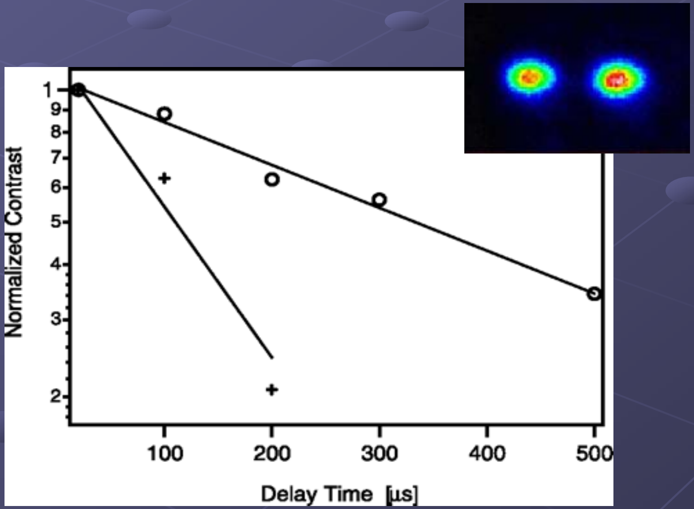
Part 2: Mostly Dynamical Decoupling

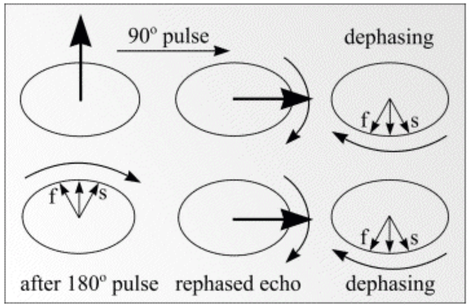
Need a way to deal with symmetry breaking…



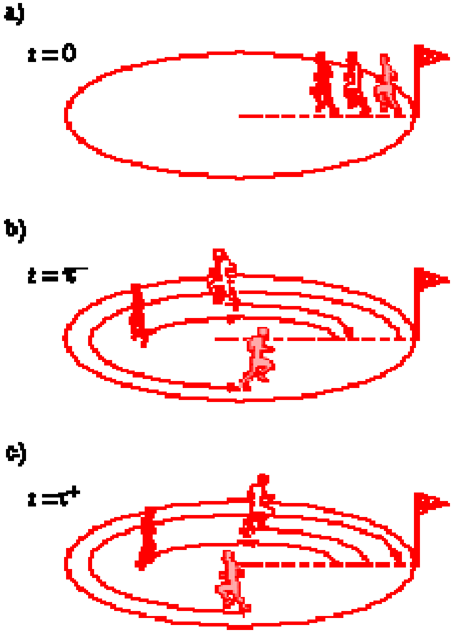
NMR to the Rescue:
Removal of Decoherence via Spin Echo=Time Reversal

From: Coherent averaging techniques
Coherent control of nuclear spin Hamiltonians in high-resolution NMR spectroscopy.

E.L. Hahn, PR **80**, 580 (1950);
U. Haeberlen & J.S. Waugh, PR **175**, 453 (1968).



Hahn spin echo idea



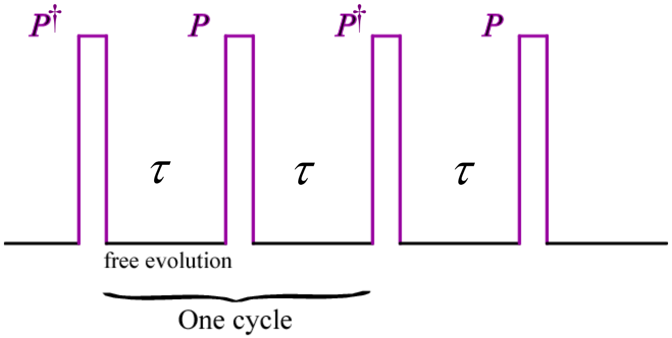
The “race-track” echo:
Effective time reversal

Dynamical Decoupling Basics

A pulse producing a unitary evolution P, such that

$$PH_{\mathrm{SB}}P^{\dagger} = -H_{\mathrm{SB}} \qquad \text{i.e., } \{P, H_{\mathrm{SB}}\} = 0$$

(CPMG, Hahn spin-echo)



Ideal (zero-width) pulses, and ignoring H_{B}:

$$P\exp(-i\tau H_{\mathrm{SB}})P^{\dagger}\exp(-i\tau H_{\mathrm{SB}}) = \exp(-i\tau PH_{\mathrm{SB}}P^{\dagger})\exp(-i\tau H_{\mathrm{SB}})$$
$$= \exp(i\tau H_{\mathrm{SB}})\exp(-i\tau H_{\mathrm{SB}}) = I$$

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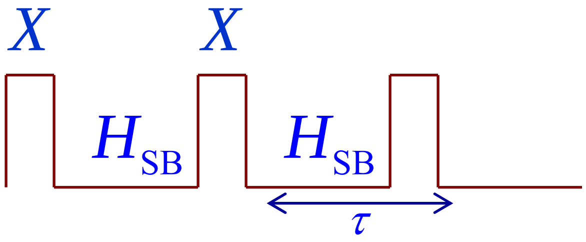

$$H_{\mathrm{SB}} = \lambda Z \otimes B$$

$$XZX = -Z \quad \Longrightarrow$$

"time reversal",

H_{SB} averaged to zero

(in 1st order Magnus expan.)

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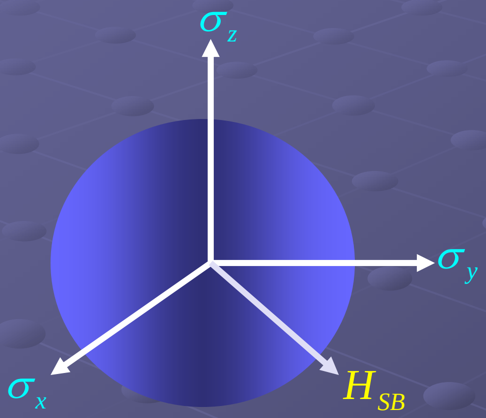
Dynamical Decoupling = Symmetrization

Viola & Lloyd Phys. Rev. A **58**, 2733 (1998); Byrd & Lidar, Q. Inf. Proc. **1**, 19 (2002)

System-bath Hamiltonian: $H_{SB} = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}$

system

bath



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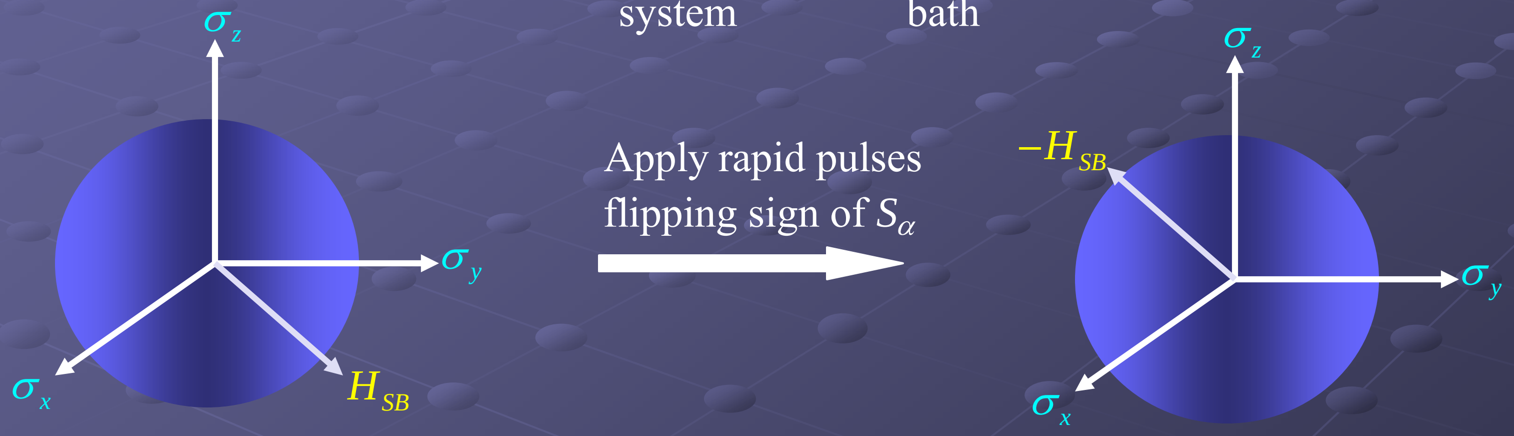

Apply rapid pulses flipping sign of S_{α}

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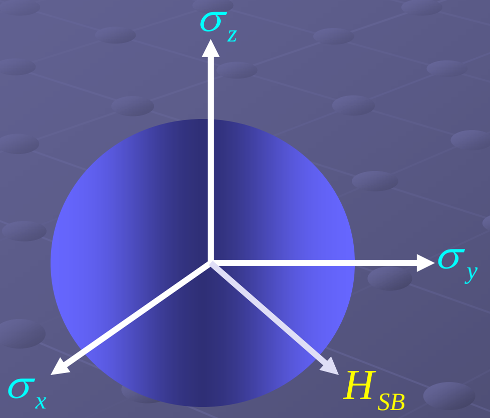

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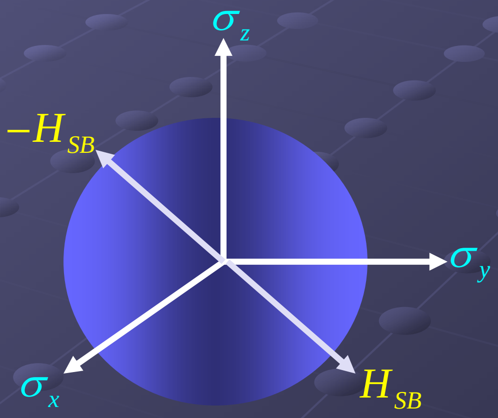

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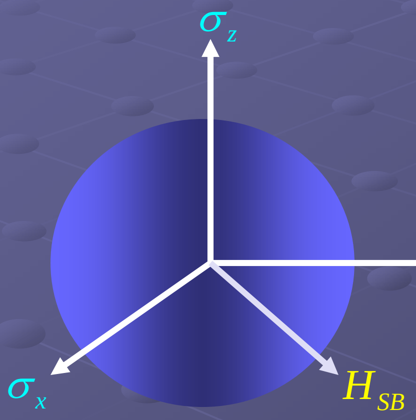

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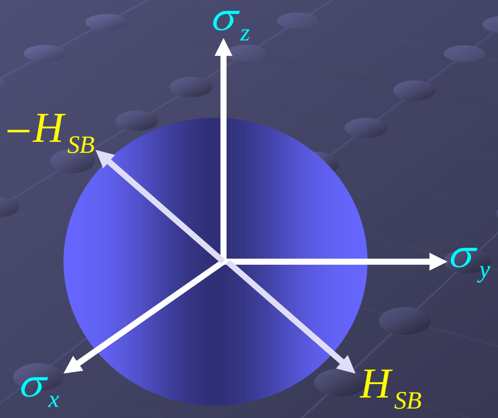

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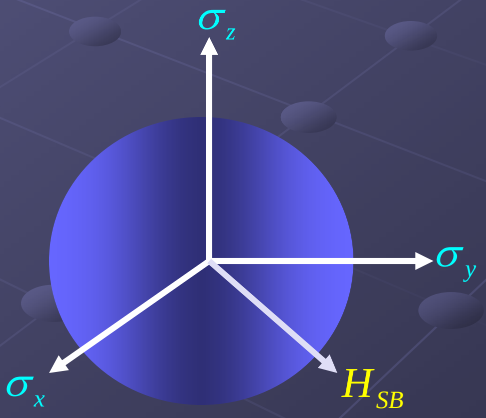

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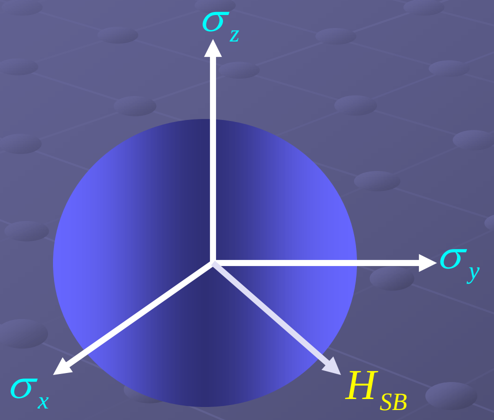

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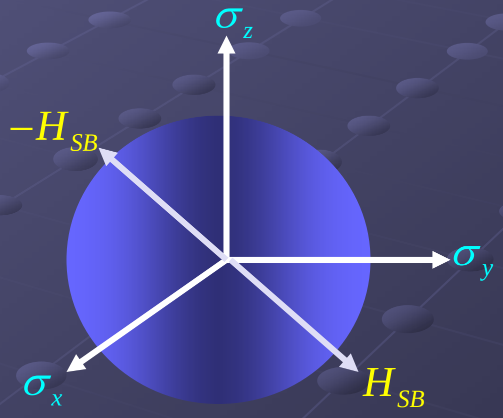

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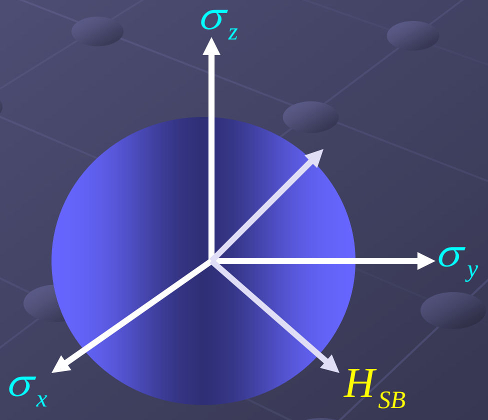

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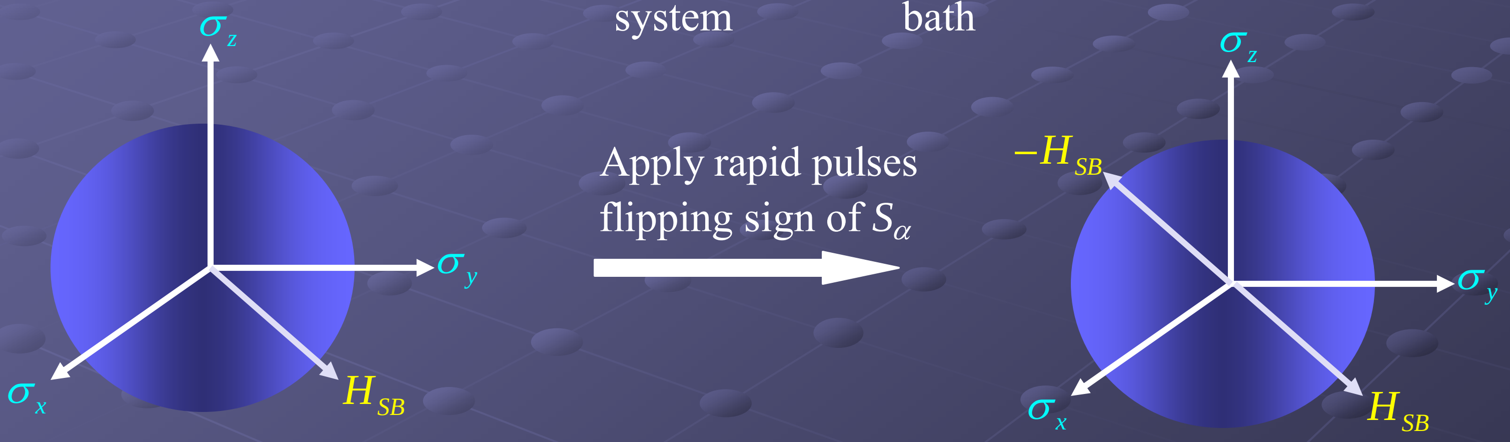

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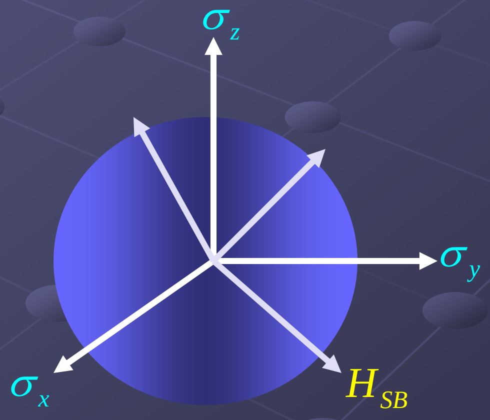

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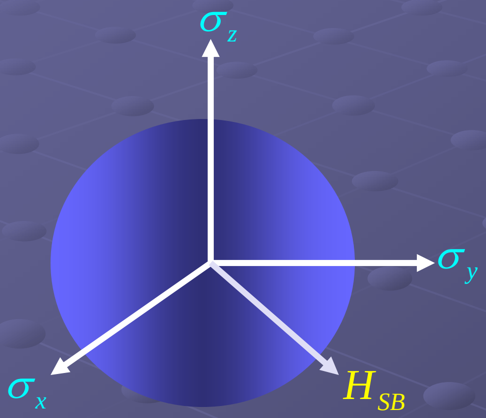

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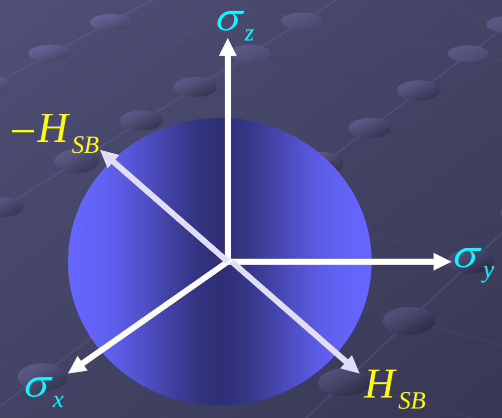

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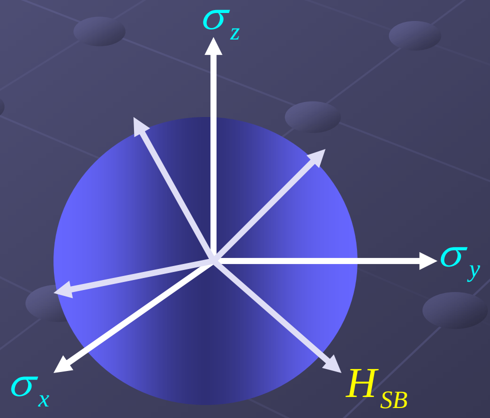

H_{SB} averaged to zero.

Dealing with Symmetry Breaking: Creating Collective Dephasing Conditions

L.-A. Wu, D.A.L., *Phys. Rev. Lett*. **88**, 207902 (2002)

General two-qubit dephasing: $H_{SB} = \sigma_1^z \otimes B_1 + \sigma_2^z \otimes B_2$

$$= \underbrace{\frac{1}{2}\left(\sigma_1^z - \sigma_2^z\right)}_{Z} \otimes \left(B_1 - B_2\right) + \frac{1}{2}\left(\sigma_1^z + \sigma_2^z\right) \otimes \left(B_1 + B_2\right)$$

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Assume: controllable exchange $X = \frac{1}{2}\left(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y\right) + \alpha \sigma_1^z \sigma_2^z.$

$$\{X, Z\} = 0 \quad \Rightarrow \quad XZX = -Z$$

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"Time reversal" Dynamical Decoupling pulse sequence:

$$\exp(-iH_{SB}t)\left[\exp(-i\frac{\pi}{2}X)\exp(-iH_{SB}t)\exp(i\frac{\pi}{2}X)\right] = \exp(-it\underbrace{\left(\sigma_1^z + \sigma_2^z\right)\otimes\left(B_1 + B_2\right)}_{\text{Collective Dephasing}})$$

$X \longleftarrow H_{SB}t \longrightarrow X \longleftarrow H_{SB}t \longrightarrow \quad = \quad 2t$ Coll.Deph.

Heisenberg is “Super-Universal”

Same method works, e.g., for *spin-coupled quantum dots* QC:

By BB pulsing of $H_{\text{Heis}} = \frac{J}{2}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z\right)$

collective decoherence conditions can be created:

$$H_{\text{SB}} = \sum_{i=1}^{n} g_i^x\sigma_i^x \otimes B_i^x + g_i^y\sigma_i^y \otimes B_i^y + g_i^z\sigma_i^z \otimes B_i^z$$

$$\rightarrow S_x \otimes B_x + S_y \otimes B_y + S_z \otimes B_z$$

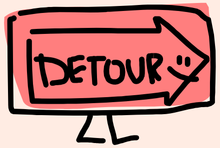
Requires sequence of 6 $\pi/2$ pulses to create collective decoherence conditions over blocks of 4 qubits. Leakage elimination requires 7 more pulses.

Details: L.-A. Wu, D.A.L., *Phys. Rev. Lett.* **88**, 207902 (2002); L.A. Wu, M.S. Byrd, D.A.L., *Phys. Rev. Lett.* **89**, 127901 (2002).

Earlier DFS work showed universal QC with Heisenberg interaction alone possible [Bacon, Kempe, D.A.L., Whaley, *Phys. Rev. Lett.* **85**, 1758 (2000)]:

All ingredients available for Heisenberg-only QC

Analysis of Dynamical Decoupling

We'll need a formal  ..

Decoherence: Isolated *vs* Open System Evolution

Isolated system: $H = H_S$

$$|\psi(t)\rangle = U_S(t)|\psi(0)\rangle \qquad \dot{U}_S = -iH_SU_S \quad U_S(0) = I$$

equivalently: $|\psi(t)\rangle\langle\psi(t)| = U_S(t)|\psi(0)\rangle\langle\psi(0)|U_S^\dagger(t)$

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Open system: $H = H_S + H_B + H_{SB}$

$$\rho_{SB}(t) = U(t)\rho_{SB}(0)U^\dagger(t) \qquad \dot{U} = -iHU \quad U(0) = I$$

$$\rho_S(t) = \mathrm{Tr}_B \rho_{SB}(t)$$

$\neq$ unitary tranformation of $\rho_S(0)$
(except when there is a decoherence-free subspace)

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decoherence: $\|\rho_S(t) - |\psi(t)\rangle\langle\psi(t)|\| > 0$

which norm?

Kolmogorov Distance and Quantum Measurements (I)

Given two classical probability distributions $\{p_i^{(1)}\}$ and $\{p_i^{(2)}\}$, their deviation is measured by the Kolmogorov distance

$$D(p^{(1)}, p^{(2)}) \equiv \frac{1}{2} \sum_i |p_i^{(1)} - p_i^{(2)}|$$

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A quantum measurement can always be described in terms of a POVM (positive operator valued measure), i.e., a set of **positive** operators E_i satisfying $\sum_i E_i = I$, where i enumerates the possible measurement outcomes.

For a system initially in the state ρ, outcome i occurs with probability

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Thus quantum measurements produce classical probability distributions.

Consider two quantum states $\rho^{(1)}[= |\psi(t)\rangle\langle\psi(t)|]$ and $\rho^{(2)}[= \rho_S(t)]$

Kolmogorov Distance and Quantum Measurements (II)

Compare measurement outcomes of same POVM on $\rho^{(1)}[= |\psi(t)\rangle\langle\psi(t)|]$ and $\rho^{(2)}[= \rho_S(t)]$:

Lemma: $\delta \equiv D(p_{\rho^{(1)}}, p_{\rho^{(2)}}) \leq \|\rho^{(1)} - \rho^{(2)}\|_{\mathrm{Tr}}$

$$\|A\|_{\mathrm{Tr}} \equiv \mathrm{Tr}\sqrt{A^\dagger A} = \sum(\text{singular values}(A))$$

The bound is tight in the sense that it is saturated for the optimal measurement designed to distinguish the two states.

Partial trace decreases trace distance

Lemma: $\|\rho_S^{(1)} - \rho_S^{(2)}\|_{\mathrm{Tr}} \leq \|\rho_{SB}^{(1)} - \rho_{SB}^{(2)}\|_{\mathrm{Tr}}$

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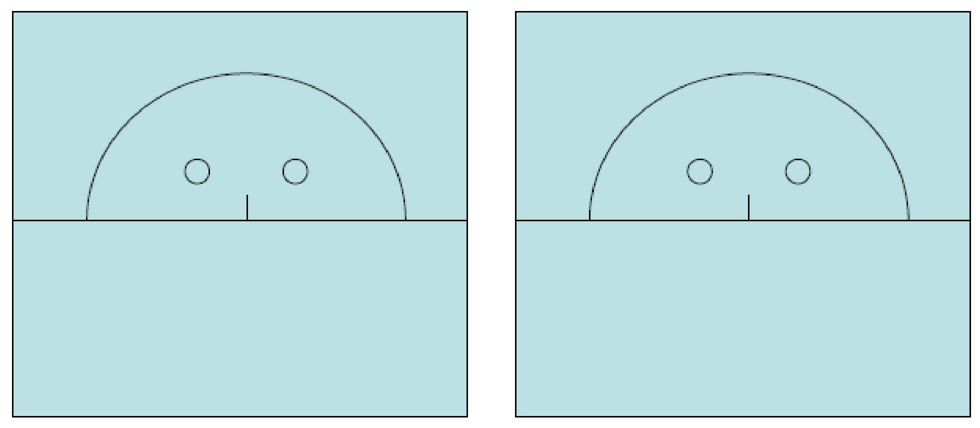

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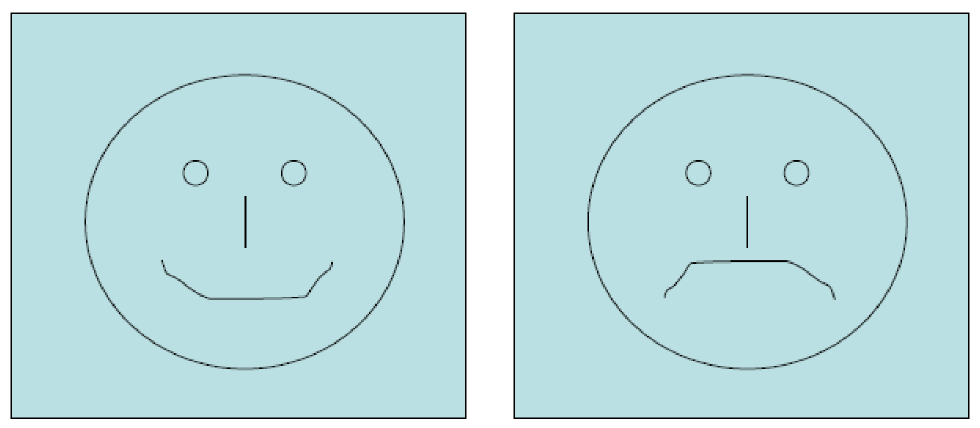

Conclusion: we can compare dynamics of ideal and actual systems over the joint system-bath space.

Ideal *vs* Actual System Evolution

Ideal system: $H = H_S + H_B$

$$\rho_{SB}^{\text{ideal}}(t) = [U_S(t) \otimes U_B(t)]\rho_{SB}(0)[U_S^\dagger(t) \otimes U_B^\dagger(t)]$$

$$\dot{U}_{S/B} = -iH_{S/B}U_{S/B} \quad U_{S/B}(0) = I$$

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Distance:

$$\|\rho_{SB}(t) - \rho_{SB}^{\text{ideal}}(t)\|_{\text{Tr}} = \|V(t)\rho_{SB}(0)V^\dagger(t) - \rho_{SB}(0)\|_{\text{Tr}}$$

$$V(t) \equiv U_S^\dagger(t) \otimes U_B^\dagger(t)U(t) \equiv \exp[-itH_{\text{eff}}(t)]$$

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$$\boxed{\text{Lemma: } \|\rho_{SB}(t) - \rho_{SB}^{\text{ideal}}(t)\|_{\text{Tr}} \leq t\|H_{\text{eff}}(t)\|_\infty}$$

follows from $\|e^{iA} - e^{iB}\|_\infty \leq \|A - B\|_\infty$; $\|A\|_\infty \equiv \sup_{|v\rangle,\langle v|v\rangle=1} \sqrt{\langle v|A^\dagger A|v\rangle} = \max \text{sing.val.}(A)$

Kolmogorov Distance Bound from Effective Hamiltonian

$$\boxed{\text{Lemma: } \delta \equiv D(p_{\rho^{(1)}}, p_{\rho^{(2)}}) \leq \|\rho^{(1)} - \rho^{(2)}\|_{\mathrm{Tr}}}$$

(trace distance bounds Kolmogorov distance)

$$\boxed{\text{Lemma: } \|\rho_S^{(1)} - \rho_S^{(2)}\|_{\mathrm{Tr}} \leq \|\rho_{SB}^{(1)} - \rho_{SB}^{(2)}\|_{\mathrm{Tr}}}$$

(partial trace decreases distinguishability)

$$V(t) \equiv U_S^\dagger(t) \otimes U_B^\dagger(t) U(t) \equiv \exp[-itH_{\mathrm{eff}}(t)]$$

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Goal: reduce effective Hamiltonian.

Method: dynamical decoupling.

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Goal: reduce effective Hamiltonian.

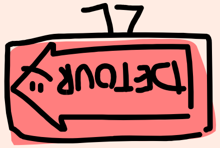
Method: dynamical decoupling.

How do we compute $H_{\text{eff}}(t)$?

The Magnus expansion

$$\begin{aligned} \dot{U} &= -iH(t)U \\ U(t) &= e^{-itH_{\text{eff}}(t)} \\ H_{\text{eff}}(t) &= \frac{1}{t}\sum_{j=1}^{\infty}\Omega_j(t) \\ \Omega_1(t) &= \int_0^t dt_1\, H(t_1) \qquad \Omega_2(t) = -\frac{i}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)] \end{aligned}$$

Analysis of Dynamical Decoupling



Dynamical Decoupling Theory

“Symmetrizing group” of pulses $\{g_i\}$ and their inverses are applied in series:

$$\boxed{(g_N^\dagger \mathbf{f} g_N)\cdots(g_2^\dagger \mathbf{f} g_2)(g_1^\dagger \mathbf{f} g_1) \approx \exp\left(-i\tau \sum_i g_i^\dagger H_{SB} g_i\right)}$$

$$\mathbf{f} \equiv \exp(-iH_{SB}\tau)$$

first order Magnus expansion

Periodic DD: periodic repetition of the universal DD pulse sequence

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Choose the pulses so that:

$$H_{SB} \mapsto H_{\text{eff}}^{(1)} \equiv \sum_i g_i^\dagger H_{SB} g_i = 0$$

Dynamical Decoupling Condition

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first order Magnus expansion

Choose the pulses so that:

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Dynamical Decoupling Condition

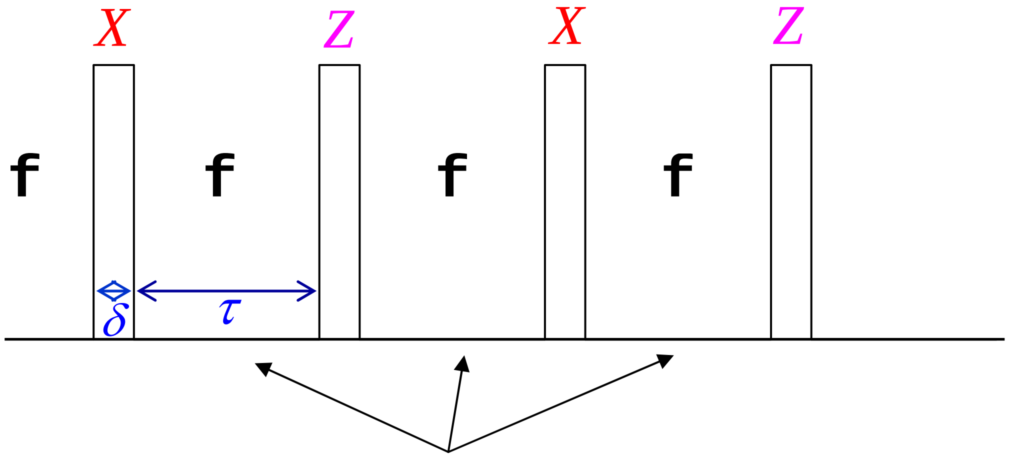
For a qubit the Pauli group $G=\{X,Y,Z,I\}$ (π pulses around all three axes) removes an *arbitrary* H_{SB}:

$$\mathbf{(XfX)(YfY)(ZfZ)(IfI) = \underline{XfZfXfZf}}$$

Periodic DD: periodic repetition of the universal DD pulse sequence

The Effective Hamiltonian

Another view of the universal decoupling sequence:



$\mathbf{f} \equiv \exp[-i\tau H_{SB}]$

$=$

$\mathbf{f'}$

$\mathbf{f'} \equiv \exp[-iTH_{eff}(T)]$

$H_{eff}(T)=0$, ideally

The Effective Hamiltonian

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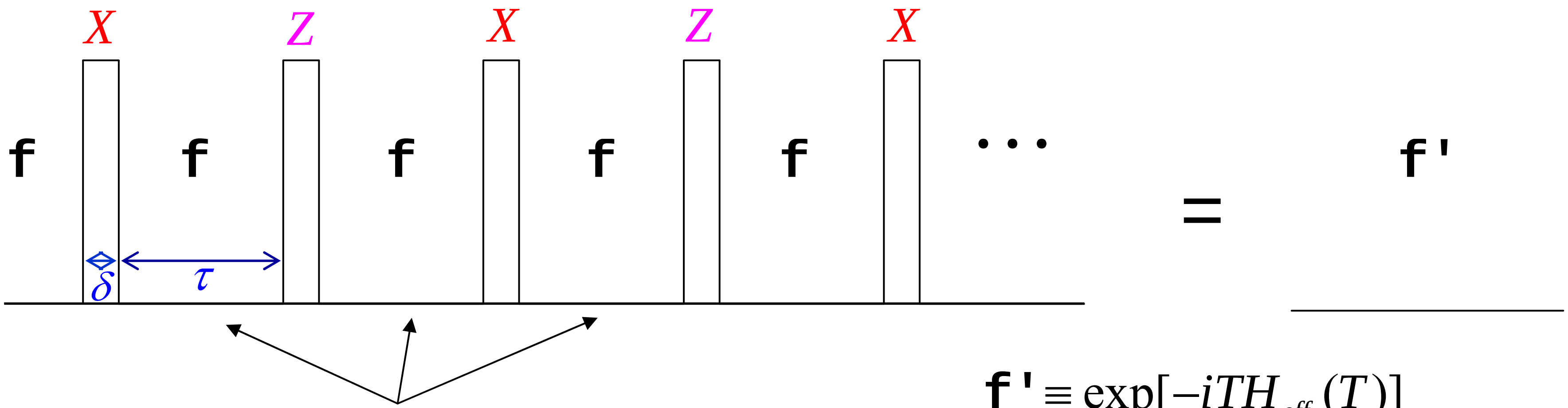

$$\mathbf{f} \equiv \exp[-i\tau H_{SB}]$$

$$\mathbf{f'} \equiv \exp[-iTH_{eff}(T)]$$

$H_{eff}(T)$=0, ideally

But, errors accumulate…: $H_{eff}(T) \neq 0$

Periodic Dynamical Decoupling

PDD Strategy: repeat the basic XfZfZfXfZ cycle with total of N pulses.

The total duration is fixed at T. N can be changed.

Pulse interval: $\tau = T/N$

Recall noise strength $\eta \equiv ||H_{\text{eff}}(T)||T$

norm of final effective system-bath Hamiltonian times the total duration.

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norm of final effective system-bath Hamiltonian times the total duration.

PDD leading order result for error:

$$\eta \propto N^{-1}$$

Can we do better?

DD as a Rescaling Transformation

$$J = \|H_{SB}\|_\infty$$

$$\beta = \|H_B\|_\infty$$

- Interaction terms are rescaled after the DD cycle

$$J = J^{(0)} \mapsto J^{(1)} \propto \max[\tau(J^{(0)})^2, \tau\beta J^{(0)}]$$

$$\beta \mapsto \beta + O((J^{(0)})^3\tau^2)$$

- We need a mechanism to continue this

Concatenated Universal Dynamical Decoupling

Nest the universal DD pulse sequence into its own free evolution periods **`f`** :

`p(1)= X f   Z f   X f   Z f`

Concatenated Universal Dynamical Decoupling

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```
p(1)= X f   Z f   X f   Z f
p(2)= X p(1)Z p(1)X p(1)Z p(1)
      p(n+1)= X p(n)Z p(n)X p(n)Z p(n)
```

Concatenated Universal Dynamical Decoupling

Nest the universal DD pulse sequence into its own free evolution periods **`f`** :

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p(2)= X p(1)Z p(1)X p(1)Z p(1)
      p(n+1)= X p(n)Z p(n)X p(n)Z p(n)
```

Level	Concatenated DD Series after multiplying Pauli matrices
1	`XfZfXfZf`
2	`fZfXfZfYfZfXfZffZfXfZfYfZfXfZf`
3	`XfZfXfZfYfZfXfZffZfXfZfYfZfXfZfZfZfXfZfYfZfXfZffZfXfZfYfZfXfZfXfZfXfZfYfZfXfZffZfXfZfYfZfXfZfZfZfXfZfYfZfXfZffZfXfZfYfZfXfZf`

Length grows exponentially; how about error reduction?

Performance of Concatenated Sequences

$$\text{error} \mapsto (\text{error})^2 \mapsto ((\text{error})^2)^2 \mapsto (((\text{error})^2)^2)^2 \mapsto \underset{k}{\cdots} \mapsto (\text{error})^{2^k}$$

Performance of Concatenated Sequences

$$\text{error} \mapsto (\text{error})^2 \mapsto ((\text{error})^2)^2 \mapsto (((\text{error})^2)^2)^2 \mapsto \underset{k}{\cdots} \mapsto (\text{error})^{2^k}$$

For fixed total time $T=N\tau$ and N zero-width (ideal) pulses:

$$\boxed{\eta \propto N^b N^{-c \log N}}$$

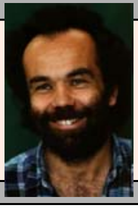
Compare to periodic DD: $\eta \propto N^{-1}$

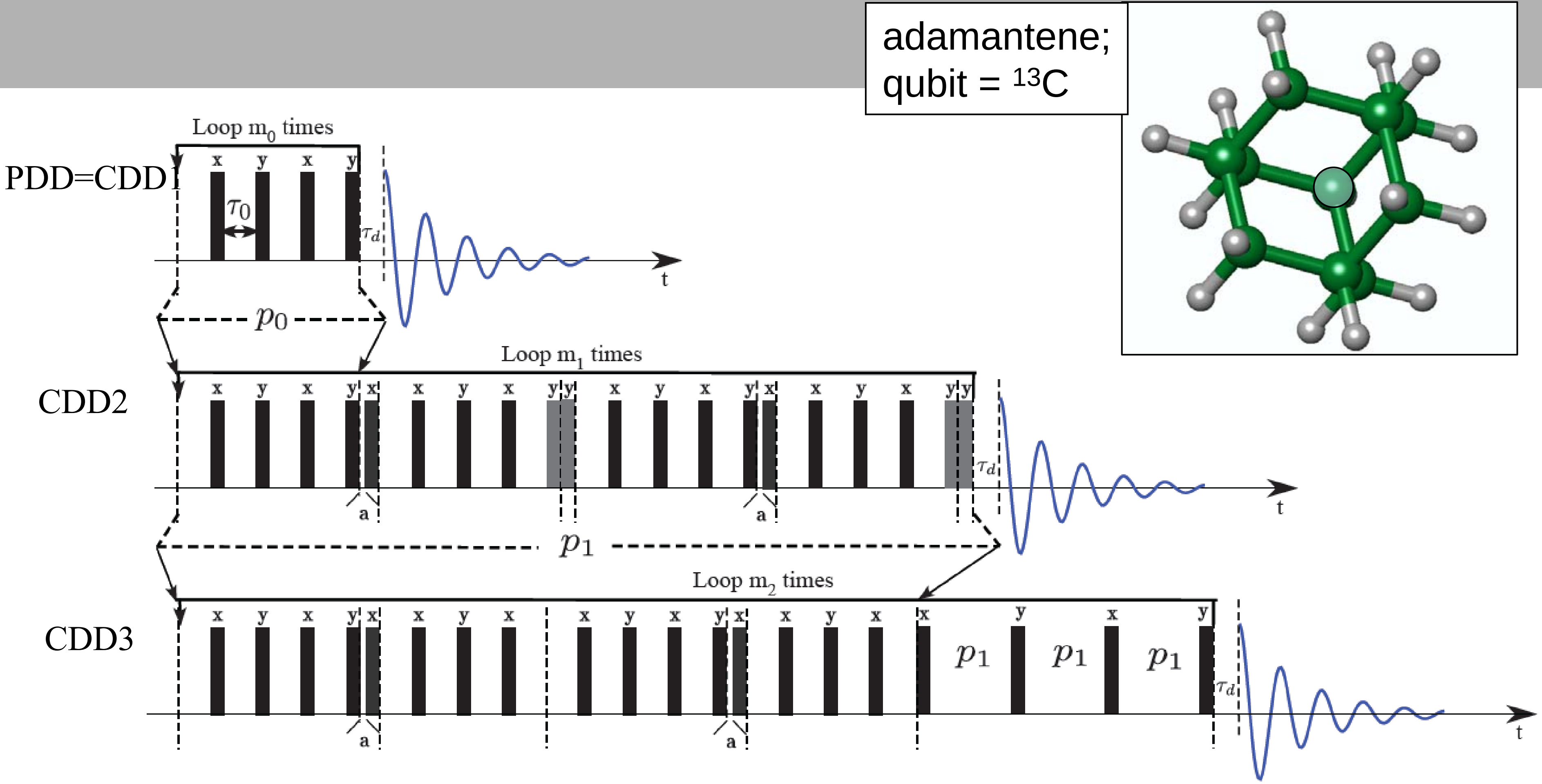
[Khodjasteh & Lidar, PRA **75**, 062310 (2007)]

Experiments

Concatenated DD on Adamantene Powder

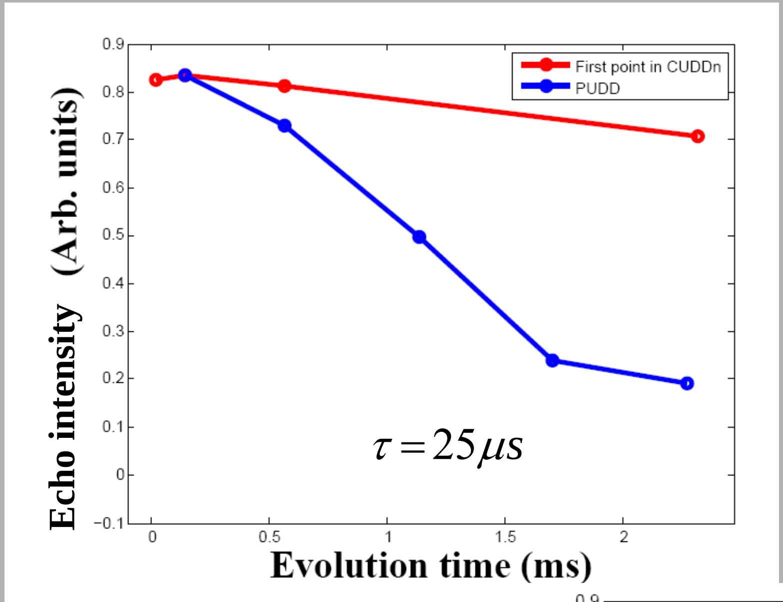
Dieter Suter, TU Dortmund

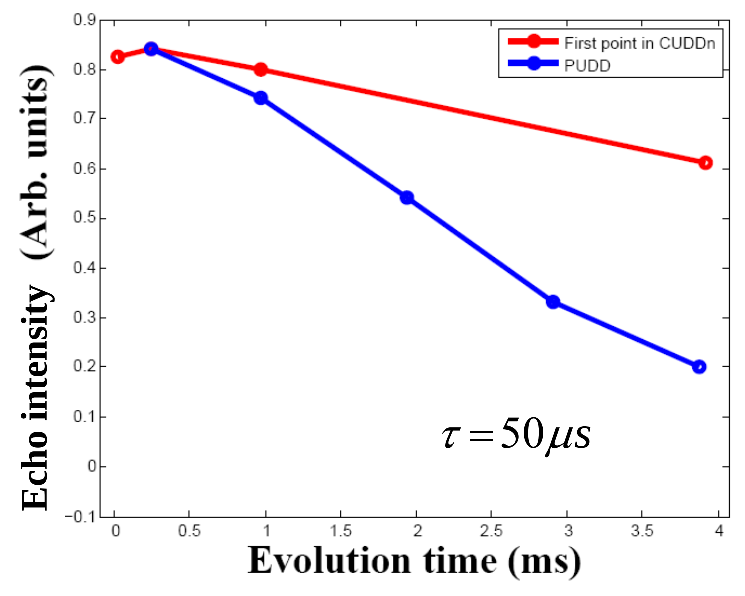




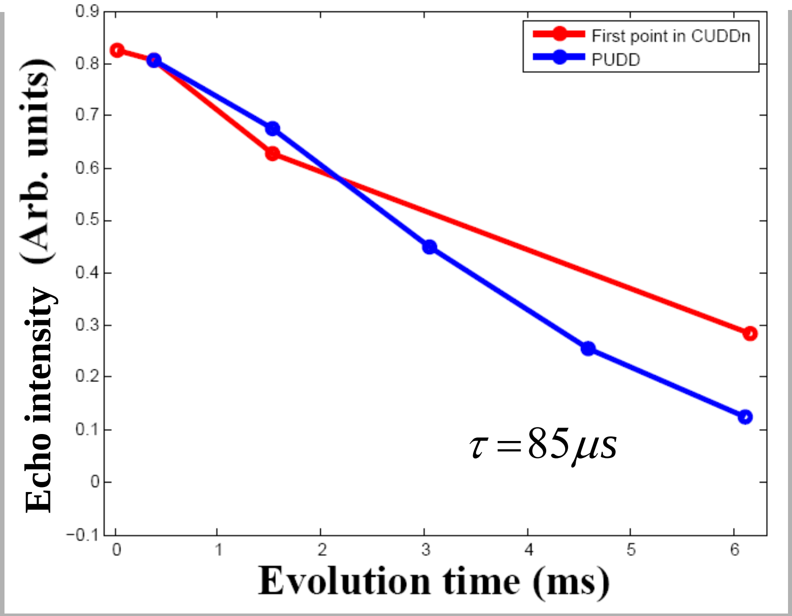
adamantene;
qubit = ^{13}C

CDD Results



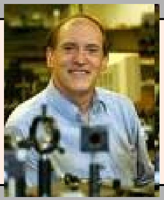


$\delta = 10.5\mu s$

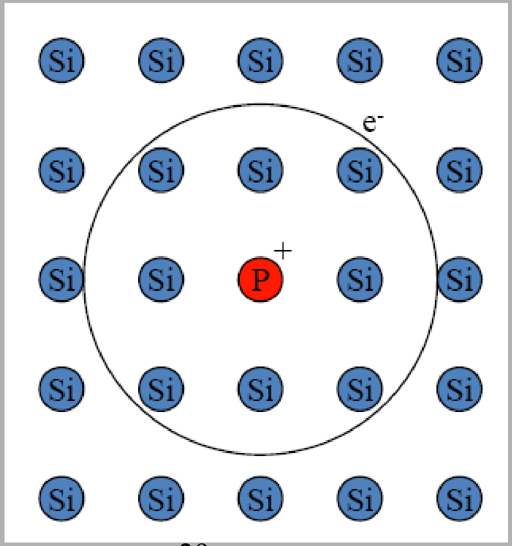


Concatenated DD for electron spin of ^{31}P donors in Si

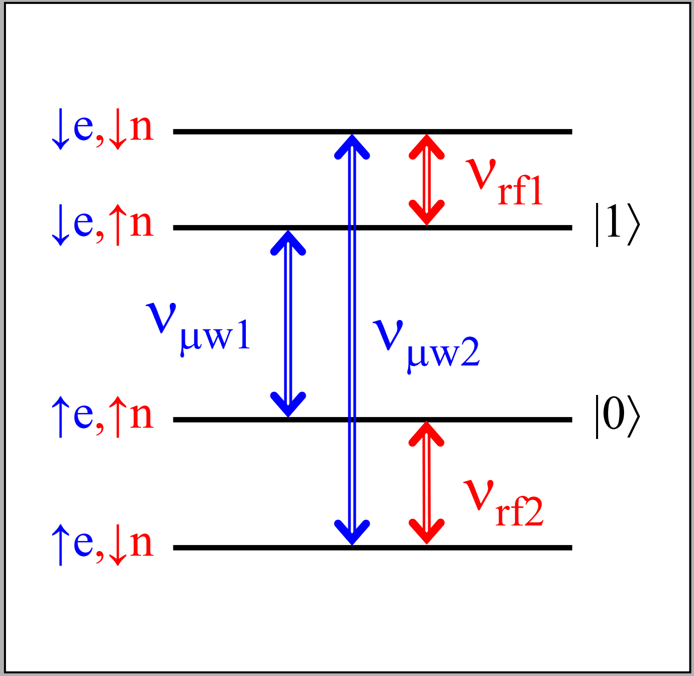
Steve Lyon, Princeton



^{31}P donor: Electron spin (S) = ½,
Nuclear spin (I) = ½



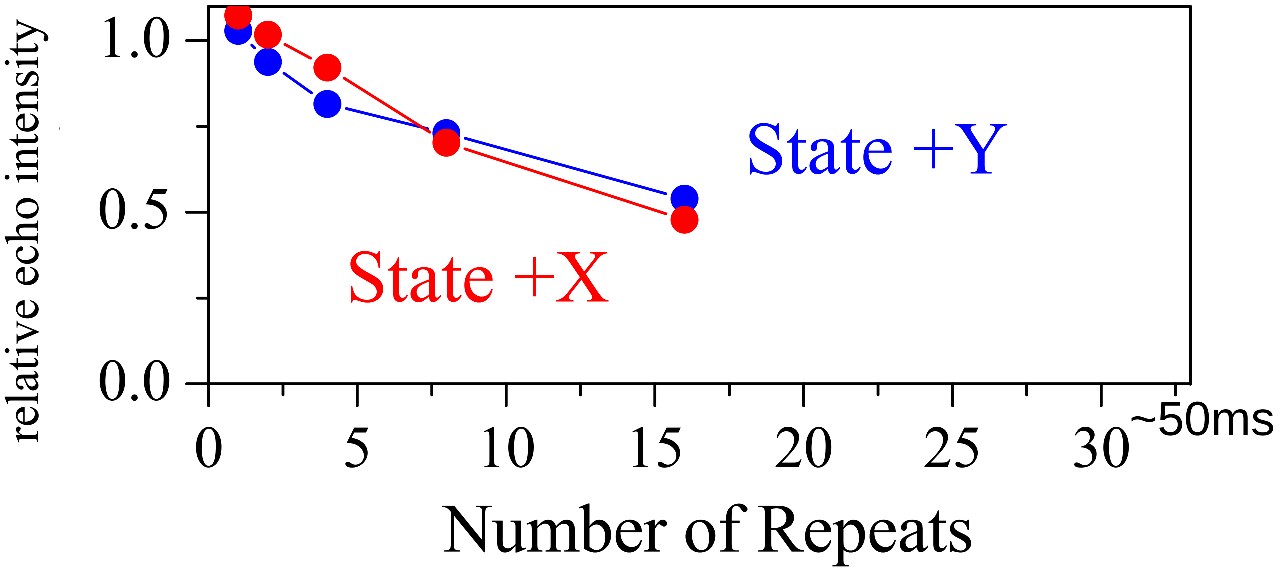
bath is ^{29}Si
~1% natural abundance



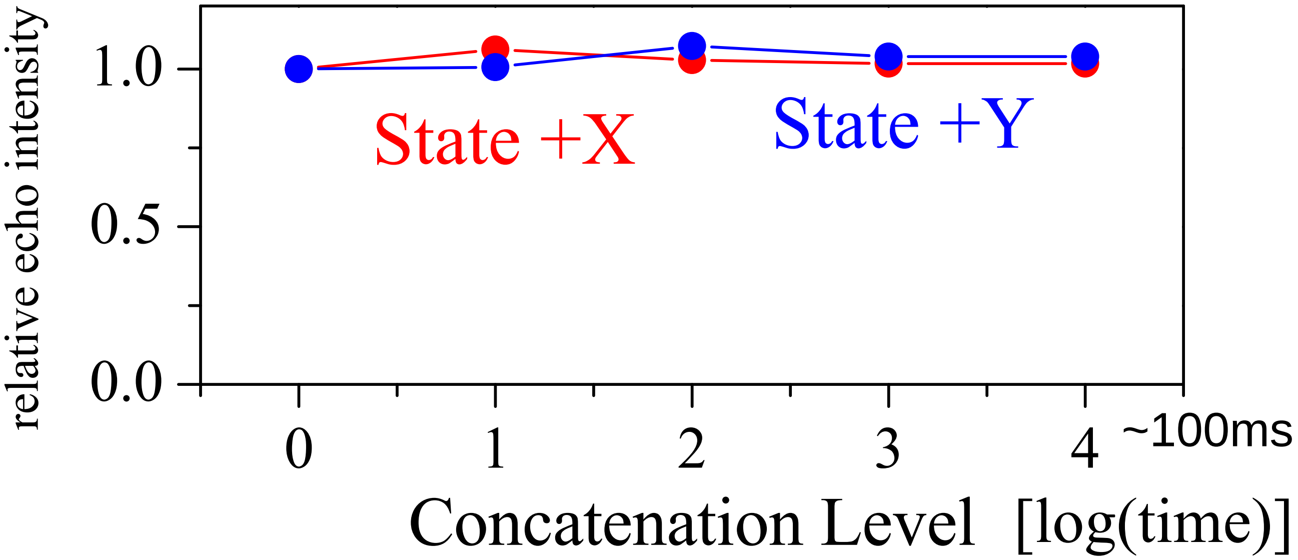
Periodic DD vs Concatenated DD

1. Periodic
(XfYfXfYf)

δ =160ns



2. Concatenated

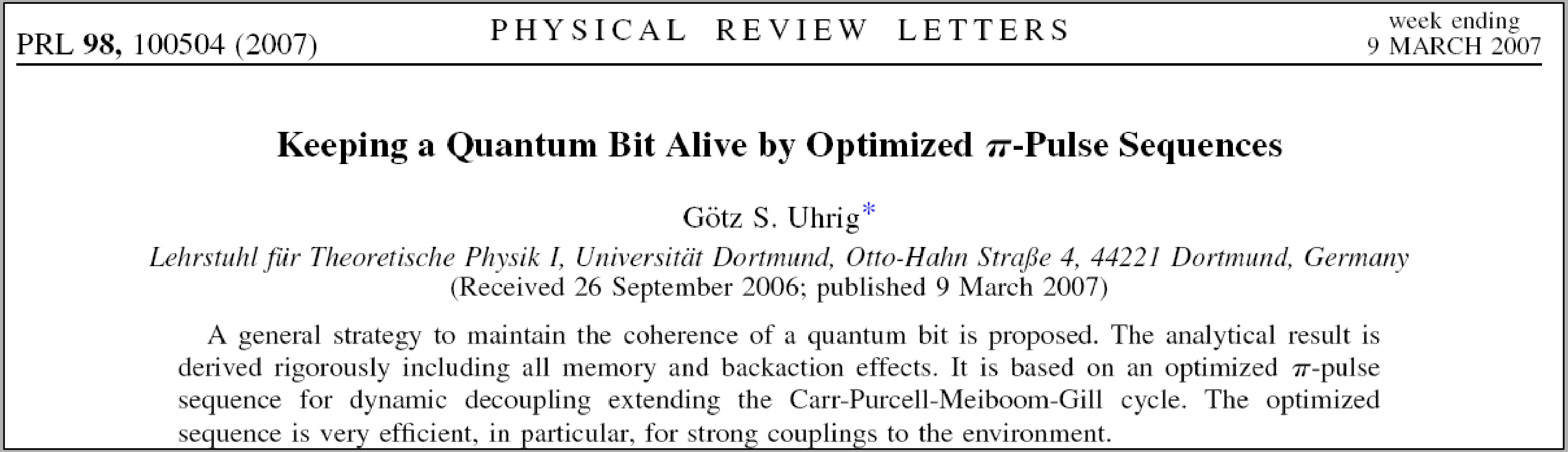


Better than Concatenated DD?

Does there exist an optimal pulse sequence?

Optimal = removes maximum decoherence
with least possible number of pulses

Better than Concatenated DD?



PRL **98,** 100504 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

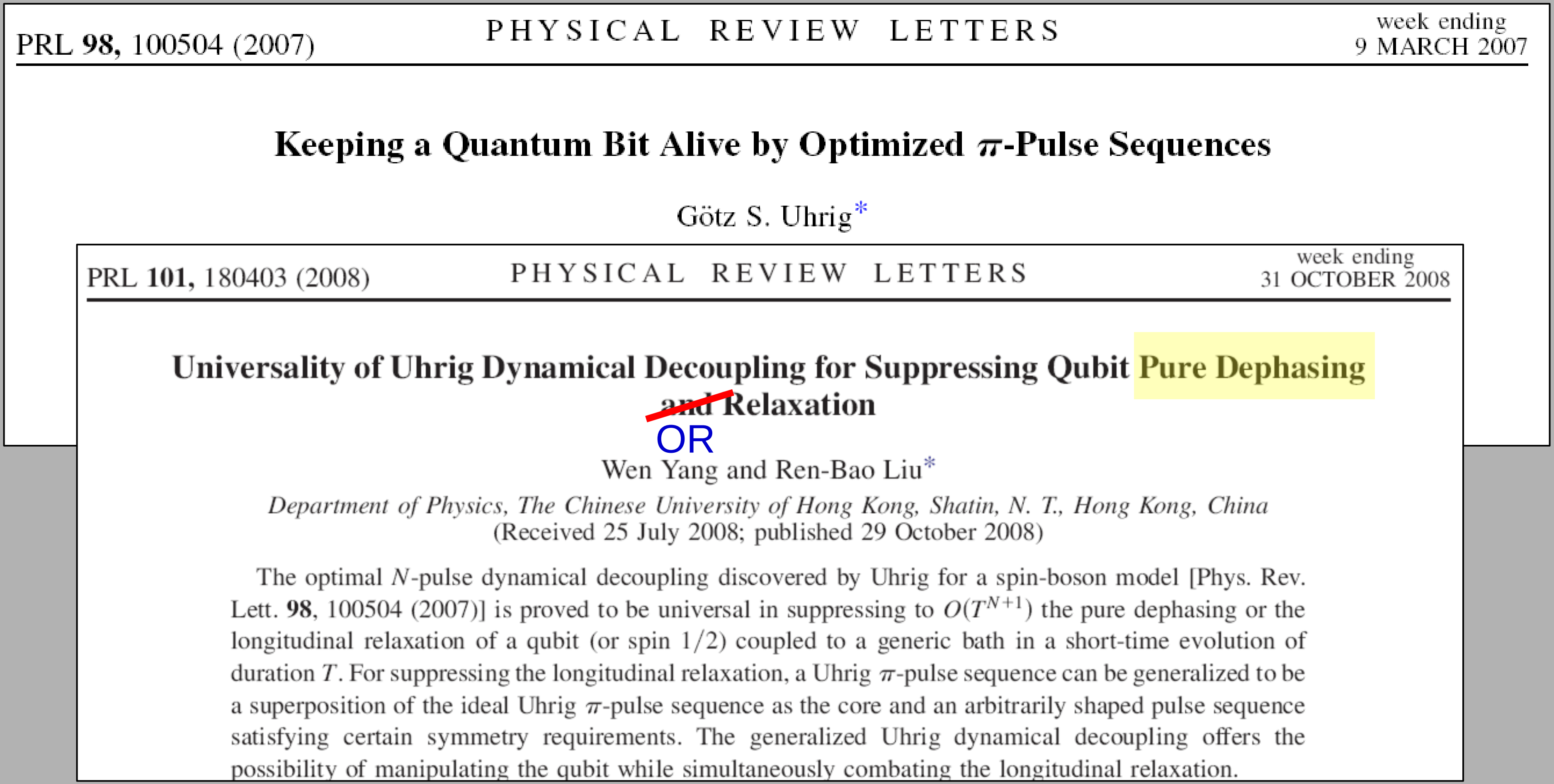
Götz S. Uhrig*

Lehrstuhl für Theoretische Physik I, Universität Dortmund, Otto-Hahn Straße 4, 44221 Dortmund, Germany

(Received 26 September 2006; published 9 March 2007)

A general strategy to maintain the coherence of a quantum bit is proposed. The analytical result is derived rigorously including all memory and backaction effects. It is based on an optimized π-pulse sequence for dynamic decoupling extending the Carr-Purcell-Meiboom-Gill cycle. The optimized sequence is very efficient, in particular, for strong couplings to the environment.

Better than Concatenated DD?



PRL **98,** 100504 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

Götz S. Uhrig[*]

PRL **101,** 180403 (2008) PHYSICAL REVIEW LETTERS week ending 31 OCTOBER 2008

Universality of Uhrig Dynamical Decoupling for Suppressing Qubit Pure Dephasing ~~and~~ OR Relaxation

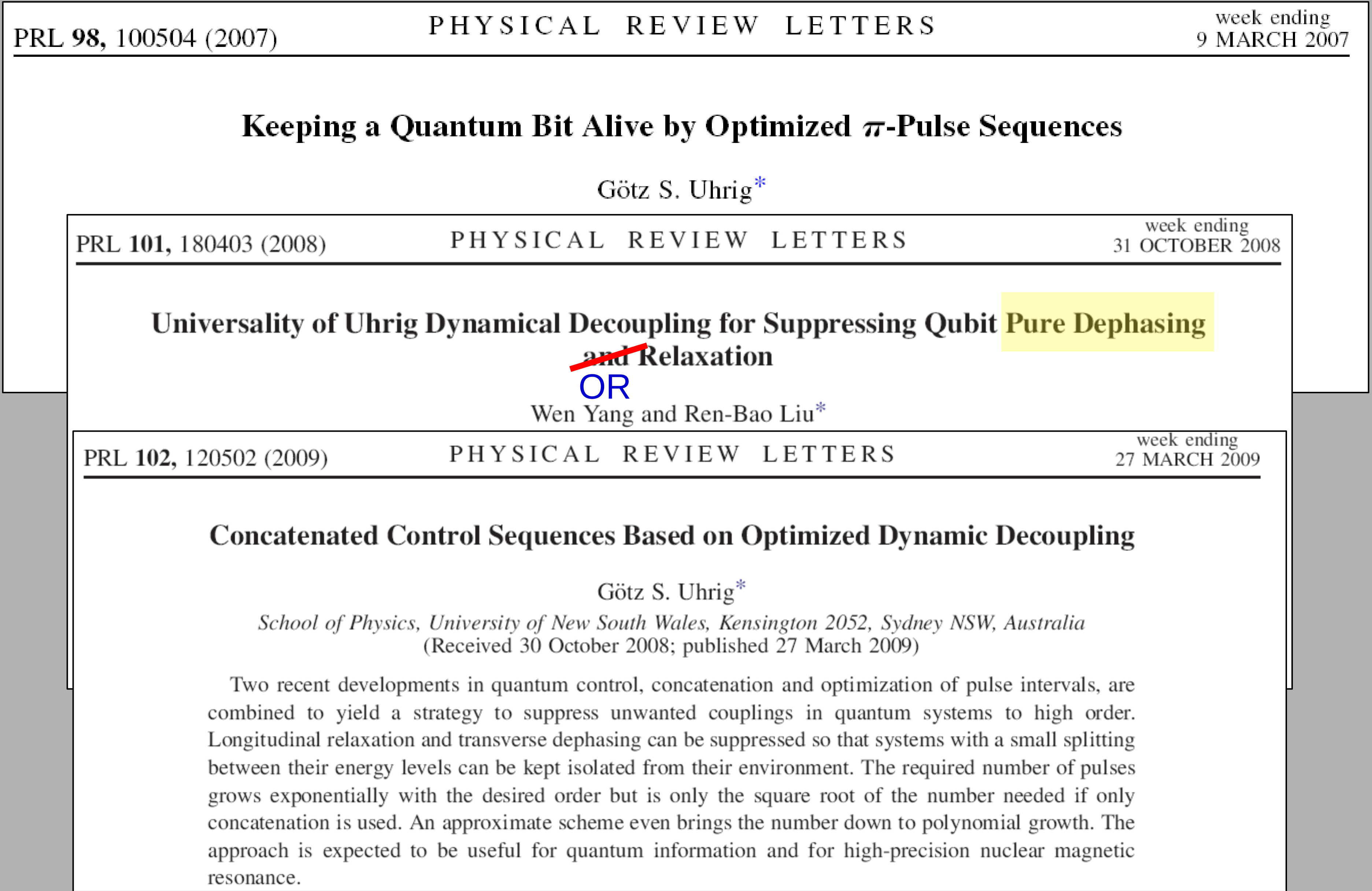
Wen Yang and Ren-Bao Liu[*]

Department of Physics, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong, China

(Received 25 July 2008; published 29 October 2008)

The optimal N-pulse dynamical decoupling discovered by Uhrig for a spin-boson model [Phys. Rev. Lett. **98**, 100504 (2007)] is proved to be universal in suppressing to $O(T^{N+1})$ the pure dephasing or the longitudinal relaxation of a qubit (or spin 1/2) coupled to a generic bath in a short-time evolution of duration T. For suppressing the longitudinal relaxation, a Uhrig π-pulse sequence can be generalized to be a superposition of the ideal Uhrig π-pulse sequence as the core and an arbitrarily shaped pulse sequence satisfying certain symmetry requirements. The generalized Uhrig dynamical decoupling offers the possibility of manipulating the qubit while simultaneously combating the longitudinal relaxation.

Better than Concatenated DD?



PRL **98,** 100504 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

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PRL **101,** 180403 (2008) PHYSICAL REVIEW LETTERS week ending 31 OCTOBER 2008

Universality of Uhrig Dynamical Decoupling for Suppressing Qubit Pure Dephasing ~~and~~ Relaxation

OR

Wen Yang and Ren-Bao Liu*

PRL **102,** 120502 (2009) PHYSICAL REVIEW LETTERS week ending 27 MARCH 2009

Concatenated Control Sequences Based on Optimized Dynamic Decoupling

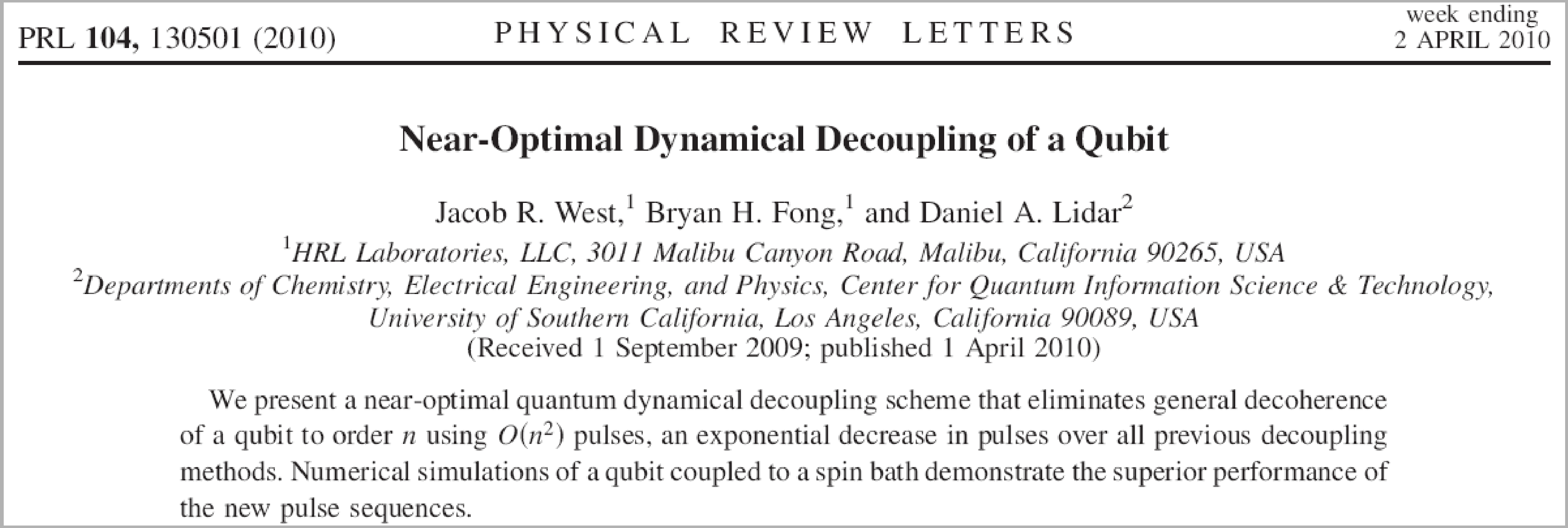
Götz S. Uhrig*

School of Physics, University of New South Wales, Kensington 2052, Sydney NSW, Australia

(Received 30 October 2008; published 27 March 2009)

Two recent developments in quantum control, concatenation and optimization of pulse intervals, are combined to yield a strategy to suppress unwanted couplings in quantum systems to high order. Longitudinal relaxation and transverse dephasing can be suppressed so that systems with a small splitting between their energy levels can be kept isolated from their environment. The required number of pulses grows exponentially with the desired order but is only the square root of the number needed if only concatenation is used. An approximate scheme even brings the number down to polynomial growth. The approach is expected to be useful for quantum information and for high-precision nuclear magnetic resonance.

Better than Concatenated DD?



PRL **104,** 130501 (2010) PHYSICAL REVIEW LETTERS week ending 2 APRIL 2010

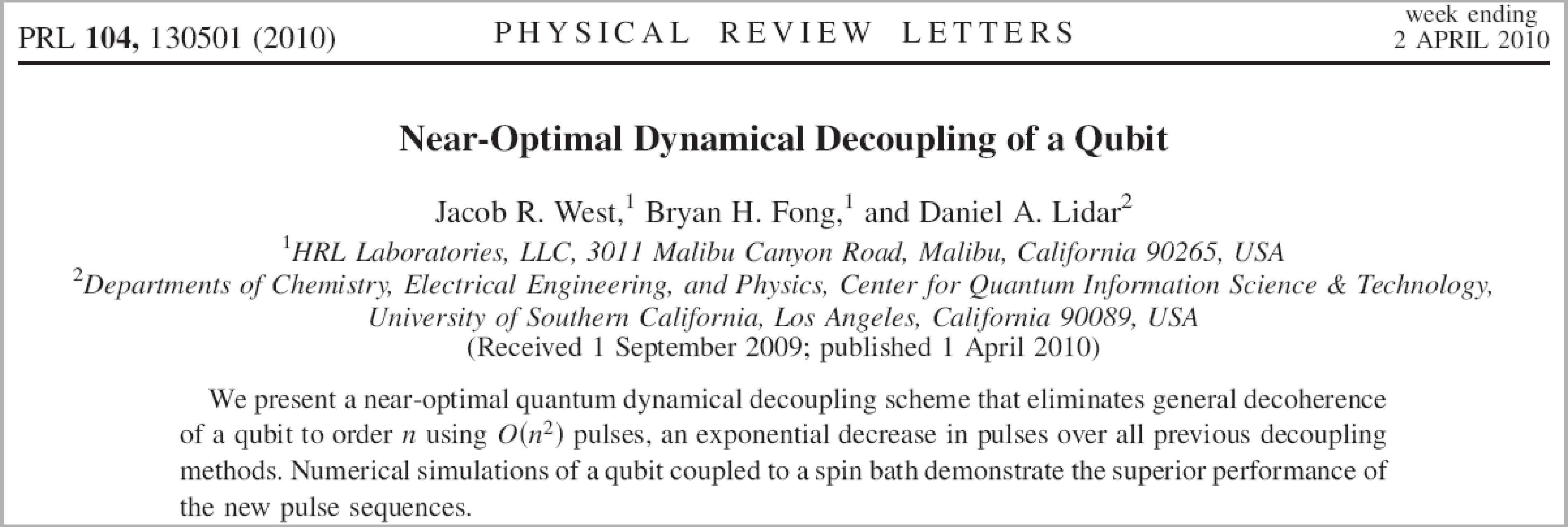
Near-Optimal Dynamical Decoupling of a Qubit

Jacob R. West,[1] Bryan H. Fong,[1] and Daniel A. Lidar[2]
[1]*HRL Laboratories, LLC, 3011 Malibu Canyon Road, Malibu, California 90265, USA*
[2]*Departments of Chemistry, Electrical Engineering, and Physics, Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, California 90089, USA*

(Received 1 September 2009; published 1 April 2010)

We present a near-optimal quantum dynamical decoupling scheme that eliminates general decoherence of a qubit to order n using $O(n^2)$ pulses, an exponential decrease in pulses over all previous decoupling methods. Numerical simulations of a qubit coupled to a spin bath demonstrate the superior performance of the new pulse sequences.

Better than Concatenated DD?



PRL **104,** 130501 (2010) PHYSICAL REVIEW LETTERS week ending 2 APRIL 2010

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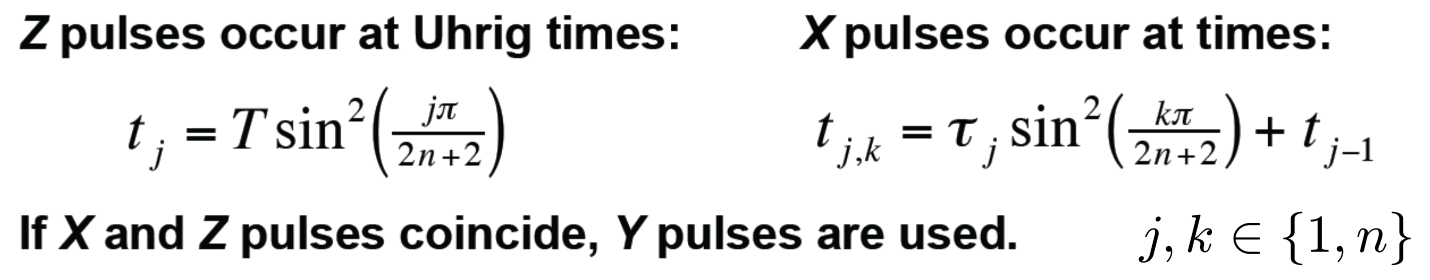
“Quadratic DD” eliminates the first *n* orders in the Dyson series of the joint system-bath propagator using n^2 pulses

Concatenated DD requires 4^n pulses to do the same, approximately

Inner workings of Quadratic DD



***Z* pulses occur at Uhrig times:**

$$t_j = T \sin^2\left(\frac{j\pi}{2n+2}\right)$$

***X* pulses occur at times:**

$$t_{j,k} = \tau_j \sin^2\left(\frac{k\pi}{2n+2}\right) + t_{j-1}$$

If *X* and *Z* pulses coincide, *Y* pulses are used.

$$j, k \in \{1, n\}$$

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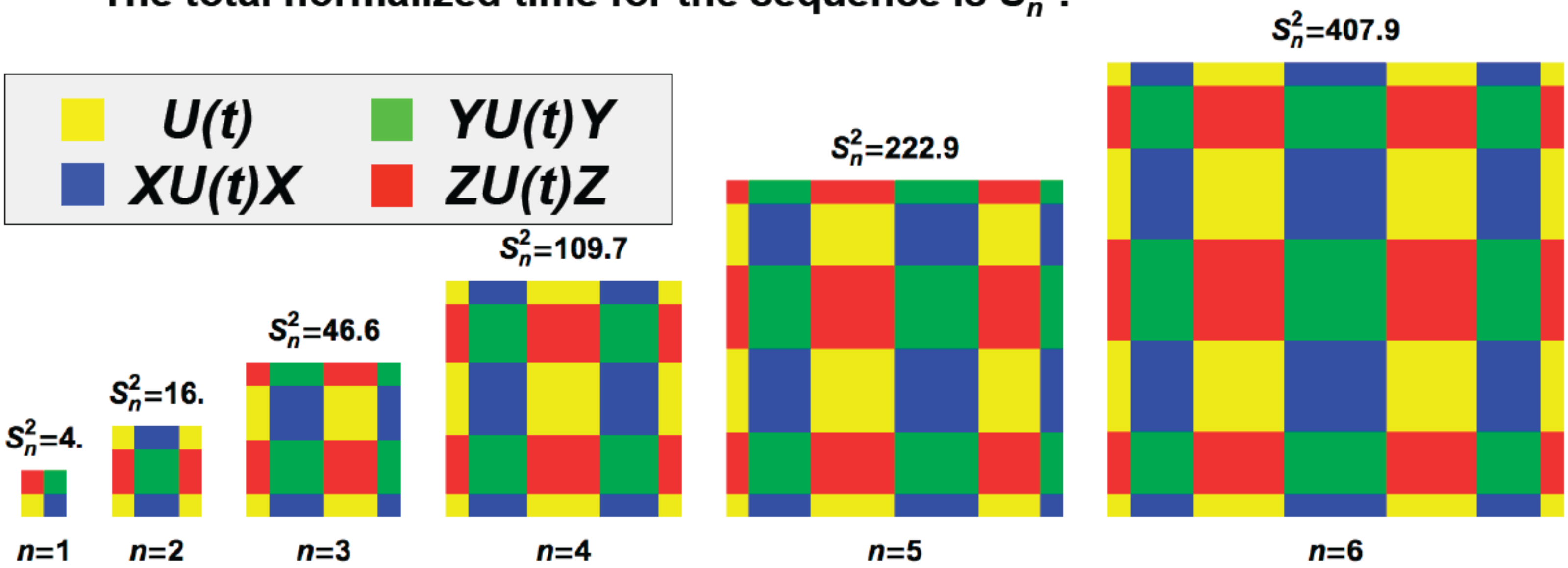
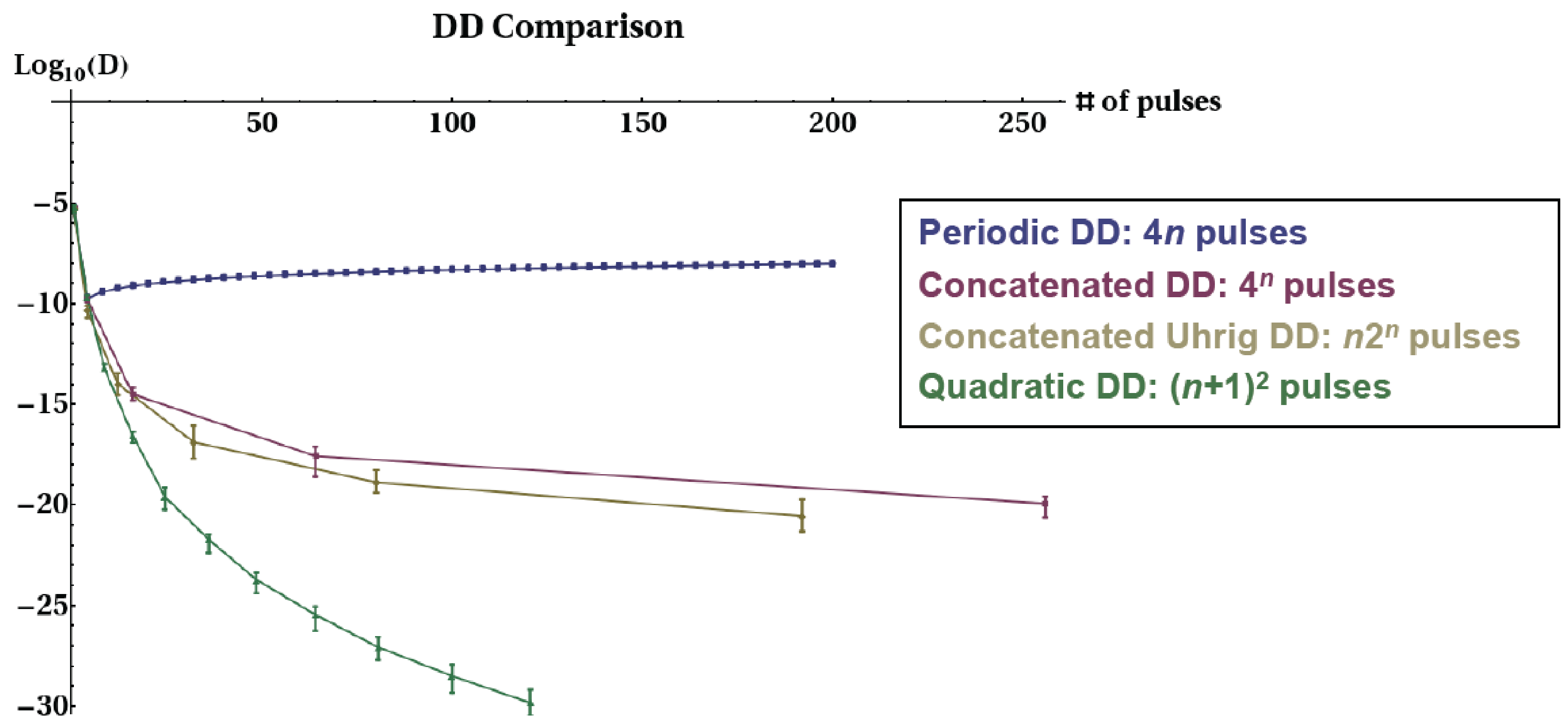
If *X* and *Z* pulses coincide, *Y* pulses are used. $j,k \in \{1,n\}$

- **This can be visualized as an outer product of Uhrig sequences:**
  - **Colors correspond to interval conjugation type (*I*,*X*,*Y*,*Z*)**
  - **Areas correspond to interval duration**
  - **Final pulse sequence is read off row-by-row down the 2D array**
- **The total normalized time for the sequence is $S_n{}^2$.**



For every value of n, the first $\sqrt{n}$ terms in the Dyson series are removed

Comparison of DD Sequences



$$H = \beta(I \otimes B_I) + J(X \otimes B_X + Y \otimes B_Y + Z \otimes B_Z)$$

$$B_\alpha = \sum_{i \neq j} \sum_{k,l=0}^{3} r_{kl}^{\alpha} (\sigma_{i,k} \otimes \sigma_{j,l})$$

$J\tau = \beta\tau = 10^{-6}$. The shortest pulse interval τ is the same in all simulations

Summary

- Symmetry as a unifying principle for both passive and active error prevention/correction strategies
- A comprehensive strategy can take advantage of a layered approach:

Hybrid Q. Error Correction: The Big Picture

DFS encoding

QECC

Hybrid Q. Error Correction: The Big Picture

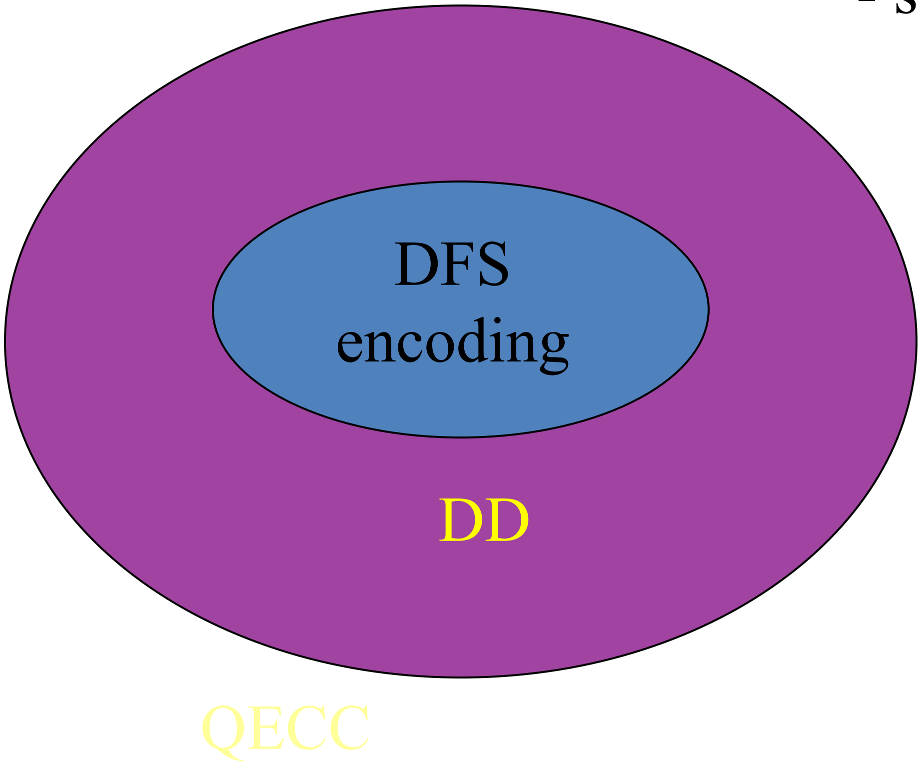
- symmetry not for free…

DFS encoding

QECC

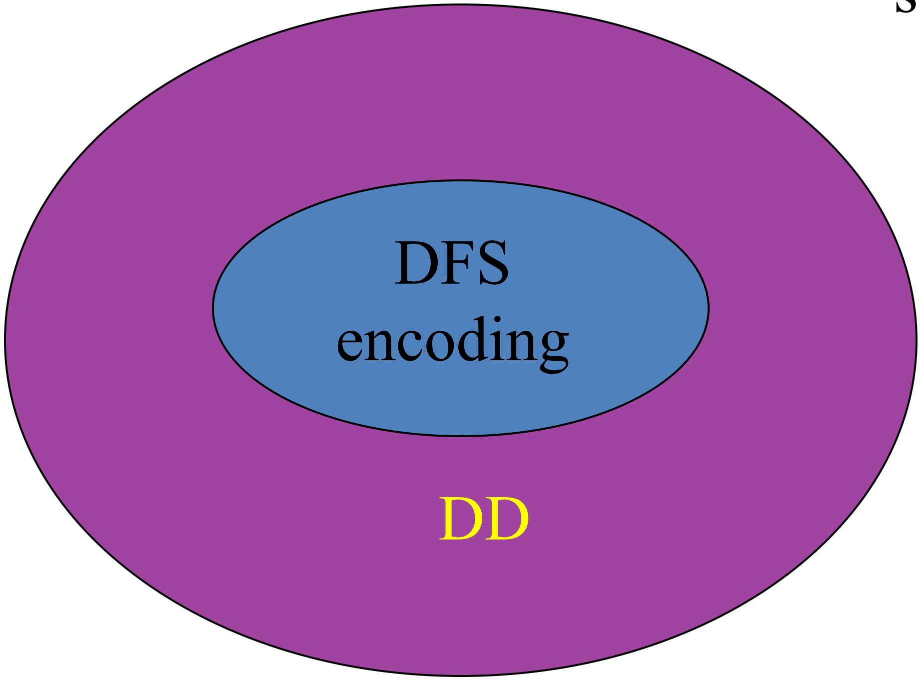
Hybrid Q. Error Correction: The Big Picture

- symmetry not for free…



Hybrid Q. Error Correction: The Big Picture

- symmetry not for free…

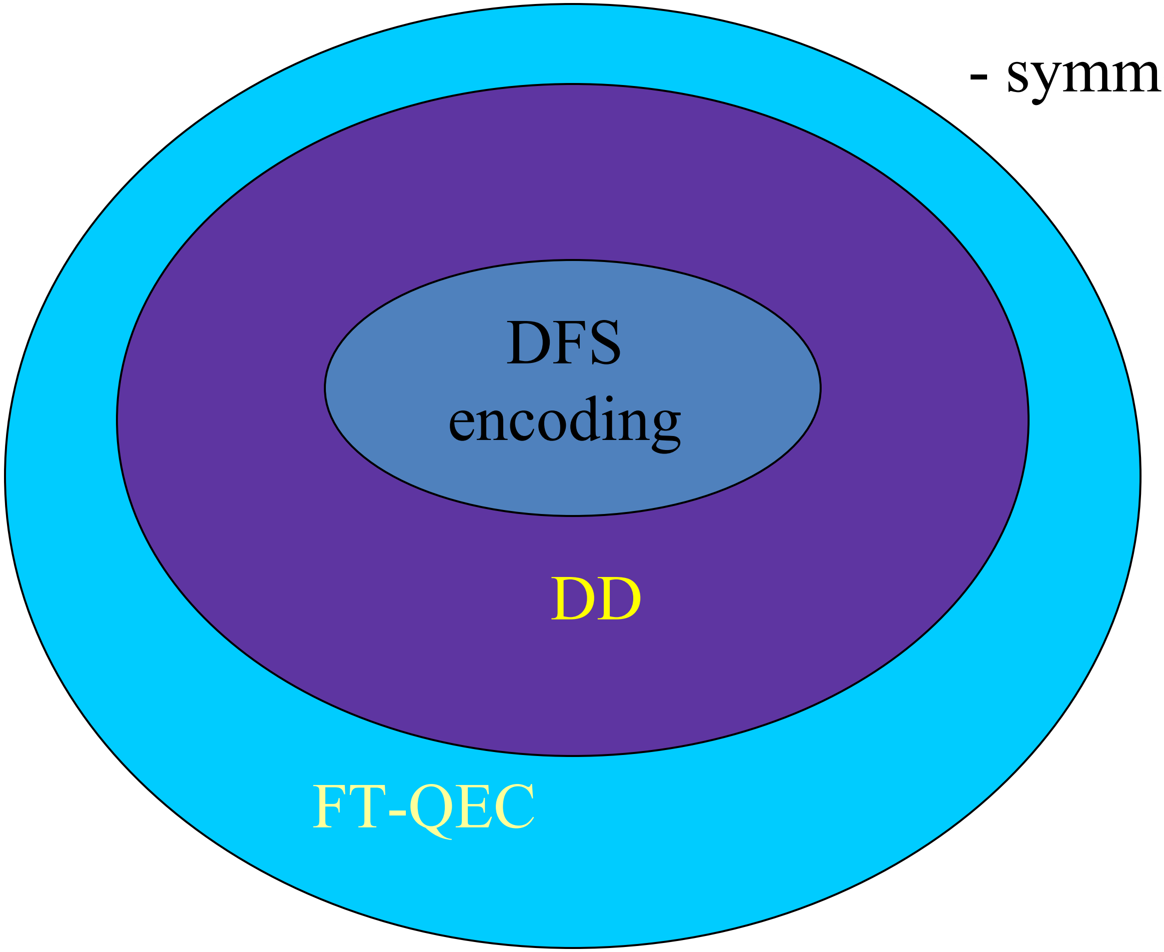


QECC

-pulse errors,

Markovian effects

Hybrid Q. Error Correction: The Big Picture



- symmetry not for free…

-pulse errors,

Markovian effects

Open Questions

- What is the optimal hybrid strategy?
- Is the fault tolerance threshold better for a hybrid strategy?

see: H.-K. Ng, D.A.L., and J. Preskill, "Combining dynamical decoupling with fault-tolerant quantum computation", arXiv:0911:3202