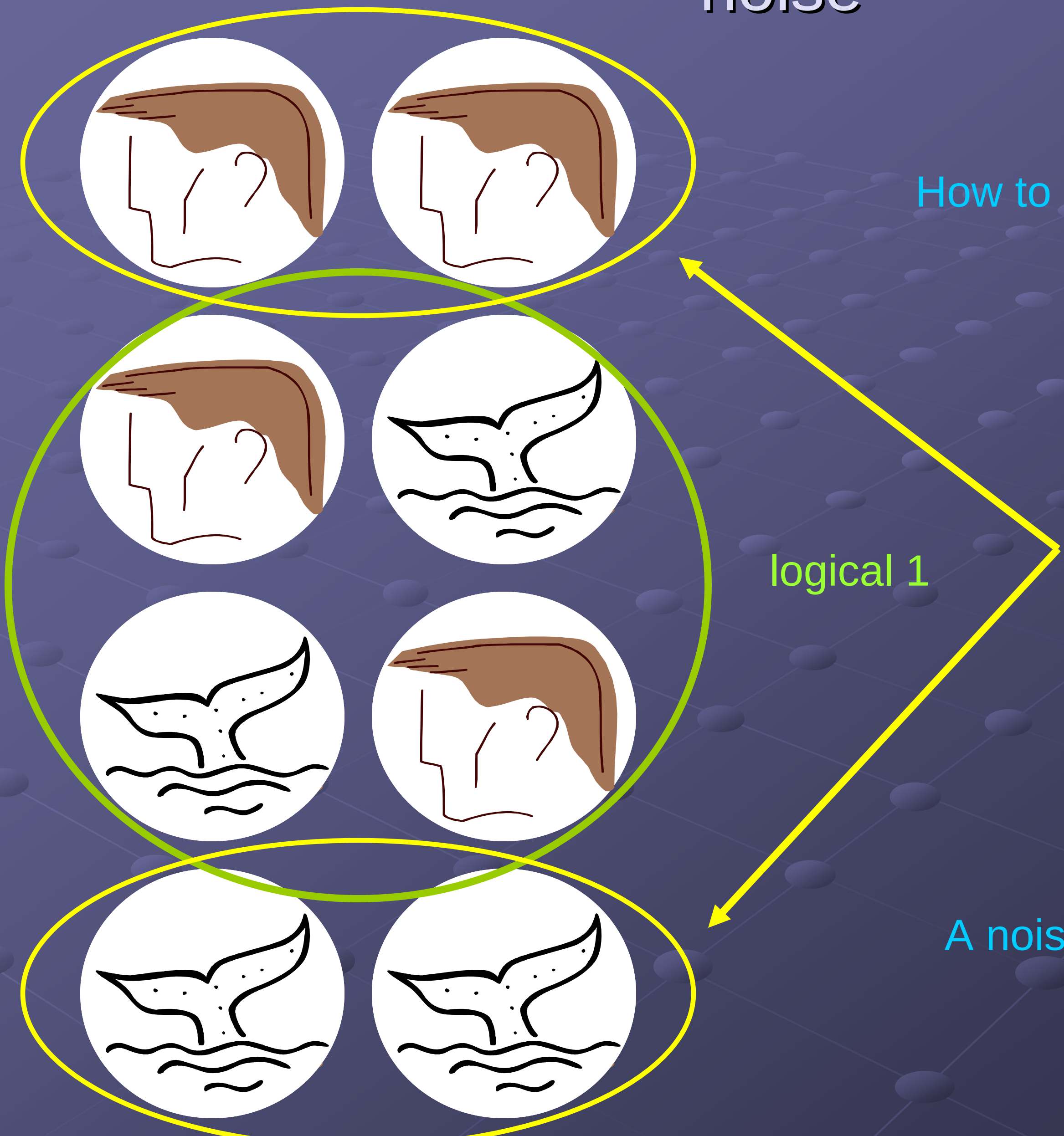

A noiseless subspace.

Unified view of Quantum Information Protection: Fixed Codes

Error Model:

Trace-preserving completely-positive (CP) maps:

$$\rho' = \sum_k E_k \rho E_k^\dagger \equiv \mathcal{E}(\rho) \qquad \sum_k E_k^\dagger E_k = I$$

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Decoherence-free subspace (DFS): $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$
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Quantum error correcting code (QEC): $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$
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active

DFS as a QEC, QEC as a DFS

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⇒ A DFS is a QEC with trivial recovery operation: $\mathcal{R} = I$

⇒ A QEC is a DFS with respect to the map $\mathcal{R} \circ \mathcal{E}$

NS as an OQEC, OQEC as an NS

Noiseless subsystem (NS): $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$, $\mathcal{A} = \mathcal{N} \otimes \mathcal{G}$
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Unitarily Invariant DFS

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More precisely:

Let the system Hilbert space $\mathcal{H}$ decompose into a direct sum as $\mathcal{H} = \mathcal{H}_{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{D}}^{\perp}$, and partition the system state ρ_S accordingly into blocks: $\rho_S = \begin{pmatrix} \rho_{\mathrm{D}} & \rho_2 \\ \rho_2^{\dagger} & \rho_3 \end{pmatrix}$.

Assume $\rho_{\mathrm{D}}(0) \neq 0$. Note imperfect initialization!

Then $\mathcal{H}_{\mathrm{D}}$ is called decoherence-free iff the initial and final DFS-blocks of ρ_S are unitarily related:

$$\rho_{\mathrm{D}}(t) = U_{\mathrm{D}} \rho_{\mathrm{D}}(0) U_{\mathrm{D}}^{\dagger},$$

where U_{D} is a unitary matrix acting on $\mathcal{H}_{\mathrm{D}}$.

U.I. DFS Conditions for CP Maps

Given a CP map:

$$\rho' = \sum_k E_k \rho E_k^\dagger \equiv \mathcal{E}(\rho) \qquad \sum_k E_k^\dagger E_k = I$$

Theorem

A necessary and sufficient condition for the existence of a DFS $\mathcal{H}_{\mathrm{D}}$ with respect to the CP map $\mathcal{E}$ is that all Kraus operators have a matrix representation of the form

$$E_k = \begin{pmatrix} c_k U_{\mathrm{D}} & 0 \\ 0 & B_k \end{pmatrix},$$

where U_{D} is unitary, c_k are scalars satisfying $\sum_k |c_k|^2 = 1$, and B_k are arbitrary operators on $\mathcal{H}_{\mathrm{D}}^{\perp}$ satisfying $\sum_k B_k^\dagger B_k = I$.

Meaning: E_k act unitarily on the DFS

U.I. DFS Conditions for Master Equations

Given a Markovian master equation:

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \frac{1}{2}\sum_\alpha 2F_\alpha \rho F_\alpha^\dagger - \rho F_\alpha^\dagger F_\alpha - F_\alpha^\dagger F_\alpha \rho$$

Theorem

A necessary and sufficient condition for the existence of a DFS $\mathcal{H}_{\mathrm{D}}$ with respect to the Markovian master equation above is that the Lindblad operators F_α and the system Hamiltonian H_S have the block-diagonal form

$$H_S = \begin{pmatrix} H_{\mathrm{D}} & 0 \\ 0 & H_{\mathrm{D}}^\perp \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} c_\alpha I & 0 \\ 0 & B_\alpha \end{pmatrix},$$

where H_{D} and $H_{\mathrm{D}}^\perp$ are Hermitian, c_α are scalars, and B_α are arbitrary operators on $\mathcal{H}_{\mathrm{D}}^\perp$.

Meaning: F_α act as identity on the DFS, while H_S preserves the DFS

Exercise

1. Prove sufficiency (easy) and necessity (not so easy) of the U.I. DFS conditions for CP maps and Markovian master equations

2. Generalize to NS, QEC, OQEC

Where is the promised symmetry?

How do we find and construct a DFS?

U.I. DFS Conditions for Hamiltonian Dynamics

Under Hamiltonian dynamics system and bath evolve jointly subject to the Schrodinger equation with the Hamiltonian $H = H_S + H_{SB} + H_B$.

Find a subspace where $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$ acts trivially,

i.e.: make $H_{SB} \propto I_S \otimes O_B$

Also, remember that H_S must preserve the DFS.

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Theorem

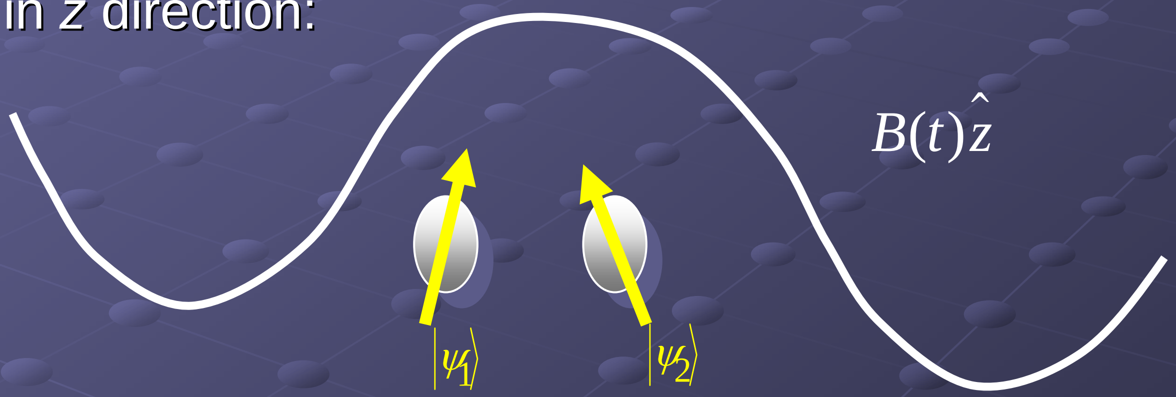
Let $A = \mathrm{alg}\{I, S_\alpha, S_\alpha^\dagger\}$.
Assume $[H_S, A] = 0$.
The dimension of the DFS $\mathcal{H}_{\mathrm{D}}$ equals the degeneracy of the 1-dimensional irreducible representation (irrep) of A.

degeneracy $\Longleftrightarrow$ symmetry

Simplest DFS Example: Collective Dephasing

DFS idea:	Find a subspace where $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$ acts trivially, i.e.: make $H_{SB} \propto I_S \otimes O_B$

Permutation symmetry in *z* direction:



Long-wavelength magnetic field *B* (environment) couples to spins

Effect: Random "**Collective Dephasing**":

$$|\psi_j\rangle = a_j|0\rangle_j + b_j|1\rangle_j \mapsto a_j|0\rangle_j + e^{i\theta} b_j|1\rangle_j$$

random but ***j*-independent** phase

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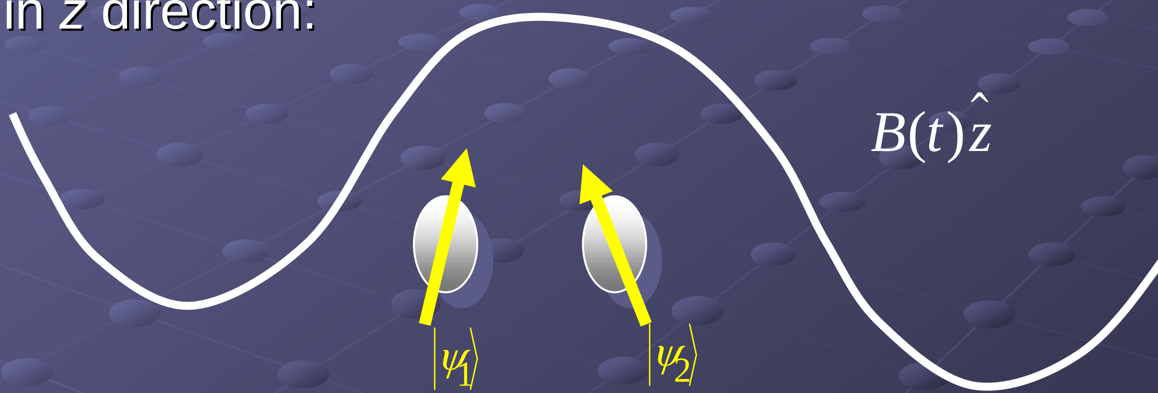
$$H_{\text{int}} = g(\sigma_1^z + \sigma_1^z) \otimes B =$$

$$\begin{array}{cl} \begin{pmatrix} -2gB & & & \\ & 0 & & \\ & & 0 & \\ & & & 2gB \end{pmatrix} & \begin{array}{l} |\downarrow\rangle_1 |\downarrow\rangle_2 \\ |\downarrow\rangle_1 |\uparrow\rangle_2 \\ |\uparrow\rangle_1 |\downarrow\rangle_2 \\ |\uparrow\rangle_1 |\uparrow\rangle_2 \end{array} \end{array}$$



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DFS encoding

$$|0\rangle_L = |0\rangle_1 \otimes |1\rangle_2$$
$$|1\rangle_L = |1\rangle_1 \otimes |0\rangle_2$$

Why it Works

Collective dephasing:

$$\left|\psi\right\rangle_j = a_j\left|0\right\rangle_j + b_j\left|1\right\rangle_j \mapsto a_j\left|0\right\rangle_j + e^{i\theta}b_j\left|1\right\rangle_j$$

Case of two qubits:

$$\left|0\right\rangle\otimes\left|0\right\rangle \mapsto \left|0\right\rangle\otimes\left|0\right\rangle \equiv \left|00\right\rangle$$

$$\left|0\right\rangle\otimes\left|1\right\rangle \mapsto \left|0\right\rangle\otimes\left(e^{i\theta}\left|1\right\rangle\right) \equiv e^{i\theta}\left|01\right\rangle$$

$$\left|1\right\rangle\otimes\left|0\right\rangle \mapsto \left(e^{i\theta}\left|1\right\rangle\right)\otimes\left|0\right\rangle \equiv e^{i\theta}\left|10\right\rangle$$

$$\left|1\right\rangle\otimes\left|1\right\rangle \mapsto \left(e^{i\theta}\left|1\right\rangle\right)\otimes\left(e^{i\theta}\left|1\right\rangle\right) \equiv e^{2i\theta}\left|11\right\rangle$$

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Global phase physically irrelevant:

$\left|\psi\right\rangle_L = a\left|0\right\rangle_L + b\left|1\right\rangle_L$ is decoherence-free:

A 2-dimensional protected subspace.

Why it Works

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Case of two qubits:

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$$|1\rangle\otimes|0\rangle \mapsto \left(e^{i\theta}|1\rangle\right)\otimes|0\rangle \equiv e^{i\theta}|10\rangle \quad \equiv |1\rangle_L$$

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pop quiz:
Are the states
$|00\rangle$ and $|11\rangle$
also in a DFS?

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Generalization: Noiseless Subsystems

[E. Knill, R. Laflamme and L. Viola, PRL 84, 2525 (2000)]

The dimension of the DFS $\mathcal{H}_{\mathrm{D}}$ equals the degeneracy of the 1-dimensional irreducible representation (irrep) of A.

1D irreps condition not needed…

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A theorem from C^* algebras:

Associative algebra $A = \text{polynomials}\{I, S_\alpha, S_\alpha^\dagger\}$

Matrix representation over $\mathbb{C}^{2^N}$:

Model of decoherence:

$$H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$$

$$\boxed{A \cong \bigoplus_J I_{n_J} \otimes M_{d_J}(\mathbb{C})}$$

dimension

multiplicity

irreducible representations

Hilbert space decomposition:

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code subsystem

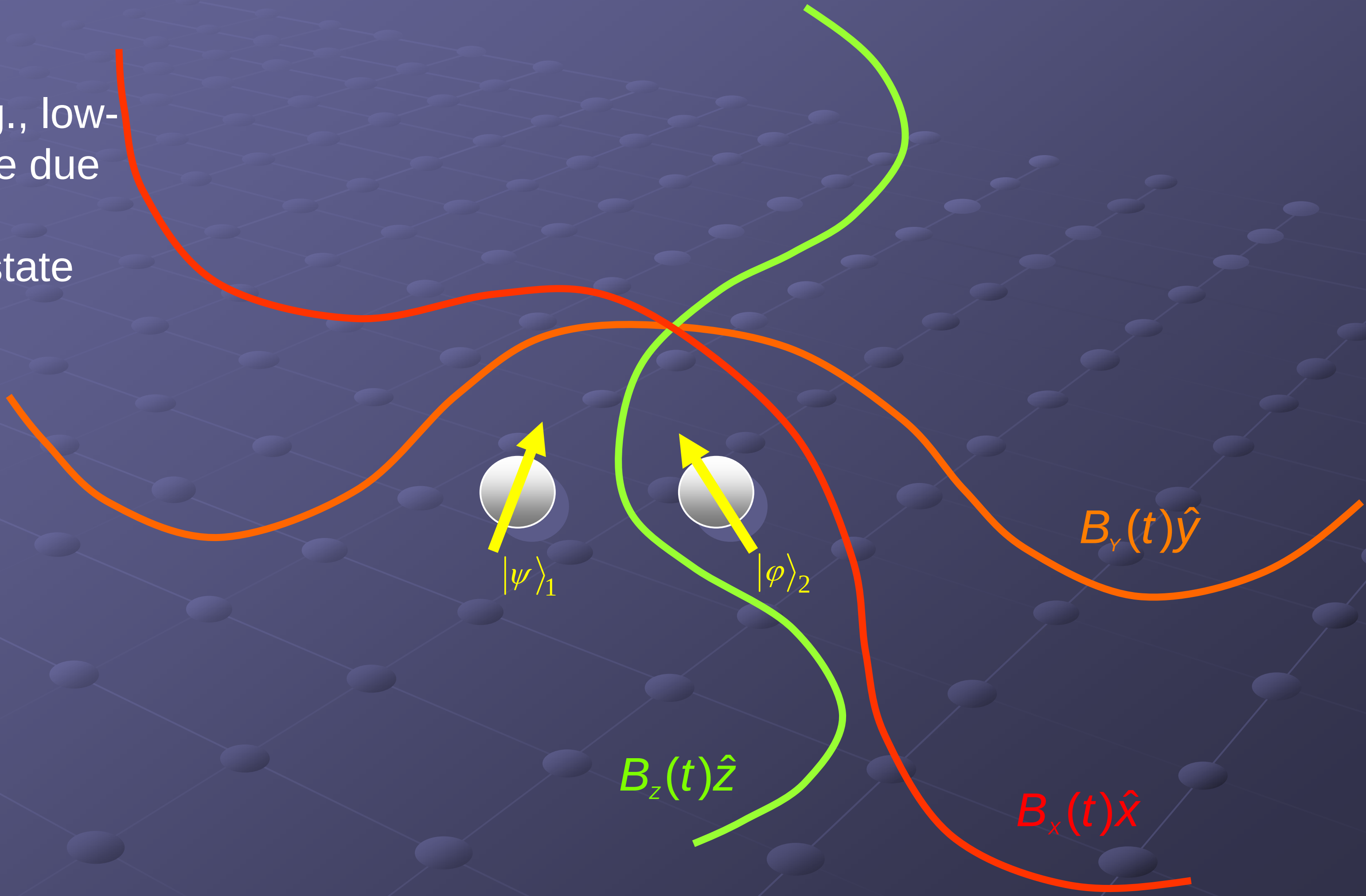
Each J labels an n_J-dimensional NS code

DFS is the case $d_J = 1$

$n_J > 1$ iff $\exists$ **symmetry** in system-env. interaction

Isotropic Quantum Errors: Collective Decoherence Model

Describes, e.g., low-T decoherence due to phonons in various solid state QC proposals



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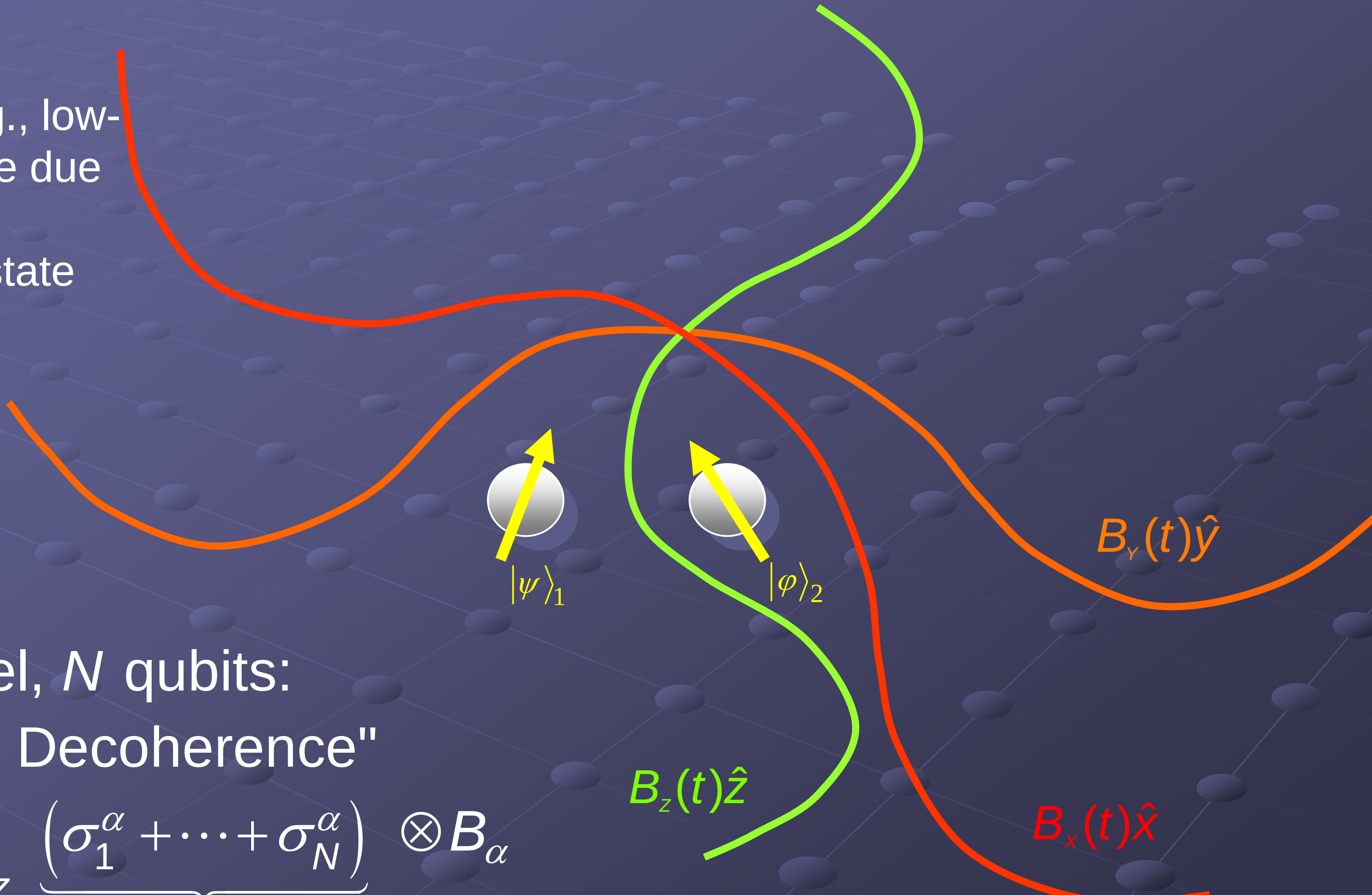

Error model, N qubits:

"Collective Decoherence"

$$H_{SB} = \sum_{\alpha=x,y,z} \underbrace{\left(\sigma_1^\alpha + \cdots + \sigma_N^\alpha\right)}_{S_\alpha = \text{total spin operator}} \otimes B_\alpha$$

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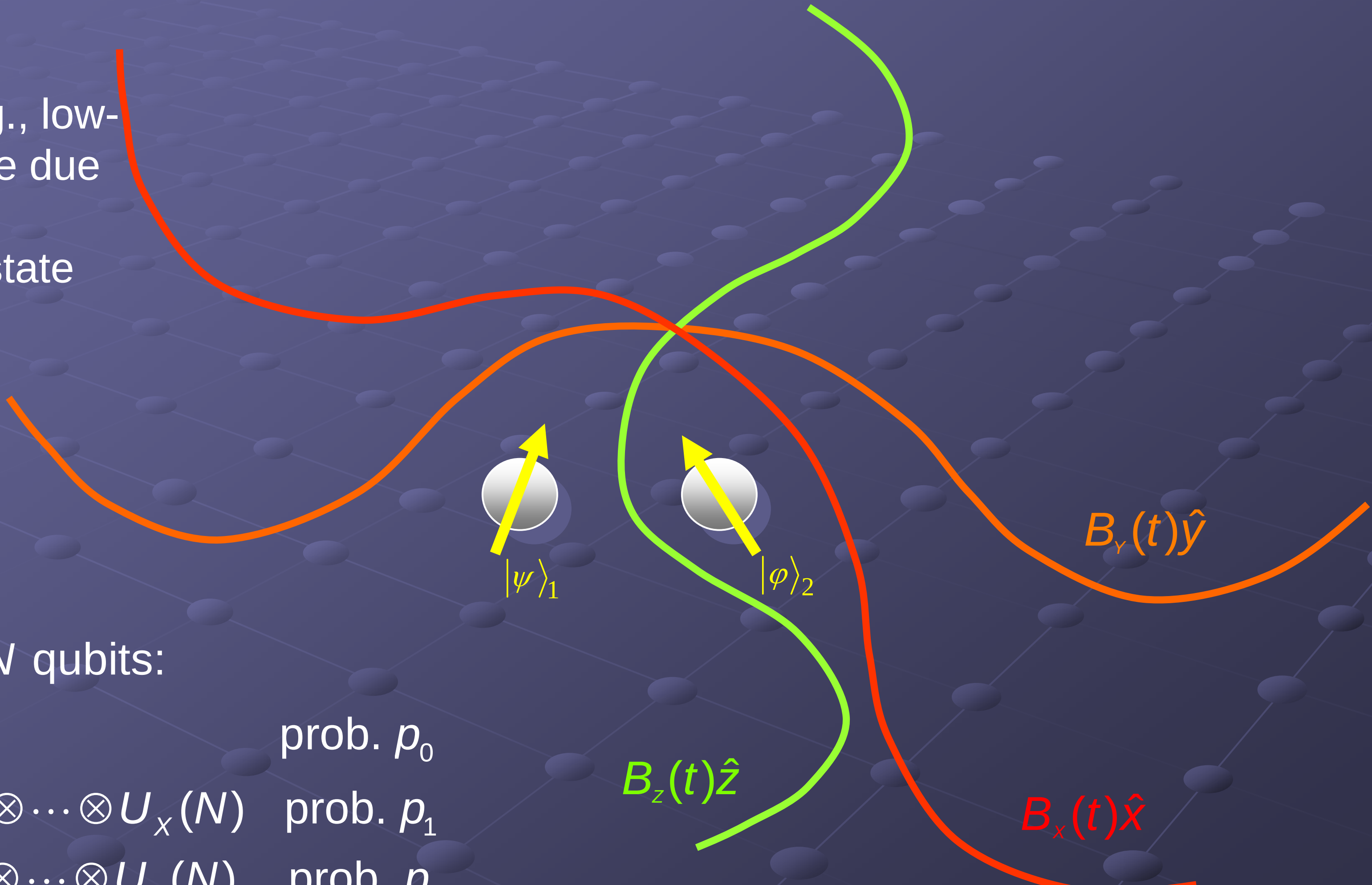


Error model, N qubits:

$$|\psi\rangle \mapsto \begin{cases} |\psi\rangle & \text{prob. } p_0 \\ U_X(1)\otimes\cdots\otimes U_X(N) & \text{prob. } p_1 \\ U_Y(1)\otimes\cdots\otimes U_Y(N) & \text{prob. } p_2 \\ U_Z(1)\otimes\cdots\otimes U_Z(N) & \text{prob. } p_3 \end{cases}$$

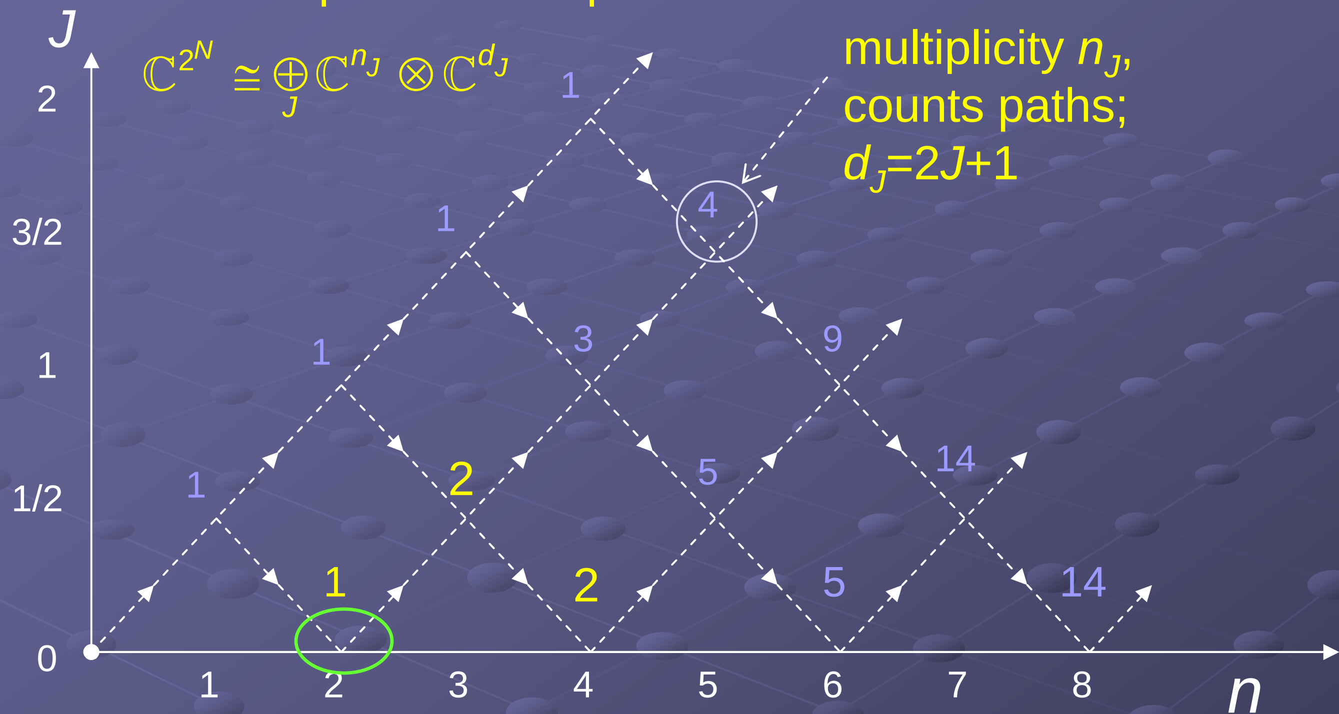
Do irreps analysis of n copies of su(2)…

All Decoherence-Free Subspaces/Subsystems for Collective Decoherence

Hilbert space decomposition:

$$\mathbb{C}^{2^N} \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$$

multiplicity n_J, counts paths; $d_J = 2J+1$



$|0_L\rangle \equiv$  $= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

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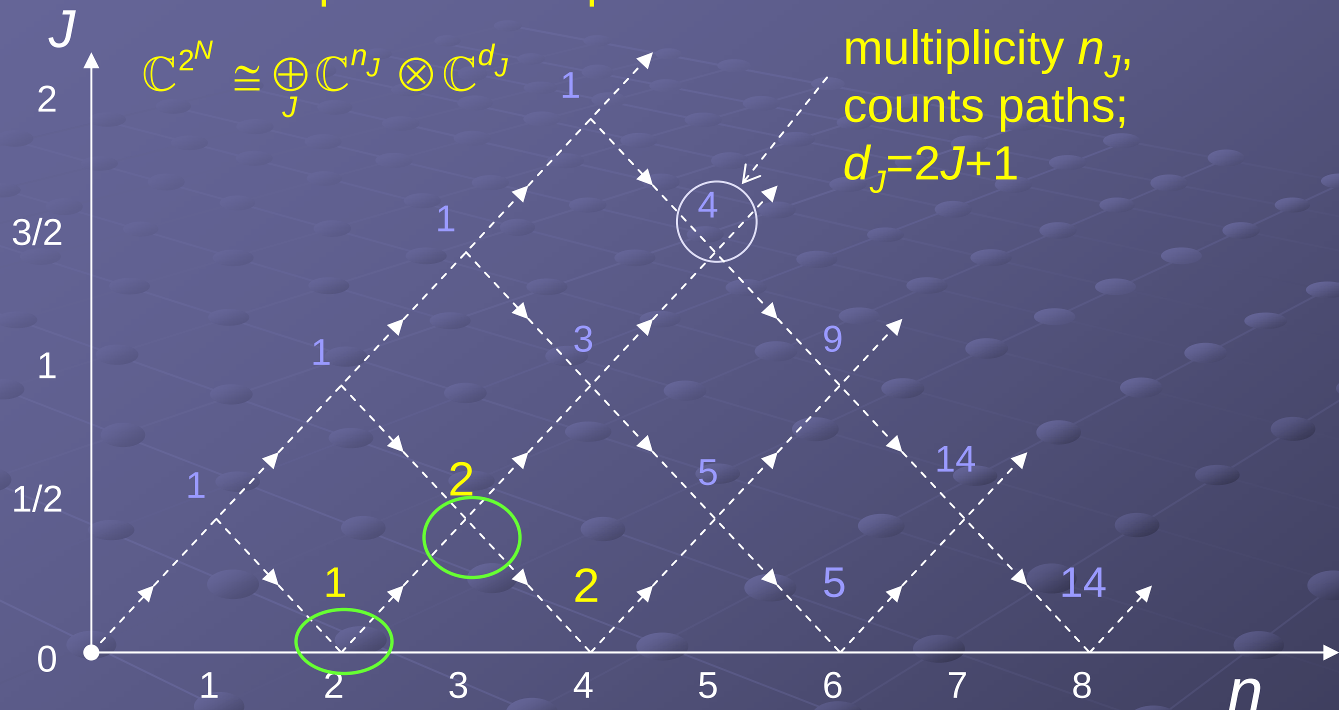
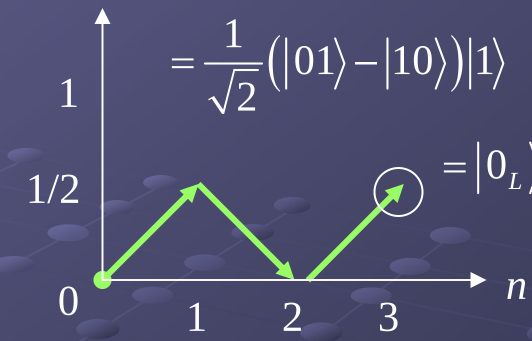


$|0_L\rangle \equiv$  $= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)|1\rangle$



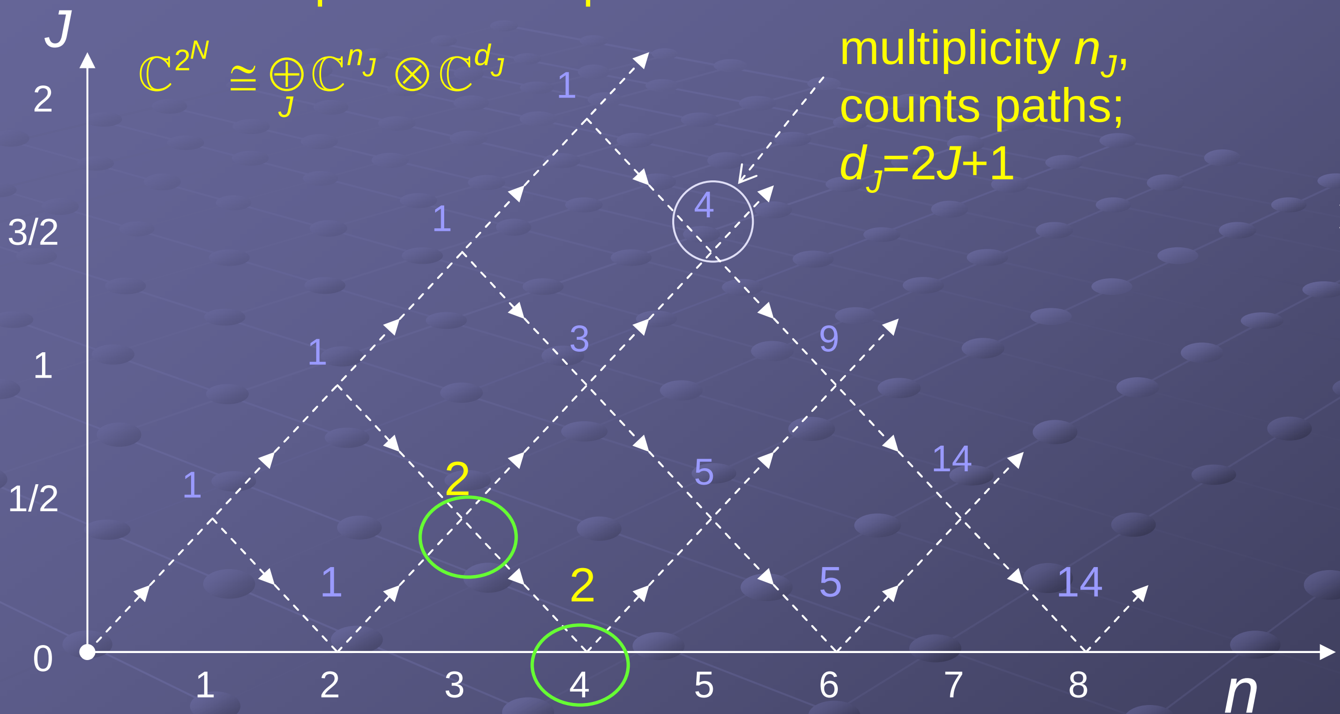
$= \sqrt{\frac{2}{3}}|110\rangle - \sqrt{\frac{1}{3}}\left(|011\rangle - |101\rangle\right)$

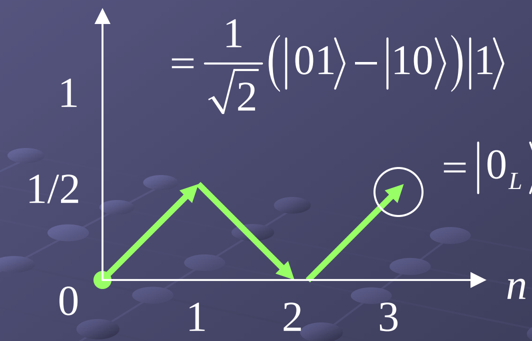
All Decoherence-Free Subspaces/Subsystems for Collective Decoherence

Hilbert space decomposition:

$$\mathbb{C}^{2^N} \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$$

multiplicity n_J, counts paths; $d_J=2J+1$





$$=\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right)|1\rangle \qquad =|0_L\rangle$$

$$=\sqrt{\frac{2}{3}}|110\rangle-\sqrt{\frac{1}{3}}\left(|011\rangle-|101\rangle\right)$$

$=|1_L\rangle$

$|0_L\rangle \equiv$

$$=\frac{1}{2}\left(|01\rangle-|10\rangle\right)\left(|01\rangle-|10\rangle\right)$$

$|1_L\rangle \equiv$

$$=\frac{1}{2\sqrt{3}}\left(2|0011\rangle+2|1100\rangle-\left(|0110\rangle+|1001\rangle+|1010\rangle+|0101\rangle\right)\right)$$

What is the “Volume” of a DFS/NS?

Degeneracy for given J, M = dimension of DFS/NS $\equiv D_J(n) = \dfrac{n!(2J+1)}{\left(n/2+J+1\right)!\left(n/2-J\right)!}$

$$\Rightarrow \text{ code rate } \equiv \frac{\text{no. of encoded qubits}}{\text{no. of physical qubits}} \overset{(J=0)}{=} \frac{\log_2 D_0(n)}{n} \xrightarrow{n\to\infty} 1-\frac{3}{2}\frac{\log_2 n}{n}$$

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DFS’s for collective decoherence asymptotically fill the Hilbert space!

Computation Inside a U.I. DFS/NS

So far have storage. What about computation?

To prevent decoherence, computation should never leave DFS/NS.

Which logic operations are compatible?

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Error algebra:

$$A \cong \bigoplus_J I_{n_J} \otimes M_{d_J}(\mathbb{C})$$

Code subsystem:

$$\mathbb{C}^{2^N} \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$$

Commutant = operators commuting with *A*

$$A' \cong \bigoplus_J M'_{n_J}(\mathbb{C}) \otimes I_{d_J}$$

The allowed logic operations!

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The allowed logic operations!

Universal quantum computation over DFS/NS is possible using "exchange Hamiltonians", e.g., Heisenberg interaction:

$$H_{\text{Heis}} = \sum_{ij} \frac{J_{ij}}{2} \left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z \right)$$

Heisenberg Computation over DFS/NS is Universal

- Heisenberg exchange interaction:

$$H_{\mathrm{Heis}} = \sum_{i,j} J_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j) \equiv \sum_{i,j} J_{ij} E_{ij}$$

- Universal over collective-decoherence DFS

[J. Kempe, D. Bacon, D.A.L., B. Whaley, Phys. Rev. A **63**, 042307 (2001)]

$$A' \cong \bigoplus_J M'_{n_J}(\mathbb{C}) \otimes I_{d_J}$$

The allowed logic operations

- Over 4-qubit DFS:

CNOT involves 14 elementary steps (D. Bacon, Ph.D. thesis)

- Implications for simplifying operation of spin-based quantum dot QCs

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$$|1_L\rangle = \frac{1}{2\sqrt{3}}\left(2|0011\rangle + 2|1100\rangle - \left(|0110\rangle + |1001\rangle + |1010\rangle + |0101\rangle\right)\right)$$

$$\bar{X} = -\frac{2}{\sqrt{3}}\left(E_{13} + \tfrac{1}{2}E_{12}\right) \qquad \bar{Z} = -E_{12}$$

$e^{i\theta\bar{X}}$ and $e^{i\theta\bar{Z}}$ generate arbitrary single encoded qubit gates

CNOT involves 42 elementary steps (D. Bacon, Ph.D. thesis)

- Implications for simplifying operation of spin-based quantum dot QCs

Experimental Verification of Decoherence-Free Subspaces

Paul G. Kwiat,[1*] Andrew J. Berglund,[1†] Joseph B. Altepeter,[1] Andrew G. White[1,2]

Using spontaneous parametric down-conversion, we produce polarization-entangled states of two photons and characterize them using two-photon tomography to measure the density matrix. A controllable decoherence is imposed on the states by passing the photons through thick, adjustable birefringent elements. When the system is subject to collective decoherence, one particular entangled state is seen to be decoherence-free, as predicted by theory. Such decoherence-free systems may have an important role for the future of quantum computation and information processing.

20 OCTOBER 2000 VOL 290 SCIENCE www.sciencemag.org

In the beginning …

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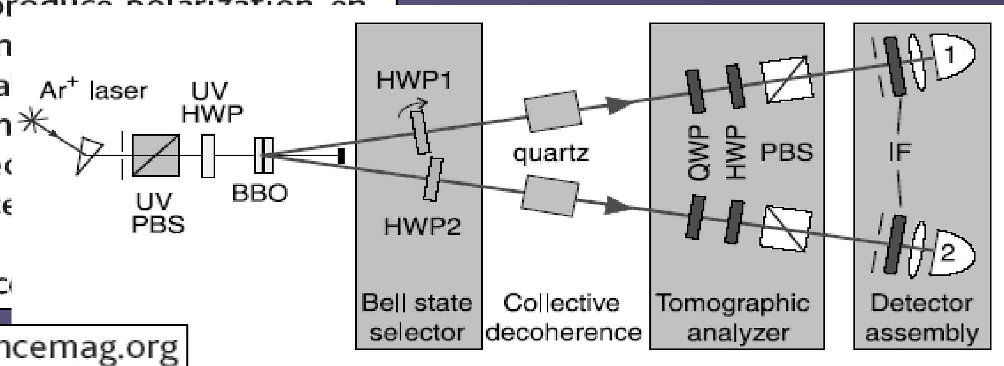

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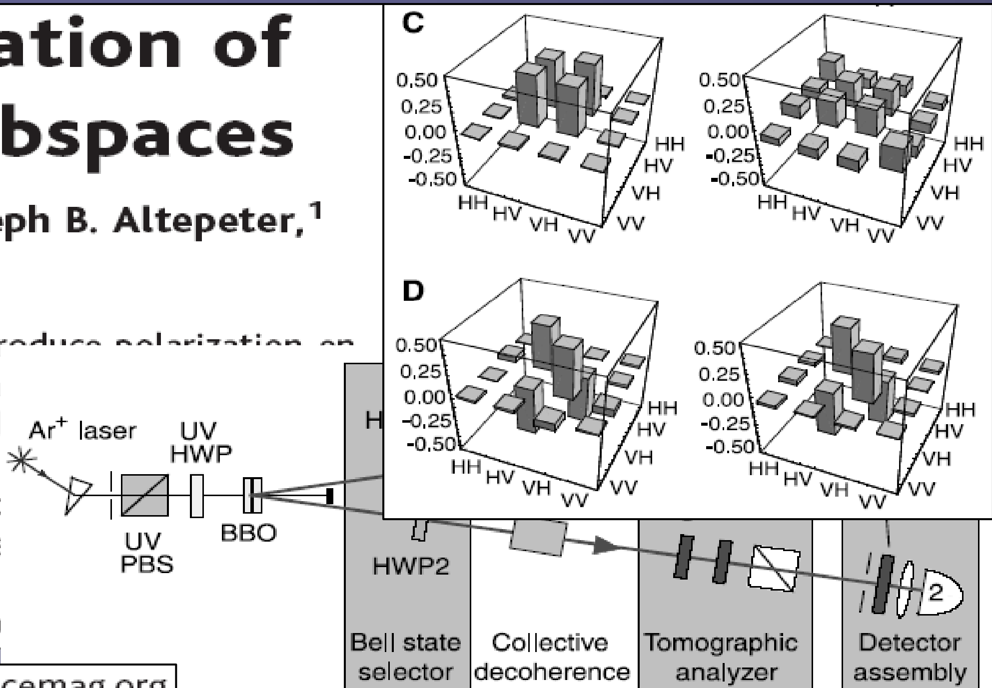

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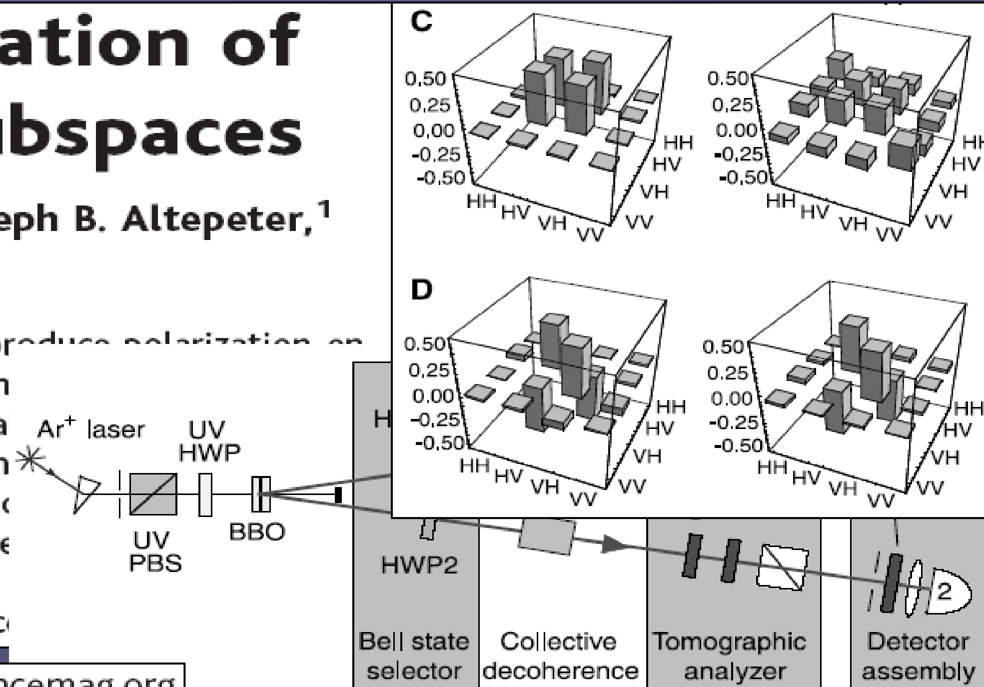


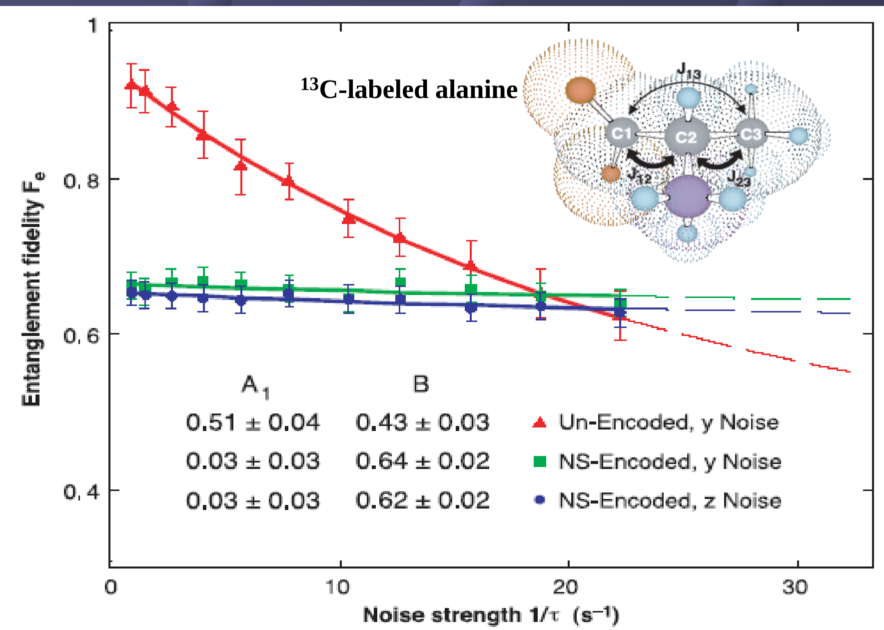
REPORTS

Experimental Realization of Noiseless Subsystems for Quantum Information Processing

Lorenza Viola,[1*†] Evan M. Fortunato,[2*] Marco A. Pravia,[2] Emanuel Knill,[1] Raymond Laflamme,[1] David G. Cory[2]

We demonstrate the protection of one bit of quantum information against all collective noise in three nuclear spins. Because no subspace of states offers this protection, the quantum bit was encoded in a proper noiseless subsystem. We therefore realize a general and efficient method for protecting quantum information. Robustness was verified for a full set of noise operators that do not distinguish the spins. Verification relied on the most complete exploration of engineered decoherence to date. The achieved fidelities show improved information storage for a large, noncommutative set of errors.

14 SEPTEMBER 2001 VOL 293 SCIENCE www.sciencemag.org



VOLUME 92, NUMBER 10 PHYSICAL REVIEW LETTERS week ending 12 MARCH 2004

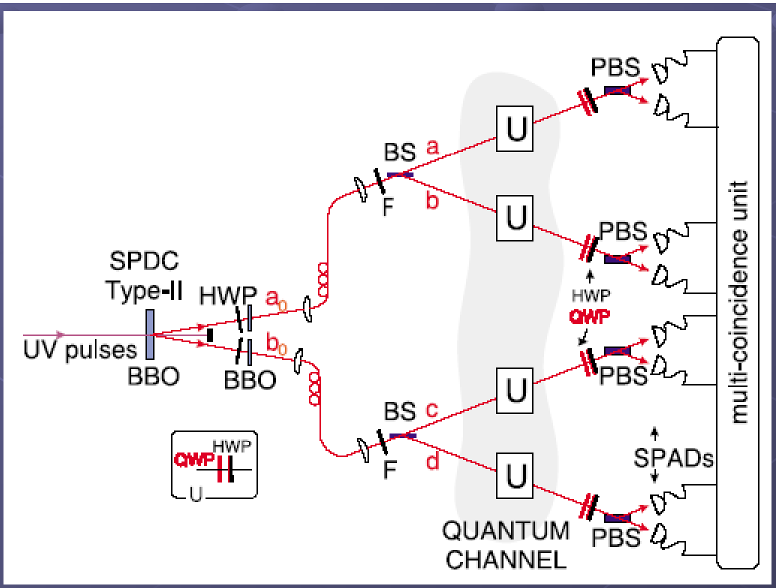
Decoherence-Free Quantum Information Processing with Four-Photon Entangled States

Mohamed Bourennane,[1,2] Manfred Eibl,[1,2] Sascha Gaertner,[1,2] Christian Kurtsiefer,[2] Adán Cabello,[3] and Harald Weinfurter[1,2]

[1]*Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*
[2]*Sektion Physik, Ludwig-Maximilians-Universität, D-80797 München, Germany*
[3]*Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain*

(Received 22 August 2003; published 9 March 2004)

Decoherence-free states protect quantum information from collective noise, the predominant cause of decoherence in current implementations of quantum communication and computation. Here we demonstrate that spontaneous parametric down conversion can be used to generate four-photon states which enable the encoding of one qubit in a decoherence-free subspace. The immunity against noise is verified by quantum state tomography of the encoded qubit. We show that particular states of the encoded qubit can be distinguished by local measurements on the four photons only.



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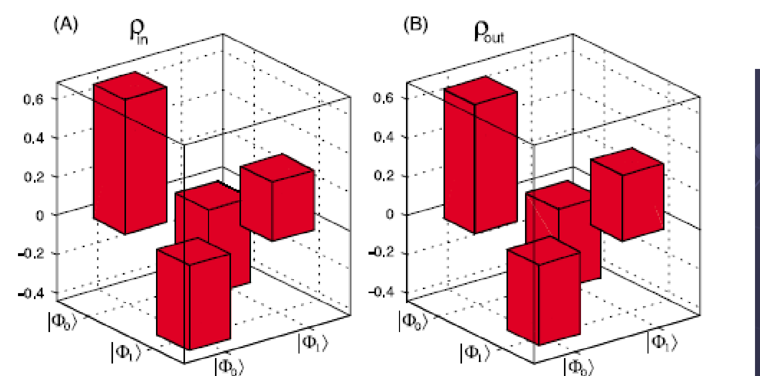

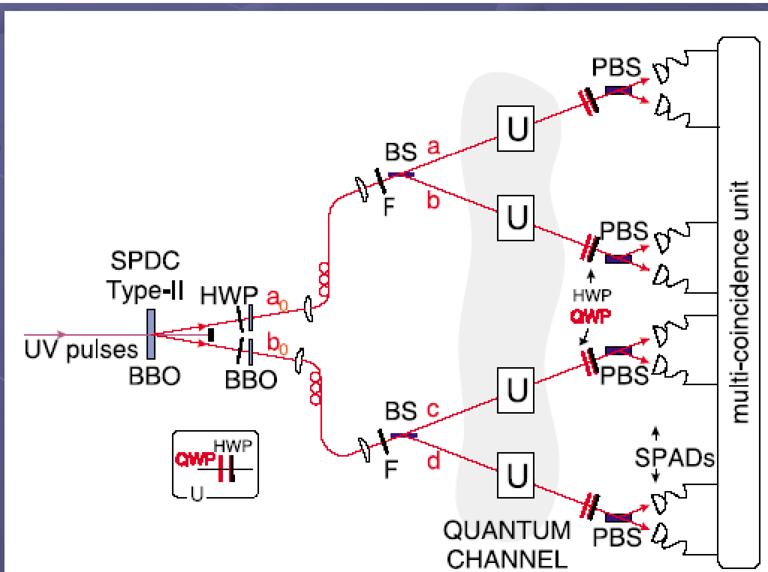

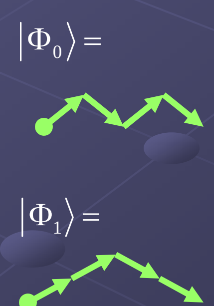

FIG. 4 (color online). Propagation of the logical qubit $|\Psi_L\rangle = (\sqrt{3}|\Phi_0\rangle - |\Phi_1\rangle)/2$: (a) and (b) show the experimentally obtained density matrices before (ρ_{in}) and after (ρ_{out}) passage through a noisy quantum channel. The encoding in a DF subspace protected the transmission, leading to a fidelity of $F_{\rho_{\mathrm{in}},\rho_{\mathrm{out}}} = 0.9958 \pm 0.0759$ in the presence of noise (overall measurement time 12 h).

What about symmetry breaking?

D.L., I.L. Chuang, K.B. Whaley, PRL **81**, 2594 (1998); D. Bacon, D.L., K.B. Whaley, PRA **60**, 1944 (1999)

Symmetry breaking:

unequal coupling constants, lowering of symmetry by a perturbation, etc.

Introduce a perturbation via $H_{SB} \mapsto H_{SB} + \epsilon \Delta H$, $\|\Delta H\| = 1$

Theory shows that fidelity depends on ϵ only to $\boxed{\text{second}}$ order.

Robustness of DFS to symmetry breaking perturbations

VOLUME 92, NUMBER 14 PHYSICAL REVIEW LETTERS week ending 9 APRIL 2004

Experimental Investigation of a Two-Qubit Decoherence-Free Subspace

J. B. Altepeter,[1,2] P. G. Hadley,[2] S. M. Wendelken,[2] A. J. Berglund,[2,*] and P. G. Kwiat[1,2,†]

[1]*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3080, USA*

[2]*Physics Division, P-23, Los Alamos National Laboratories, Los Alamos, New Mexico 87545, USA*

(Received 31 May 2003; published 9 April 2004)

We thoroughly explore the phenomenon of a decoherence-free subspace (DFS) for two-qubit systems. Specifically, we both collectively and noncollectively decohere entangled polarization-encoded two-qubit states using thick birefringent crystals. These results characterize the basis-dependent effect of decoherence on the four Bell states, the robustness of the DFS state against perturbations in the assumption of collective decoherence, and the existence of a DFS for each type of stable noncollective decoherence. Finally, we investigate the effects of collective and noncollective dissipation.

Robustness of DFS to symmetry breaking perturbations

VOLUME 92, NUMBER 14 PHYSICAL REVIEW LETTERS week ending 9 APRIL 2004

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J. B. Altepeter,[1,2] P. G. Hadley,[2] S. M. Wendelken,[2] A. J.

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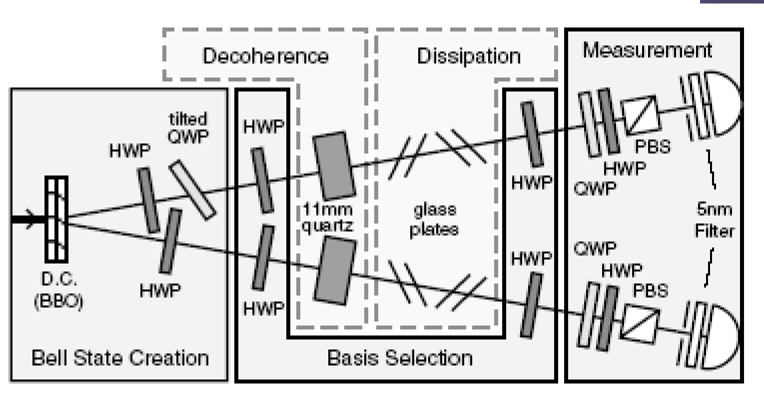
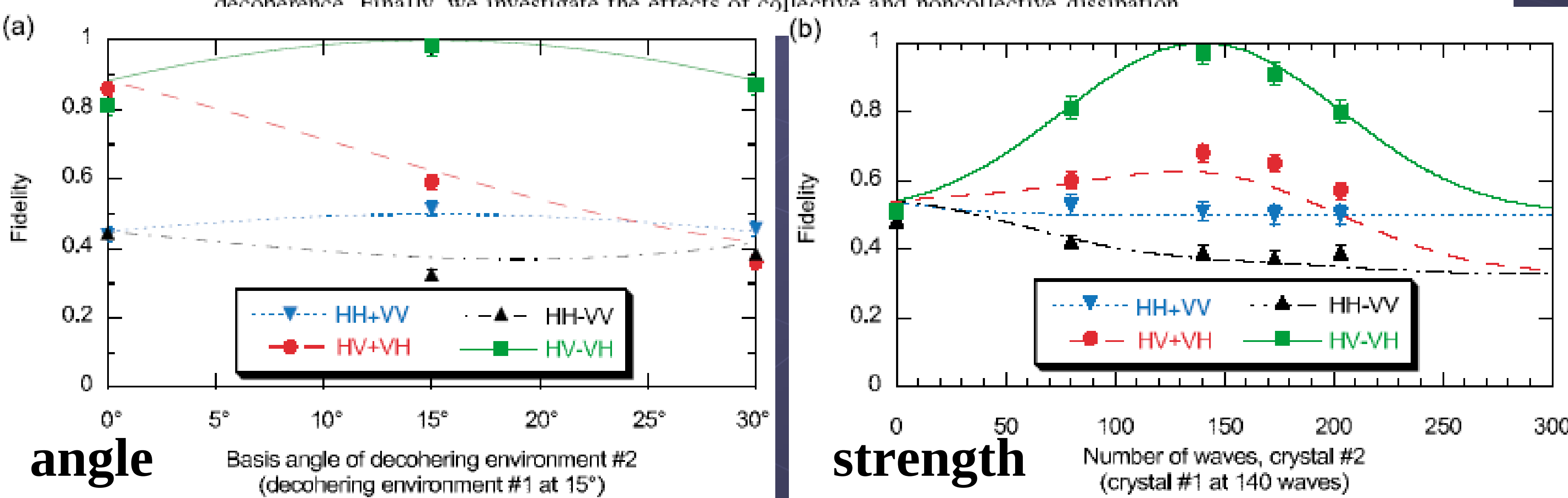
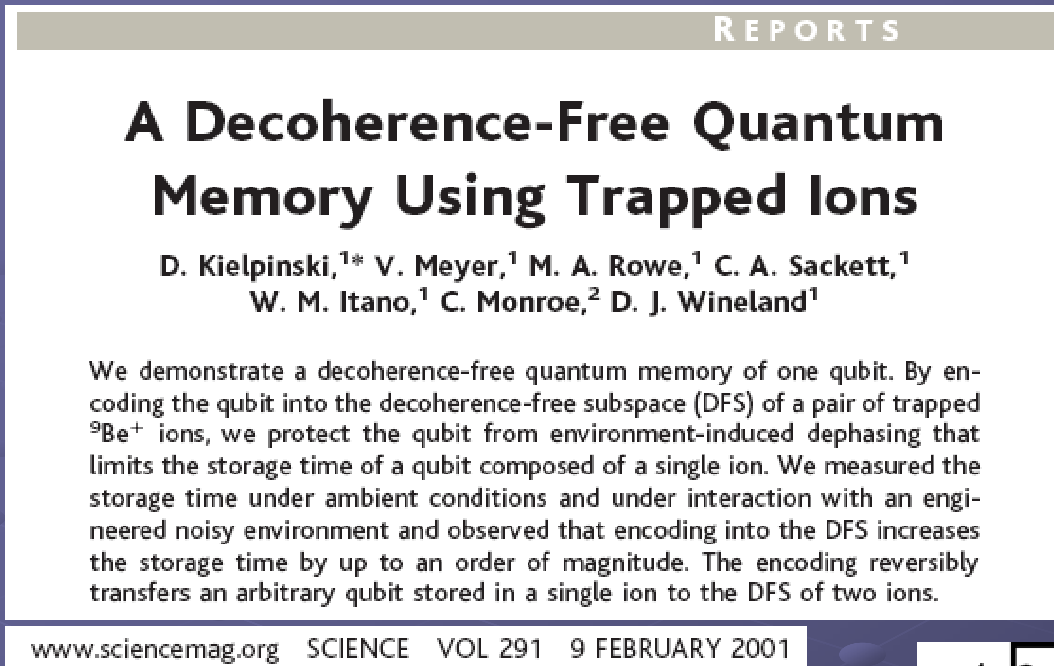




Strong Symmetry Breaking



REPORTS

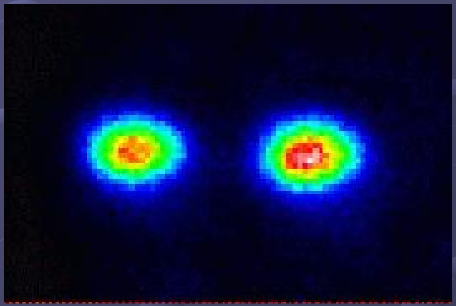
A Decoherence-Free Quantum Memory Using Trapped Ions

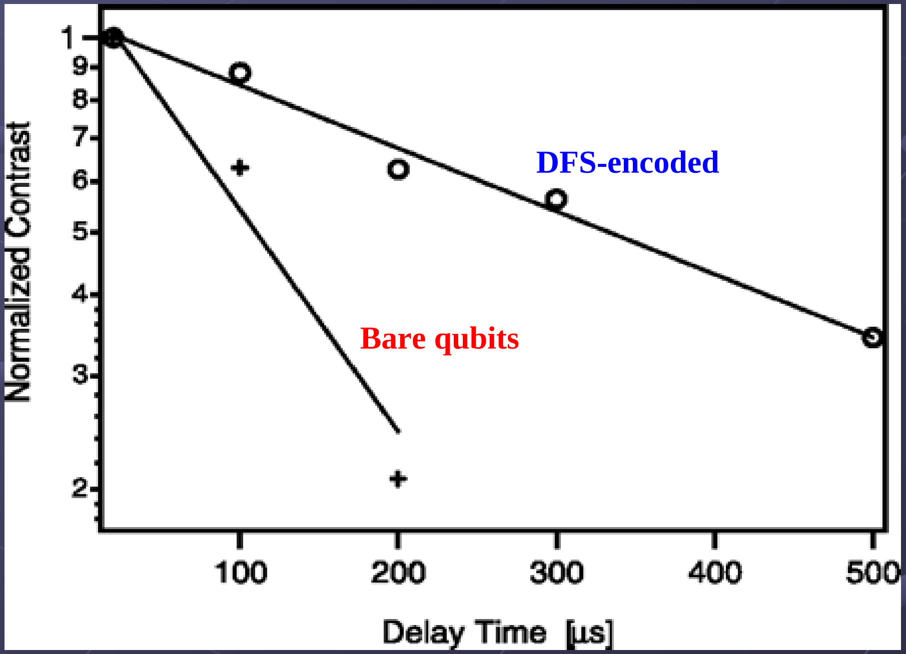
D. Kielpinski,[1*] V. Meyer,[1] M. A. Rowe,[1] C. A. Sackett,[1] W. M. Itano,[1] C. Monroe,[2] D. J. Wineland[1]

We demonstrate a decoherence-free quantum memory of one qubit. By encoding the qubit into the decoherence-free subspace (DFS) of a pair of trapped $^9Be^+$ ions, we protect the qubit from environment-induced dephasing that limits the storage time of a qubit composed of a single ion. We measured the storage time under ambient conditions and under interaction with an engineered noisy environment and observed that encoding into the DFS increases the storage time by up to an order of magnitude. The encoding reversibly transfers an arbitrary qubit stored in a single ion to the DFS of two ions.

www.sciencemag.org SCIENCE VOL 291 9 FEBRUARY 2001

Bare qubit:
two hyperfine states of trapped $^9Be^+$ ion





Chief decoherence sources:

(i) fluctuating long-wavelength ambient magnetic fields;

(ii) heating of ion CM motion during computation: a symmetry-breaking process

DFS encoding: into pair of ions

$$|0\rangle_L = |0\rangle_1 \otimes |1\rangle_2 \qquad |1\rangle_L = |1\rangle_1 \otimes |0\rangle_2$$

Need a way to deal with symmetry breaking…

Intermission & Bathroom Break