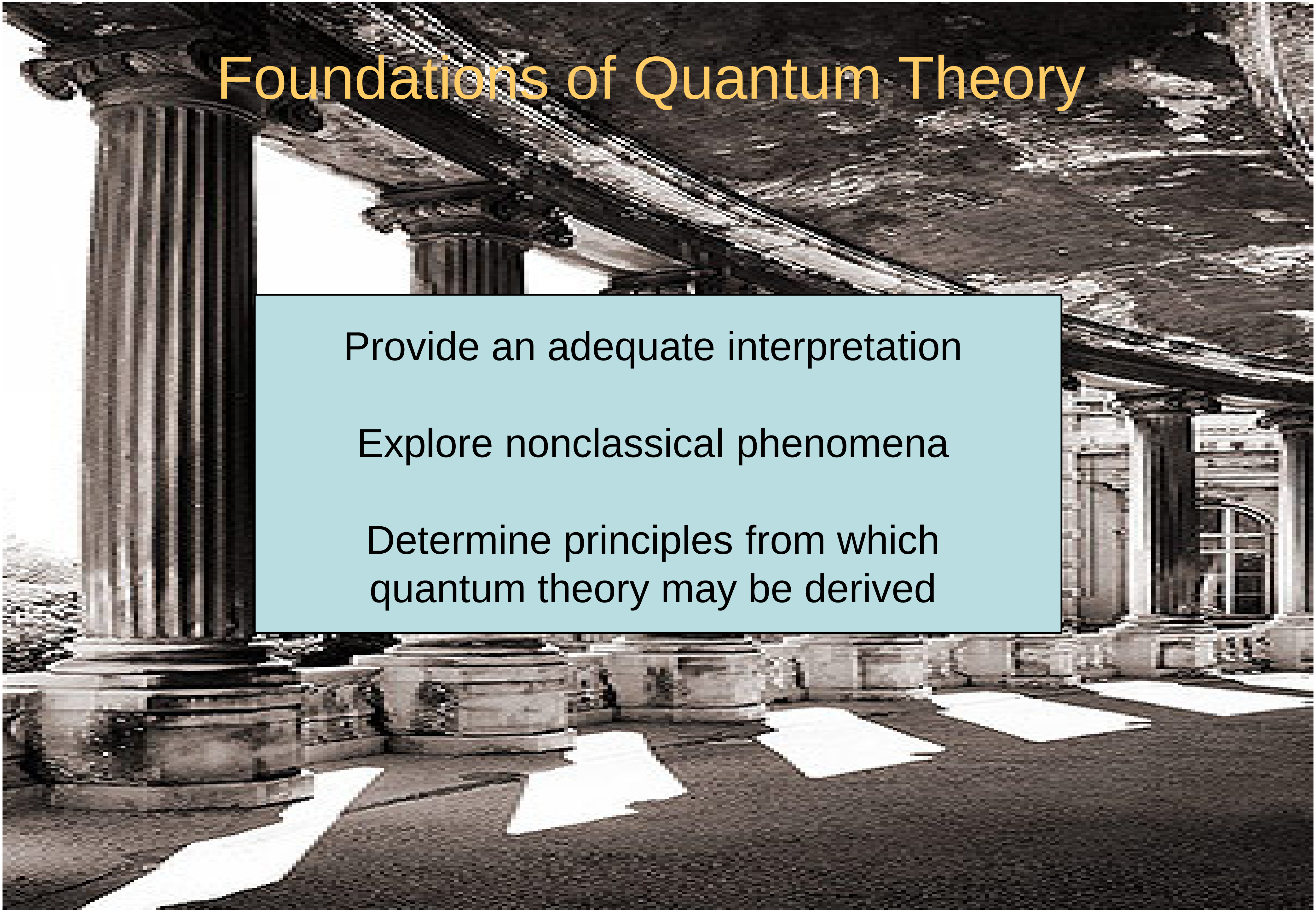
Foundations of Quantum Theory



Provide an adequate interpretation

Explore nonclassical phenomena

Determine principles from which quantum theory may be derived

What's the problem?

“Orthodox” postulates of quantum theory

Representational completeness of ψ. The rays of Hilbert space correspond one-to-one with the physical states of the system.

Measurement. If the Hermitian operator *A* with spectral projectors $\{P_k\}$ is measured, the probability of outcome *k* is $\langle\psi|P_k|\psi\rangle$. These probabilities are objective -- indeterminism.

Evolution of isolated systems. It is unitary, $|\psi\rangle \to U|\psi\rangle = e^{-\frac{i}{\hbar}Ht}|\psi\rangle$ therefore deterministic and continuous.

Evolution of systems undergoing measurement. If Hermitian operator *A* with spectral projectors $\{P_k\}$ is measured and outcome *k* is obtained, the physical state of the system changes discontinuously,

$$|\psi\rangle \to |\psi_k\rangle = \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}$$

First problem: the term “measurement” is not defined in terms of the more primitive “physical states of systems”. Isn’t a measurement just another kind of physical interaction?

Two strategies:

(1) Realist strategy: Eliminate measurement as a primitive concept and describe everything in terms of physical states

(2) Operational strategy: Eliminate “the physical state of a system” as a primitive concept and describe everything in terms of operational concepts

“It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? ”

- John Bell

“In a strict sense, quantum theory is a set of rules allowing the computation of probabilities for the outcomes of tests which follow specified preparations.”

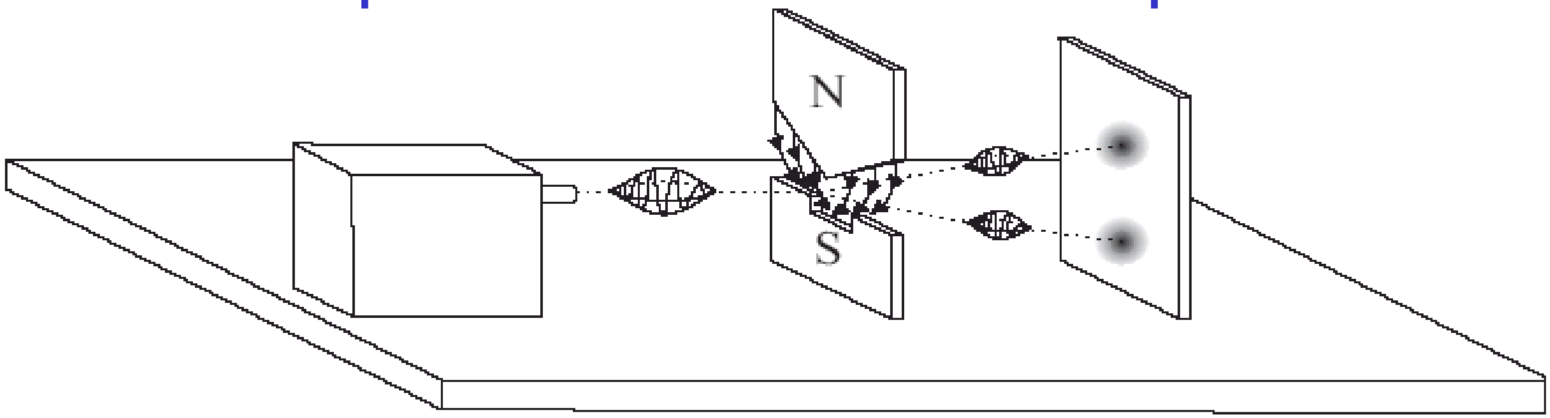
- Asher Peres

The realist strategy

Inconsistencies of the orthodox interpretation

By the collapse postulate (applied to the system)	By unitary evolution postulate (applied to isolated system that includes the apparatus)
Indeterministic and discontinuous evolution	Deterministic and continuous evolution
Determinate properties	Indeterminate properties

The quantum measurement problem



If the measurement apparatus is treated externally

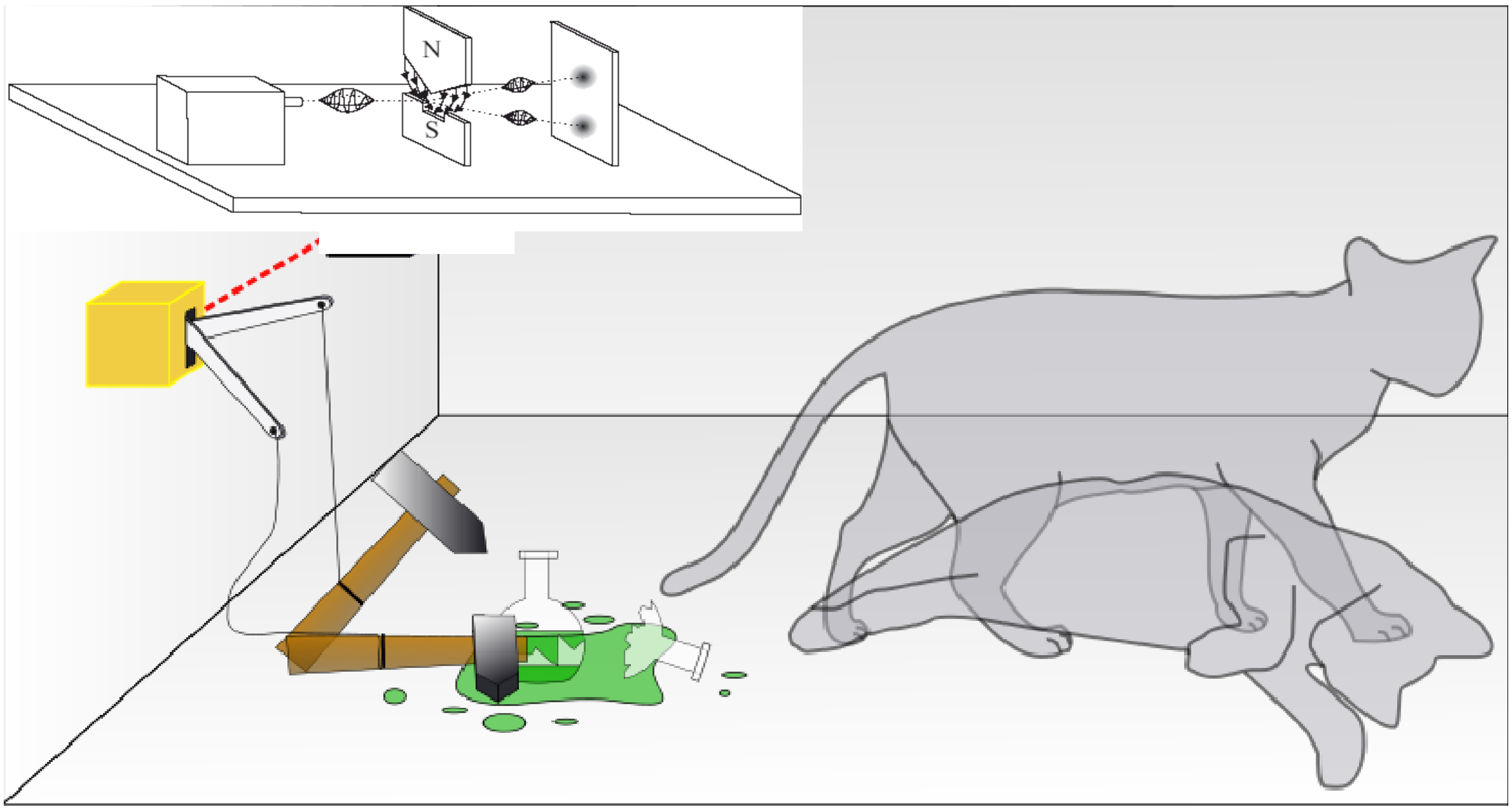
$$a|\uparrow\rangle + b|\downarrow\rangle \;\; \to |\uparrow\rangle \text{ with probability } |a|^2$$
$$\to |\downarrow\rangle \text{ with probability } |b|^2$$

If the measurement apparatus is treated internally

$$|\uparrow\rangle \otimes |\text{``ready''}\rangle \;\; \to U(|\uparrow\rangle \otimes |\text{``ready''}\rangle) = |\uparrow\rangle \otimes |\text{``up''}\rangle$$
$$|\downarrow\rangle \otimes |\text{``ready''}\rangle \;\; \to U(|\downarrow\rangle \otimes |\text{``ready''}\rangle) = |\downarrow\rangle \otimes |\text{``down''}\rangle$$

U is a linear operator $U(a|\psi\rangle + b|\phi\rangle) = aU|\psi\rangle + bU|\phi\rangle$

$$(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |\text{``ready''}\rangle \to U[a|\uparrow\rangle \otimes |\text{``ready''}\rangle + b|\downarrow\rangle \otimes |\text{``ready''}\rangle]$$
$$= a|\uparrow\rangle \otimes |\text{``up''}\rangle + b|\downarrow\rangle \otimes |\text{``down''}\rangle$$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False starts on the measurement problem

- Interpret coherent superposition as disjunction

$$a|\uparrow\rangle \otimes |\text{“up”}\rangle + b|\downarrow\rangle \otimes |\text{“down”}\rangle$$

Means either $|\uparrow\rangle \otimes |\text{“up”}\rangle$

or $|\downarrow\rangle \otimes |\text{“down”}\rangle$

with probabilities $|a|^2$ and $|b|^2$ respectively

This is a denial of the representational completeness of ψ

False starts on the measurement problem

- Interpret the reduced density operator as a proper mixture

$$a|\uparrow\rangle \otimes |\text{“up”}\rangle + b|\downarrow\rangle \otimes |\text{“down”}\rangle$$

$$\rho = |a|^2 |\text{“up”}\rangle\langle\text{“up”}| + |b|^2 |\text{“down”}\rangle\langle\text{“down”}|$$

Either contradicts original assignment of entangled state
Or is a denial of the representational completeness of ψ

False starts on the measurement problem

- Appeal to environment-induced decoherence

$$(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |\text{``ready''}\rangle \otimes |E_0\rangle$$

$$\rightarrow (a|\uparrow\rangle \otimes |\text{``up''}\rangle + b|\downarrow\rangle \otimes |\text{``down''}\rangle) \otimes |E_0\rangle$$

$$\rightarrow a|\uparrow\rangle \otimes |\text{``up''}\rangle \otimes |E_1\rangle + b|\downarrow\rangle \otimes |\text{``down''}\rangle \otimes |E_2\rangle$$

$$\rho = |a|^2 |\text{``up''}\rangle\langle\text{``up''}| + |b|^2 |\text{``down''}\rangle\langle\text{``down''}|$$

This doesn't help

False starts on the measurement problem

- Appeal to differences in the state of the apparatus

$$(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |\text{“ready(1)”}\rangle \rightarrow |\uparrow\rangle \otimes |\text{“up”}\rangle$$

$$(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |\text{“ready(2)”}\rangle \rightarrow |\downarrow\rangle \otimes |\text{“down”}\rangle$$

But for the interaction to be considered a measurement, we require

$$|\uparrow\rangle \otimes |\text{“ready(1)”}\rangle \rightarrow |\uparrow\rangle \otimes |\text{“up”}\rangle$$

$$|\downarrow\rangle \otimes |\text{“ready(1)”}\rangle \rightarrow |\downarrow\rangle \otimes |\text{“down”}\rangle$$

And by linearity

$$(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |\text{“ready(1)”}\rangle \rightarrow a|\uparrow\rangle \otimes |\text{“up”}\rangle + b|\downarrow\rangle \otimes |\text{“down”}\rangle$$

The postulated evolution does not correspond to a proper measurement

Responses to the measurement problem

1. Deny universality of quantum dynamics
   - Quantum-classical hybrid models
   - Collapse models

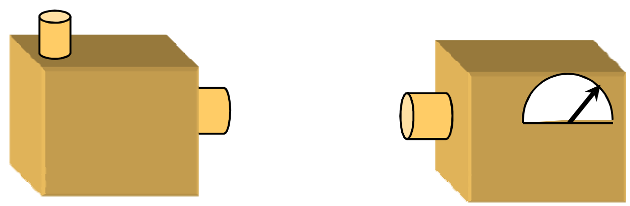
2. Deny representational completeness of ψ
   - ψ-ontic hidden variable models (e.g. Bohmian mechanics)
   - ψ-epistemic hidden variable models

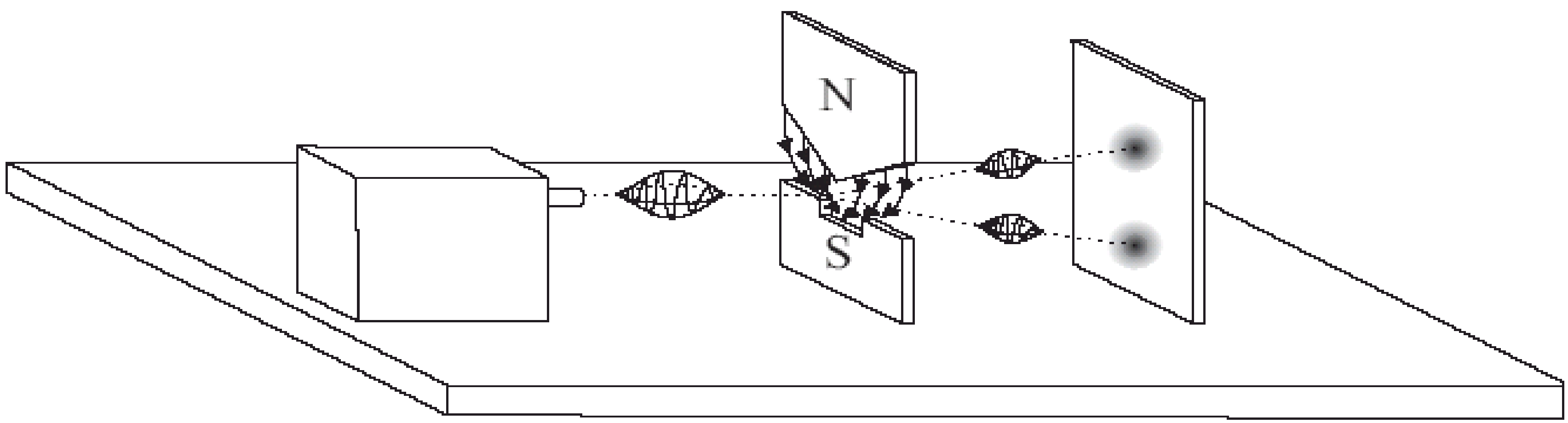
3. Deny that there is a unique outcome
   - Everett's relative state interpretation (many worlds)

4. Deny some aspect of classical logic or classical probability theory
   - Quantum logic and quantum Bayesianism

5. Deny some other feature of the realist framework?

The operational strategy

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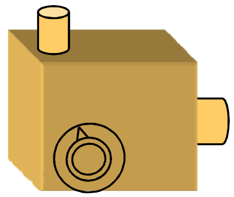
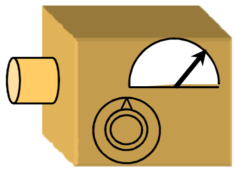
Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

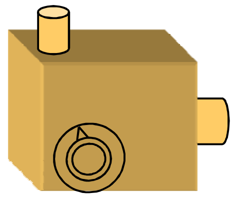
Hermitian operator

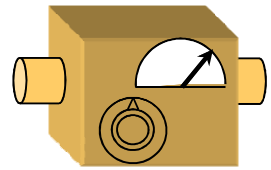
A

$A = \sum_k a_k P_k$

$$Pr(k|\mathrm{P},\mathrm{M}) = \langle\psi|P_k|\psi\rangle$$

Operational Quantum Mechanics





Preparation

P

Measurement

M

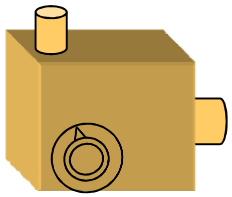
Effective preparation

P_k

Update map

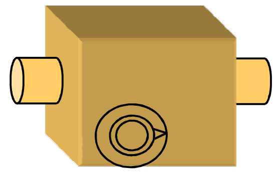
$$|\psi\rangle \to |\psi_k\rangle = \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}$$

Operational Quantum Mechanics



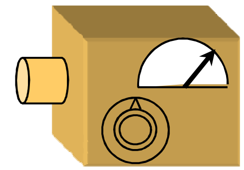
Preparation

P



Transformation

T



Measurement

M

Vector

$|\psi\rangle$

Unitary map

U

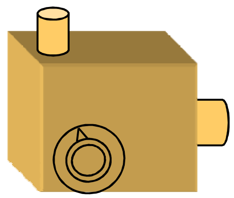
Hermitian operator

A

$A = \sum_k a_k P_k$

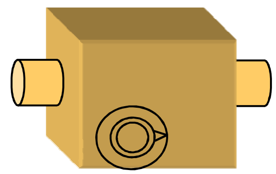
$$Pr(k|\mathrm{P},\mathrm{T},\mathrm{M}) = \langle\psi|U^\dagger P_k U|\psi\rangle$$

Operational Quantum Mechanics



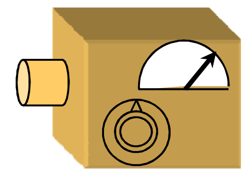
Preparation

P



Transformation

T



Measurement

M

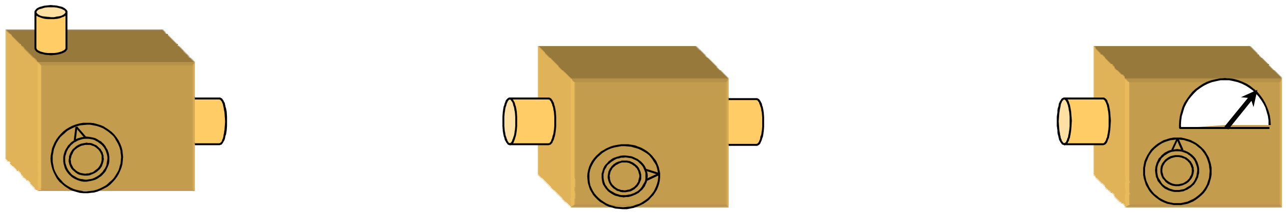
Effective preparation

P′

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

$$Pr(k|\mathrm{P}', \mathrm{M}) = \langle\psi'|P_k|\psi'\rangle = \langle\psi|U^\dagger P_k U|\psi\rangle$$

Operational Quantum Mechanics



Preparation P

Transformation T

Measurement M

Effective Measurement M′

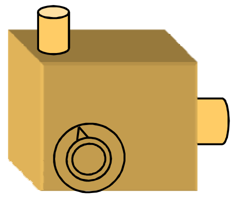
$$A \rightarrow A' = U^\dagger A U$$

$$A' = \sum_k a_k P'_k$$

$$Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|P'_k|\psi\rangle = \langle\psi|U^\dagger P_k U|\psi\rangle$$

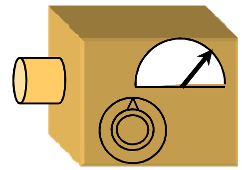
The real formalism of operational quantum theory

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

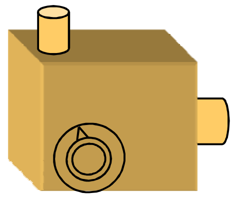
ρ

Position operator valued measure (POVM)

$\{E_k\}$

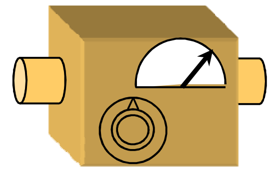
$$Pr(k|\mathsf{P},\mathsf{M}) = \mathsf{Tr}(\rho E_k)$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Effective preparation

P_k

Update map

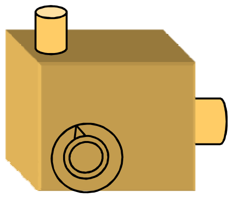
$$\rho \longrightarrow \rho_k = \frac{\mathcal{T}_k(\rho)}{\mathrm{Tr}[\mathcal{T}_k(\rho)]}$$

where $\mathcal{T}_k^\dagger(I) = E_k$

Trace-decreasing completely positive linear map

$\mathcal{T}_k$

Operational Quantum Mechanics

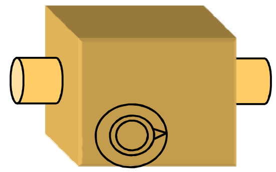


Preparation

P

Density operator

ρ

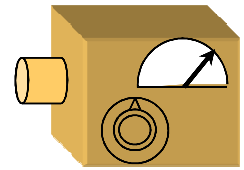


Transformation

T

Trace-preserving completely positive linear map (CP map)

$\mathcal{T}$



Measurement

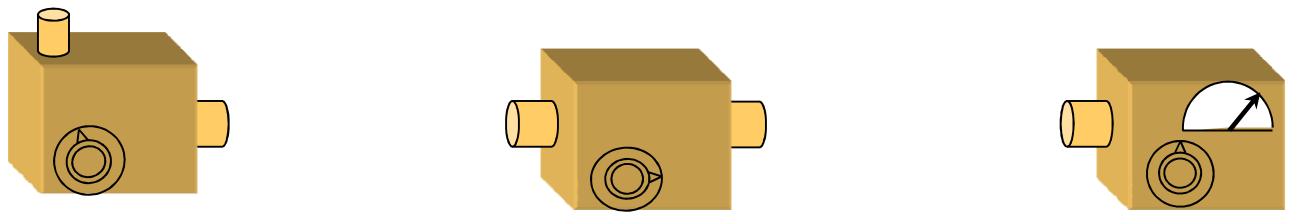
M

Positive operator-valued measure (POVM)

$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \mathrm{Tr}[E_k\mathcal{T}(\rho)]$$

Operational Quantum Mechanics



Preparation | Transformation | Measurement

P | T | M

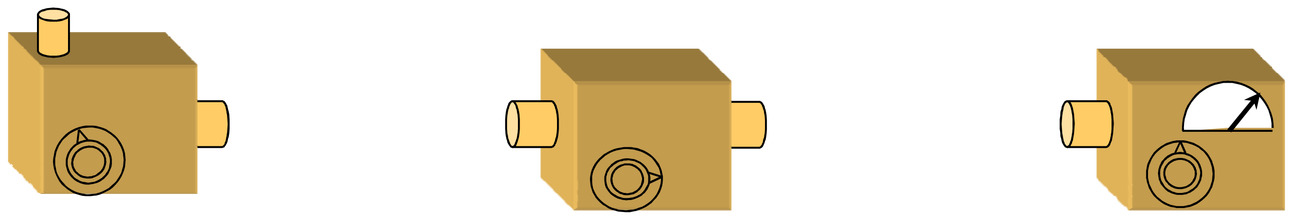
Effective preparation

P'

$$\rho \rightarrow \rho' = \mathsf{T}(\rho)$$

$$Pr(k|\mathsf{P}',\mathsf{M}) = \mathsf{Tr}(E_k\rho') = \mathsf{Tr}(E_k\mathcal{T}(\rho))$$

Operational Quantum Mechanics



Preparation P

Transformation T

Measurement M

Effective Measurement M′

$$E_k \rightarrow E'_k = \mathcal{T}^{\dagger}(E_k)$$

$$Pr(k|\mathsf{P}', \mathsf{M}) = \mathrm{Tr}(E'_k \rho) = \mathrm{Tr}(\mathcal{T}^{\dagger}(E_k)\rho)$$

Operational postulates of quantum theory

Every preparation P is associated with a density operator ρ

Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $$Pr(k|\mathsf{P},\mathsf{M}) = \mathsf{Tr}(\rho E_k)$$

Every transformation is associated with a trace-preserving completely-positive linear map $$\rho\ !\ \ \rho' = \mathsf{T}\left(\rho\right)$$

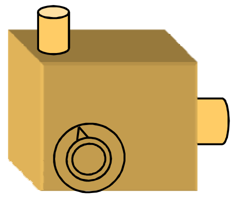
Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that

$$\rho \rightarrow \rho_k = \frac{\mathcal{T}_k(\rho)}{\mathsf{Tr}[\mathcal{T}_k(\rho)]}$$

No mention of “physical states” or their evolution

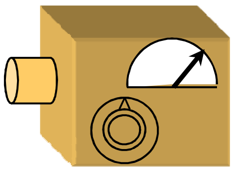
How density operators and POVMs arise in the operational approach

Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

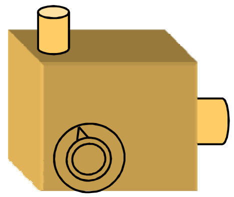
Hermitian operator

A

$A = \sum_k a_k \Pi_k$

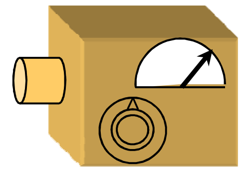
$$Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|\Pi_k|\psi\rangle$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

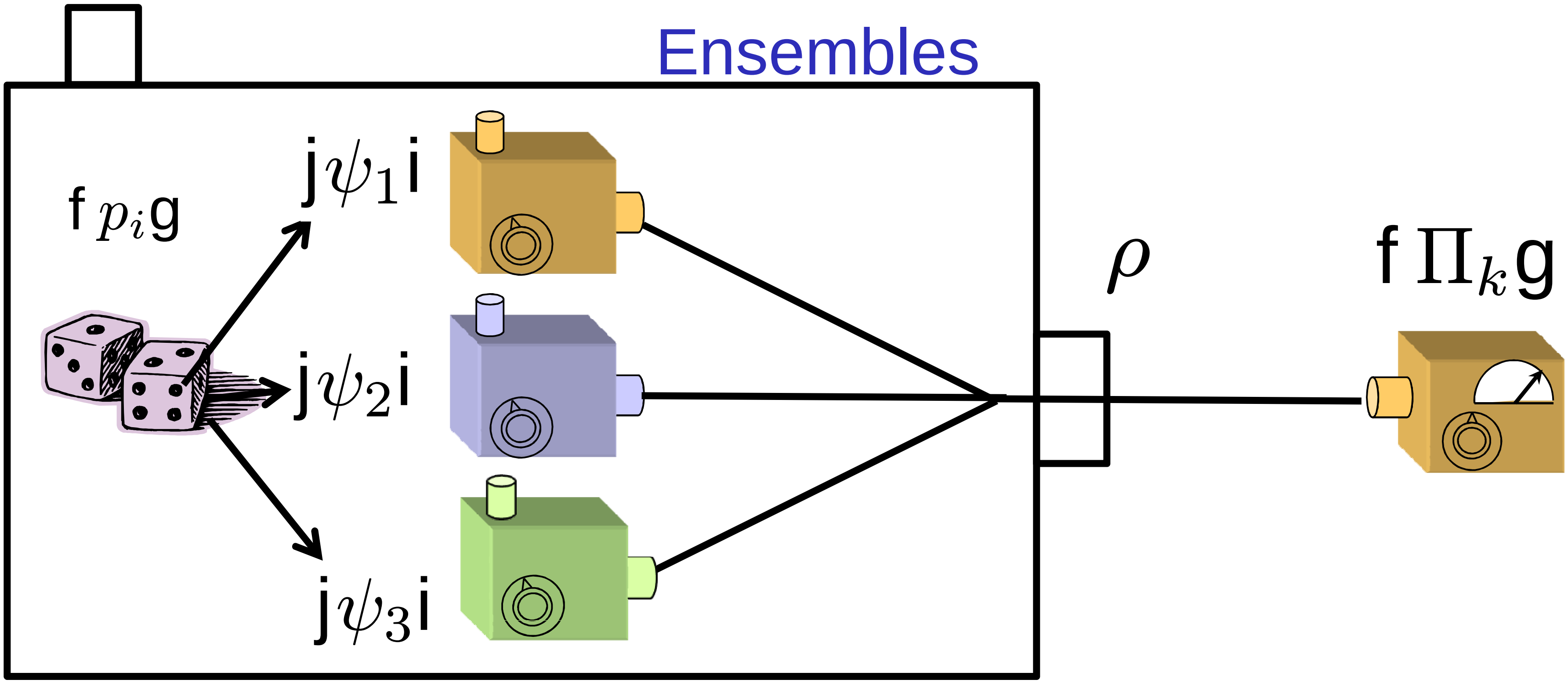
ρ

Hermitian operator

A

$A = \sum_k a_k \Pi_k$

$$Pr(k|\mathsf{P}, \mathsf{M}) = \mathrm{Tr}(\rho \Pi_k)$$



$$
\begin{aligned}
p(k) &= \sum_i p(k|i)\,p(i) \\
&= \sum_i \langle\psi_i|\Pi_k|\psi_i\rangle p_i \\
&= \sum_i \mathrm{Tr}(\Pi_k|\psi_i\rangle\langle\psi_i|)\,p_i \\
&= \mathrm{Tr}(\Pi_k \sum_i p_i|\psi_i\rangle\langle\psi_i|)
\end{aligned}
$$

$$p(k) = \mathrm{Tr}(\Pi_k \rho)$$

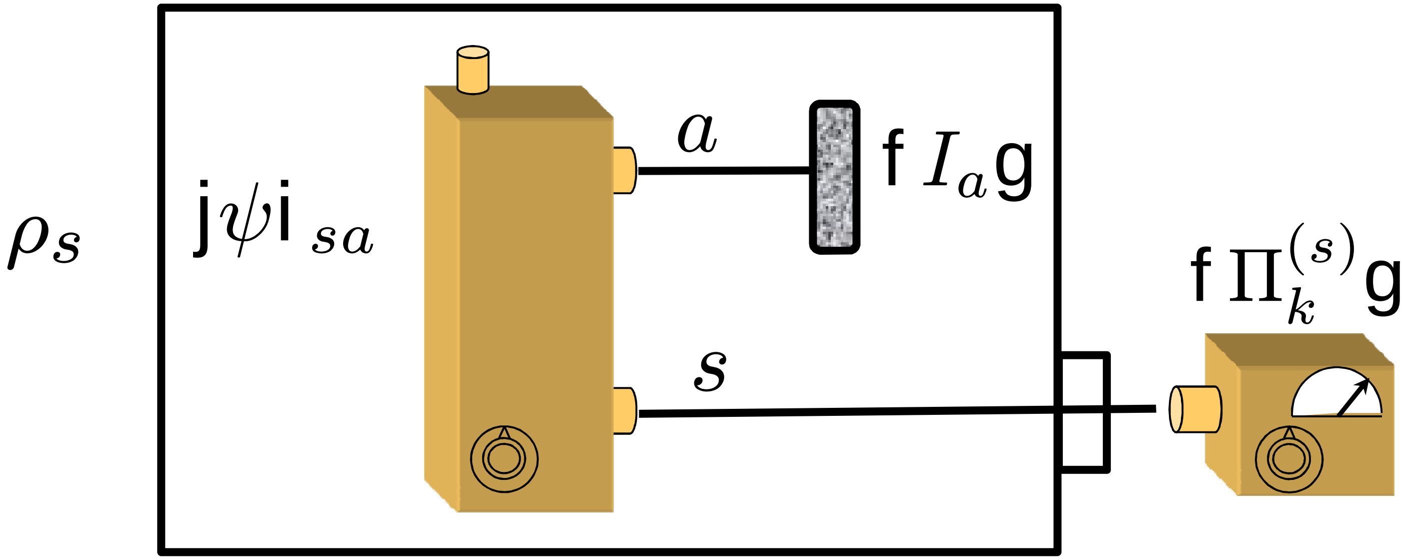
where $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$

"Density operator"

Positive $\langle\psi|\rho|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in H$

Unit trace $\mathrm{Tr}(\rho) = 1$

Reduced density operators



$$p(k) = \mathrm{Tr}_{sa}[(\Pi_k^{(s)} \otimes I_a)\,|\psi\rangle_{sa}\langle\psi|]$$
$$= \mathrm{Tr}_s[\Pi_k^{(s)}(\mathrm{Tr}_a(|\psi\rangle_{sa}\langle\psi|))]$$

$$p(k) = \mathrm{Tr}(\Pi_k^{(s)}\rho_s)$$

where $\rho_s = \mathrm{Tr}_a(|\psi\rangle_{sa}\langle\psi|)$

$$\rho = |\psi\rangle\langle\psi| \quad \leftrightarrow$$ Pure preparation

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad \leftrightarrow$$ Mixed preparation

Multiplicity of convex decompositions

$$\frac{1}{2}I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

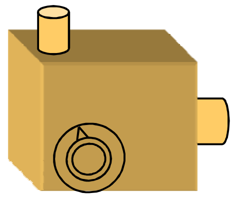
$$\frac{1}{2}I = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

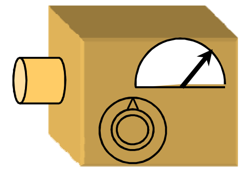
Multiplicity of purifications

$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)]$$

$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)]$$

Operational Quantum Mechanics





Preparation

P

Measurement

M

Density operator

ρ

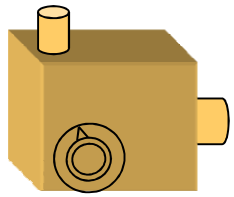
Hermitian operator

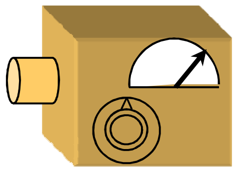
A

$A = \sum_k a_k \Pi_k$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho \Pi_k)$$

Operational Quantum Mechanics





Preparation

P

Measurement

M

Density operator

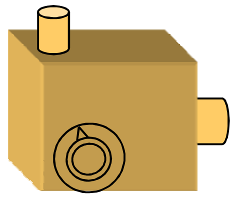
ρ

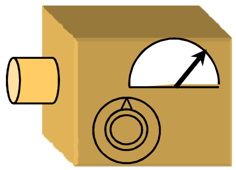
Projection valued measure (PVM)

$\{\Pi_k\}$

$$Pr(k|\mathsf{P}, \mathsf{M}) = \mathrm{Tr}(\rho \Pi_k)$$

Operational Quantum Mechanics





Preparation

P

Measurement

M

Density operator

ρ

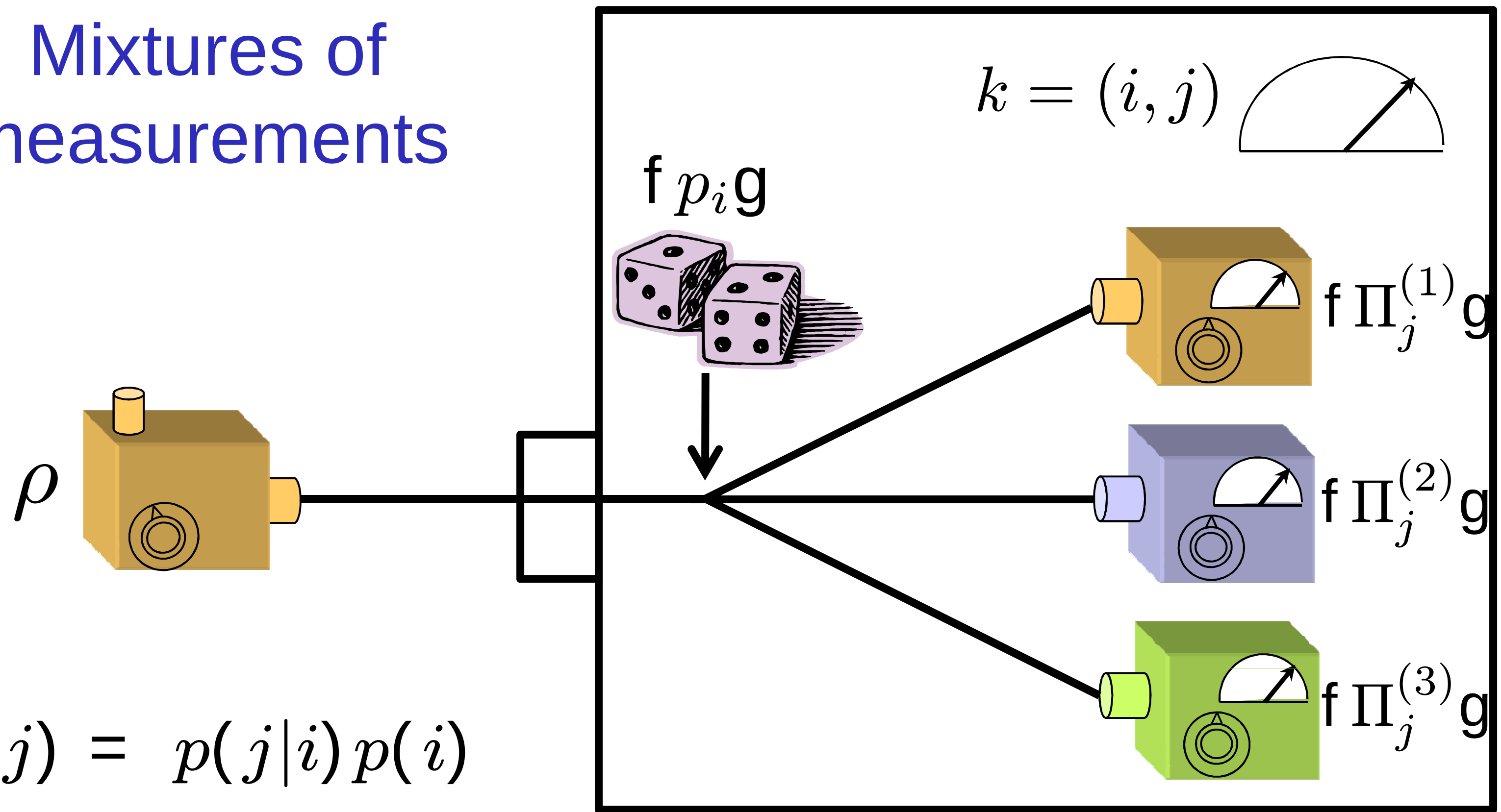
Position operator valued measure (POVM)

$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho E_k)$$

Standard Measurements	**Generalized Measurements**
$\{\Pi_i\}$	$\{E_d\}$
$\langle\psi\|\Pi_i\|\psi\rangle \geq 0\,,\ \forall\|\psi\rangle$	$\langle\psi\|E_d\|\psi\rangle \geq 0\,,\ \forall\|\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_d E_d = I$
$P(i) = \mathrm{tr}(\rho\Pi_i)$	$P(d) = \mathrm{tr}(\rho E_d)$
$\Pi_i\Pi_j = \delta_{ij}\,\Pi_i$	———

Mixtures of measurements



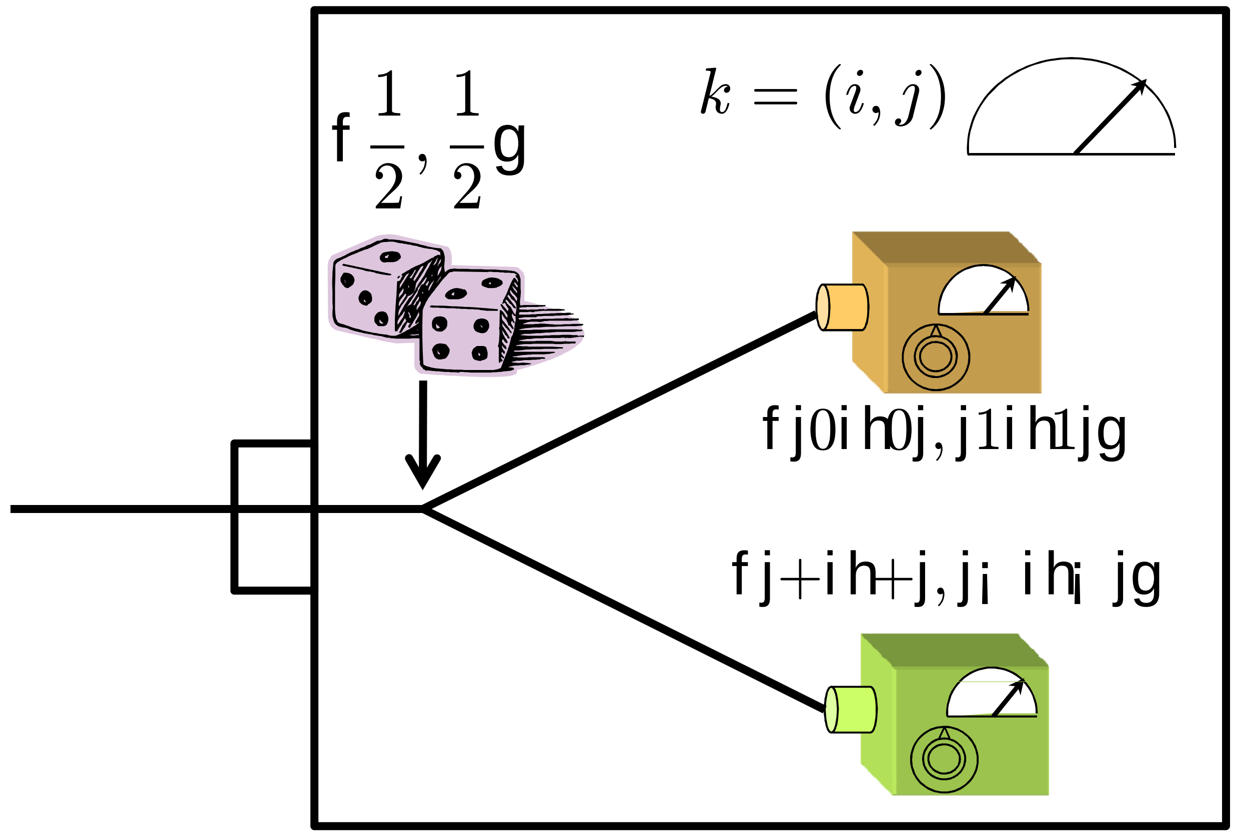
$$
\begin{aligned}
p(i,j) &= p(j|i)\,p(i) \\
&= \mathrm{Tr}(\Pi_j^{(i)}\rho)\,p_i \\
&= \mathrm{Tr}(\underbrace{p_i\Pi_j^{(i)}}_{E_{i,j}}\rho)
\end{aligned}
$$

$$p(k) = \mathrm{Tr}(E_k\rho)$$

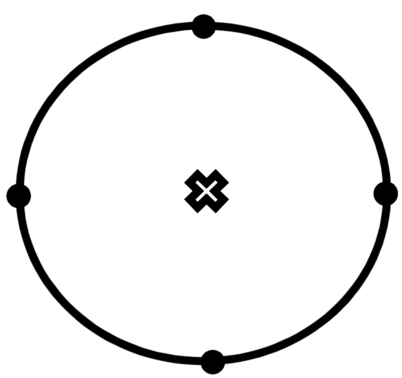
Positive $\langle\psi|E_k|\psi\rangle \geq 0 \quad \forall|\psi\rangle \in H$

Sum to identity $\sum_k E_k = I$

$\{E_k\}$ "Positive operator valued measure (POVM)"



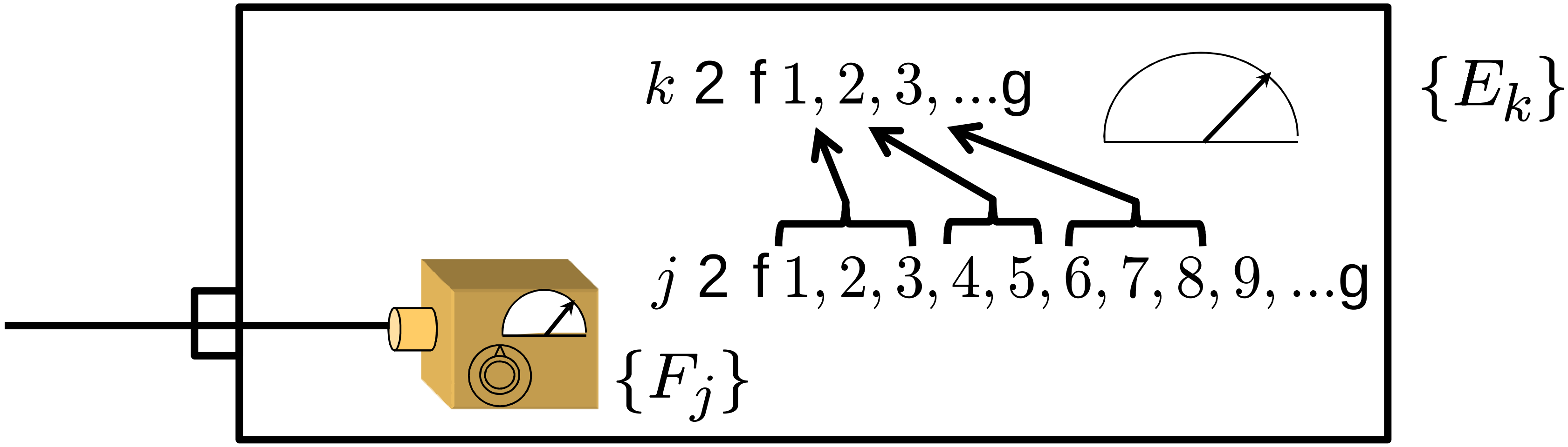
$$\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -|\}$$



Recall

$$\tfrac{1}{4}|0\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1| + \tfrac{1}{4}|+\rangle\langle +| + \tfrac{1}{4}|-\rangle\langle -| = \tfrac{1}{2}I$$

Coarse-graining



$$p(k) = \sum_{j \in S_k} p(j)$$

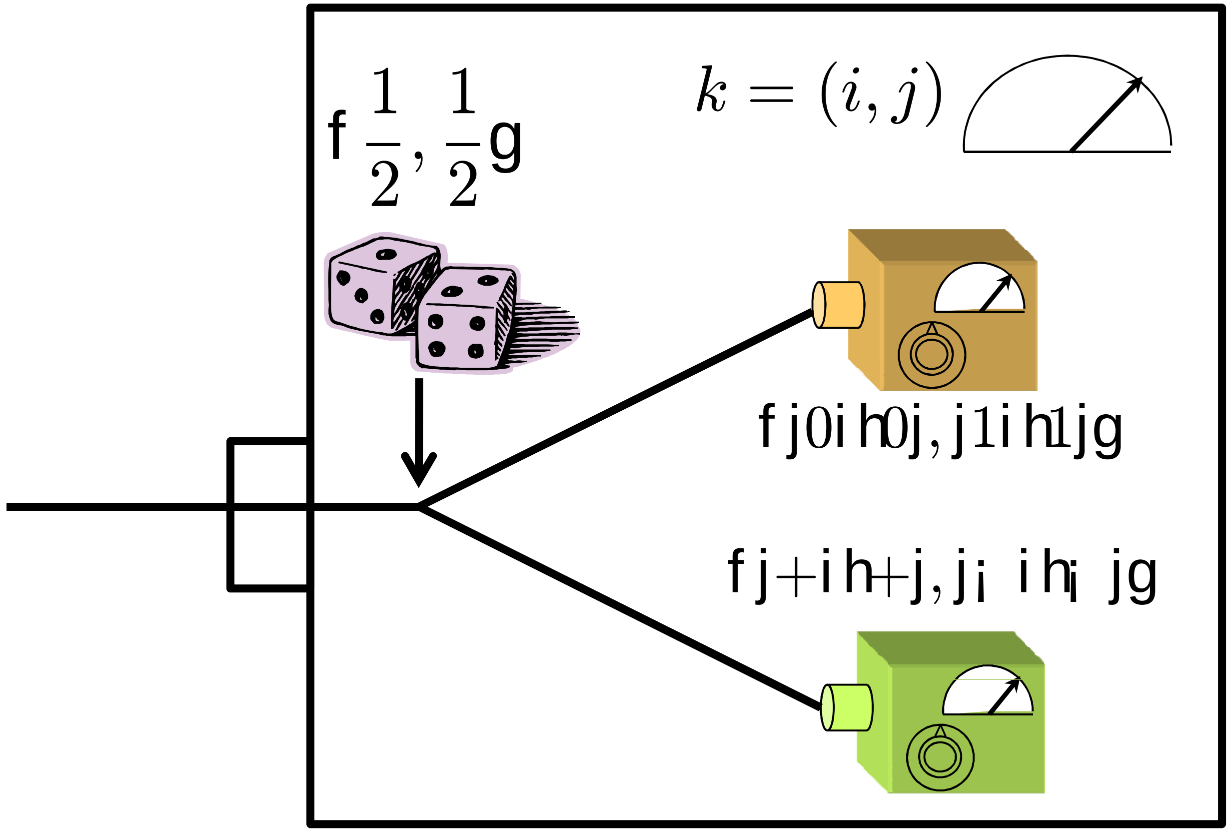
$$\mathrm{Tr}(E_k \rho) = \sum_{j \in S_k} \mathrm{Tr}(F_j \rho) \quad \forall \rho$$

$$= \mathrm{Tr}[(\sum_{j \in S_k} F_j)\,\rho] \quad \forall \rho$$

$$E_k = \sum_{j \in S_k} F_j$$

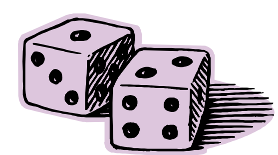
Note: the E_k need not be rank 1

Example

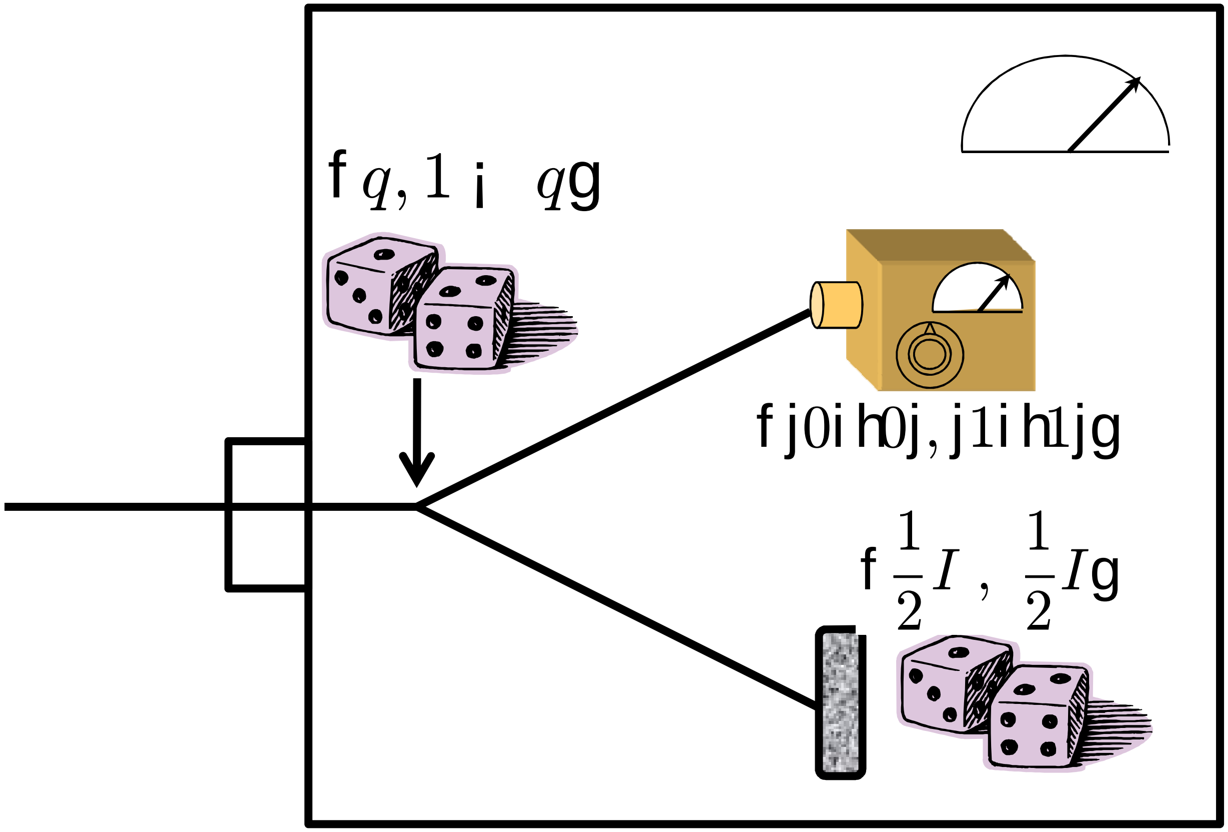


f $\frac{1}{2}$j0i h0j , $\frac{1}{2}$j1i h1j , $\frac{1}{2}$j+i h+j , $\frac{1}{2}$j¡ i h¡ jg

f $\frac{1}{2}$j0i h0j $+$ $\frac{1}{2}$j1i h1j , $\frac{1}{2}$j+i h+j $+$ $\frac{1}{2}$j¡ i h¡ jg $=$ f $\frac{1}{2}I$, $\frac{1}{2}I$g



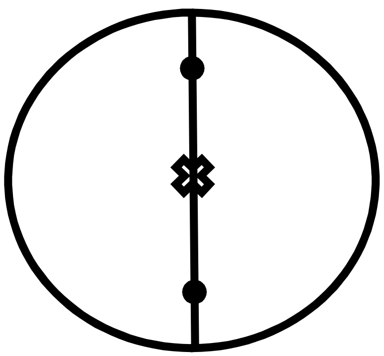
Another example



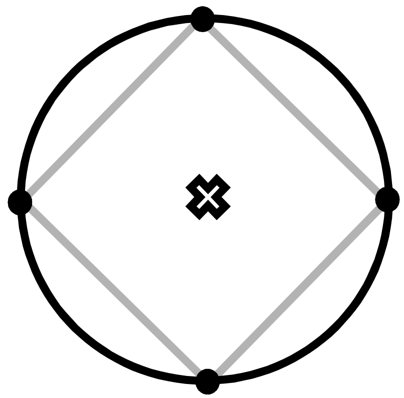
$$\{q|0\rangle\langle 0|,\ q|1\rangle\langle 1|,\ (1-q)\frac{1}{2}I,\ (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I,\ \ q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

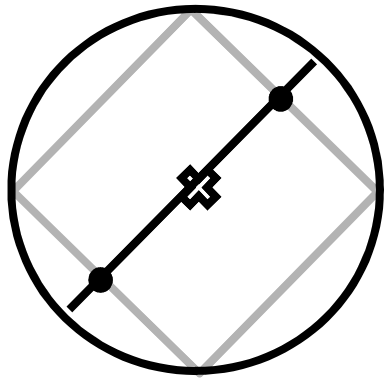
$$= \{\frac{1+q}{2}|0\rangle\langle 0| + \frac{1-q}{2}|1\rangle\langle 1|,\ \frac{1-q}{2}|0\rangle\langle 0| + \frac{1+q}{2}|1\rangle\langle 1|\}$$



Noisy S·z



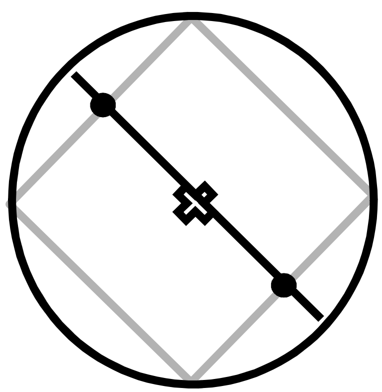
$$\left\{\frac{1}{2}|0\rangle\langle 0|,\ \frac{1}{2}|1\rangle\langle 1|,\ \frac{1}{2}|+\rangle\langle +|,\ \frac{1}{2}|-\rangle\langle -|\right\}$$



$$\left\{\frac{1}{2}|0\rangle\langle 0|,\ \frac{1}{2}|1\rangle\langle 1|,\ \frac{1}{2}|+\rangle\langle +|,\ \frac{1}{2}|-\rangle\langle -|\right\}$$

$$\{E_0, E_1\}$$

Noisy $S\cdot n$



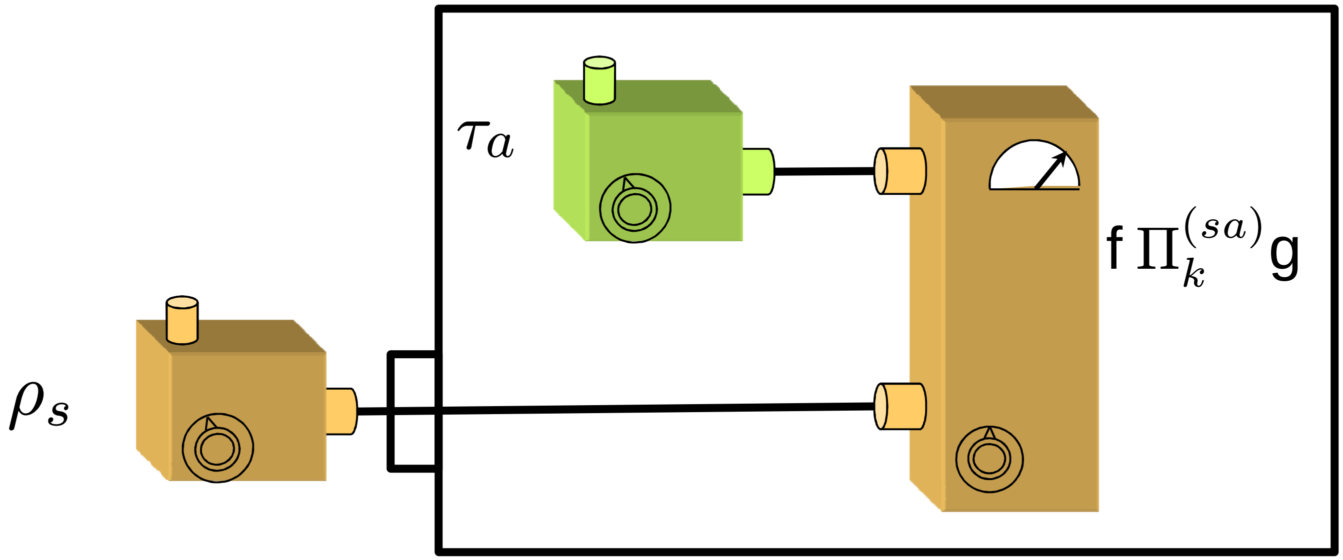
$$\left\{\frac{1}{2}|0\rangle\langle 0|,\ \frac{1}{2}|1\rangle\langle 1|,\ \frac{1}{2}|+\rangle\langle +|,\ \frac{1}{2}|-\rangle\langle -|\right\}$$

$$\{F_0, F_1\}$$

Noisy $S\cdot n^{\perp}$

Note: General conditions for joint measurability of POVMs are not known

Measurement by coupling to an ancilla

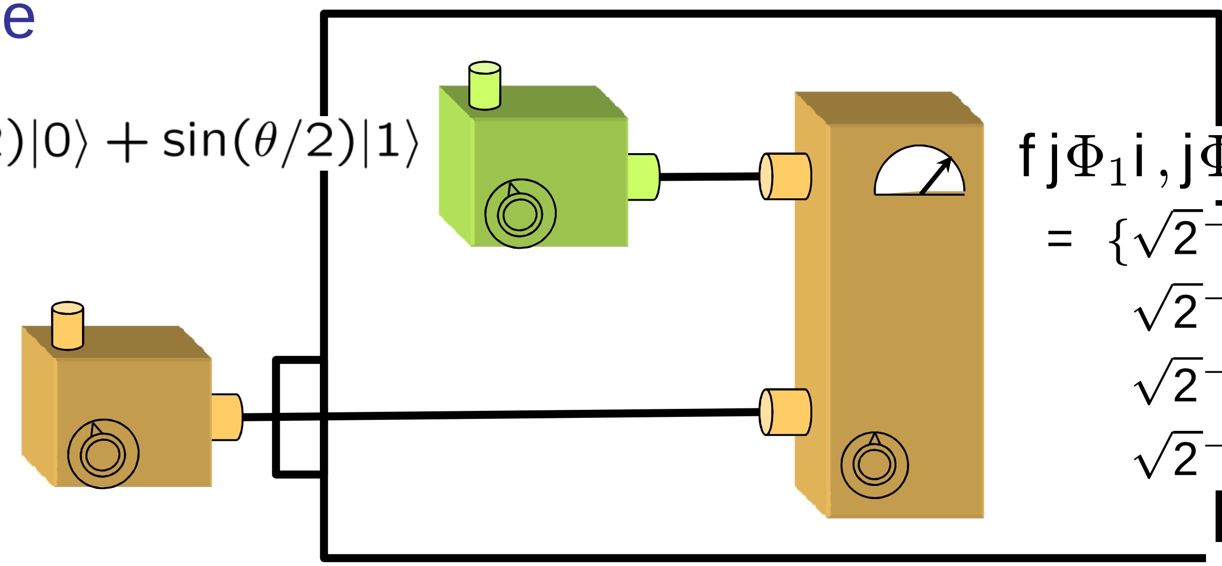


$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
$$= \mathrm{Tr}_s[\underbrace{\mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)}_{E_k^{(s)}}\,\rho_s]$$

Example

$|\theta\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$

ρ_s

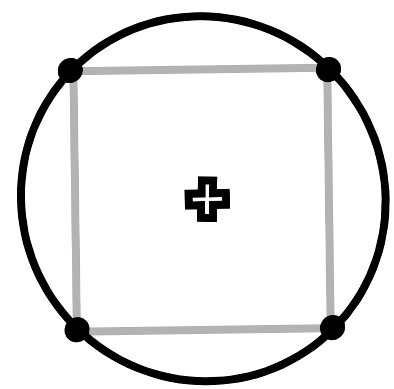


$\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle, |\Phi_4\rangle\}$
$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$
$\sqrt{2}^{-1}(|0\rangle|0\rangle - 1\rangle|1\rangle),$
$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$
$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$

$$E_k^{(s)} = \mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a) = \langle\theta|_a|\Phi_k\rangle_{sa}\langle\Phi_k|_{sa}|\theta\rangle_a$$

$$\langle\theta|_a|\Phi_{1(2)}\rangle_{sa} = \sqrt{2}^{-1}[\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] =$$

$$\langle\theta|_a|\Phi_{3(4)}\rangle_{sa} = \sqrt{2}^{-1}[\sin(\theta/2)|0\rangle_s \pm \cos(\theta/2)|1\rangle_s] =$$

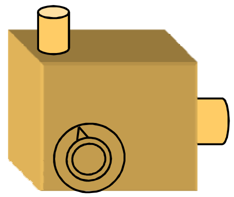


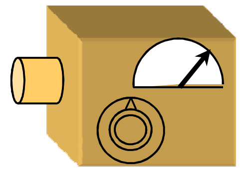
$$\{E_k\} = \{\frac{1}{2}|\theta\rangle\langle\theta|, \frac{1}{2}|-\theta\rangle\langle-\theta|, \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta|, \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta|\}$$

$$\theta = \pi/4$$

Naimark’s theorem: Every POVM can be implemented by coupling to an ancilla and implementing a projective measurement

Operational Quantum Mechanics





Preparation

P

Measurement

M

Density operator

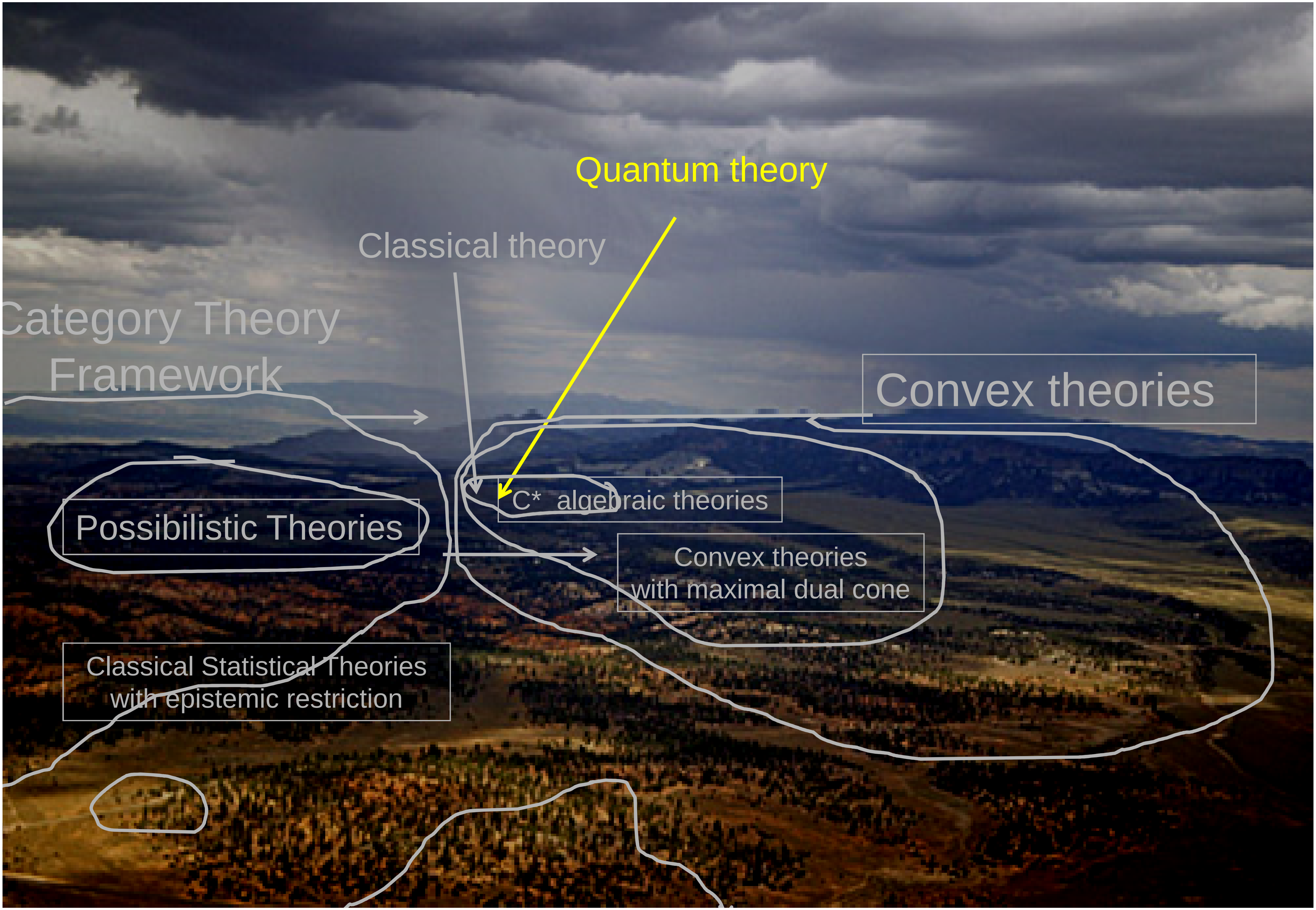
ρ

Position operator valued measure (POVM)

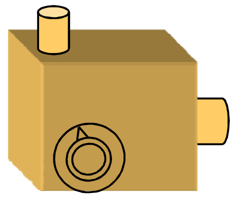
$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho E_k)$$

Towards an operational axiomatization of quantum theory

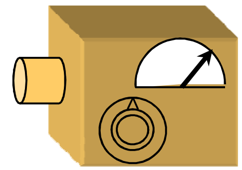

Quantum theory
Classical theory
Category Theory
Framework
Convex theories
C* algebraic theories
Possibilistic Theories
Convex theories
with maximal dual cone
Classical Statistical Theories
with epistemic restriction

A framework for convex operational theories



Preparation

P



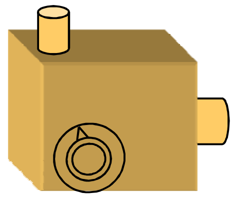
Measurement

M

$$\mathbf{s}_P = \begin{pmatrix} \Pr(1|M,P) \\ \Pr(2|M,P) \\ \Pr(1|M',P) \\ \Pr(2|M',P) \\ \Pr(3|M',P) \\ \vdots \end{pmatrix} \qquad \mathbf{r}_{M,k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

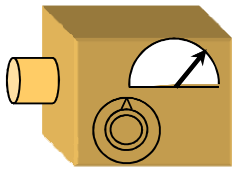
$$Pr(k|\mathrm{P},\mathrm{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$$

A framework for convex operational theories



Preparation

P



Measurement

M

Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)

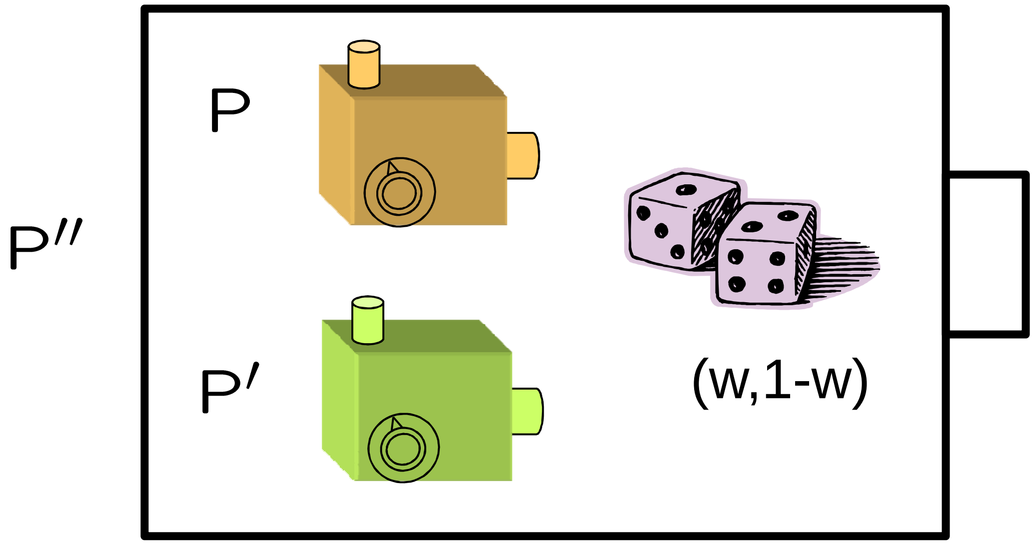
$$\mathbf{s}_P = \begin{pmatrix} \mathrm{Pr}(\text{pass}|M_1, P) \\ \mathrm{Pr}(\text{pass}|M_2, P) \\ \vdots \\ \mathrm{Pr}(\text{pass}|M_K, P) \end{pmatrix}$$

"operational state"

$$Pr(k|\mathrm{P},\mathrm{M}) = f_{M,k}(\mathbf{s}_P)$$

What can we say about *f*?

Operational states form a convex set



$$\forall M, k : p(k|\mathsf{M}, \mathsf{P}'') = w\, p(k|\mathsf{M}, \mathsf{P}) + (1-w)\, p(k|\mathsf{M}, \mathsf{P}')$$

$$f(\mathbf{s}_{\mathsf{P}''}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$$

Also true for fiducial mmts, so $\mathbf{s}_{\mathsf{P}''} = w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}$

Closed under convex combination

$$f(w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$$

Convex linear

Convex linearity implies linearity

If f is convex linear on opt'l states

$$\mathbf{s} = \sum_i w_i \mathbf{s}_i \quad \Rightarrow \quad f(\mathbf{s}) = \sum_i w_i f(\mathbf{s}_i) \qquad 0 \leq w_i \leq 1 \text{ and } \sum_i w_i = 1$$

Then f is linear on opt'l states

$$\mathbf{s} = \sum_i \alpha_i \mathbf{s}_i \quad \Rightarrow \quad f(\mathbf{s}) = \sum_i \alpha_i f(\mathbf{s}_i) \qquad \alpha_i \in \mathbb{R} \text{ and } \sum_i \alpha_i = 1$$

Proof:

$$\mathbf{s} = \sum_i \alpha_i \mathbf{s}_i$$

$$\mathbf{s} + \sum_{j \in I_-} |\alpha_j| \mathbf{s}_j = \sum_{i \in I_+} |\alpha_i| \mathbf{s}_i$$

Note that:

$$1 = \sum_i \alpha_i$$

$$1 + \sum_{j \in I_-} |\alpha_j| \; = \; \sum_{i \in I_+} |\alpha_i| \; \equiv \; \mathcal{N}$$

Thus:

$$\frac{1}{\mathcal{N}} \mathbf{s} + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} \mathbf{s}_j = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} \mathbf{s}_i$$

$$\frac{1}{\mathcal{N}} f(\mathbf{s}) + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} f(\mathbf{s}_j) = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} f(\mathbf{s}_i)$$

$$f(\mathbf{s}) = \sum_i \alpha_i f(\mathbf{s}_i)$$

Convex linearity implies linearity

If f is convex linear on opt'l states

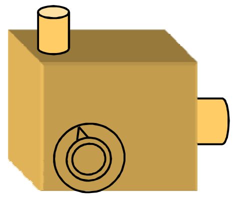
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i)$ $\quad 0 \leq w_i \leq 1$ and $\sum_i w_i = 1$

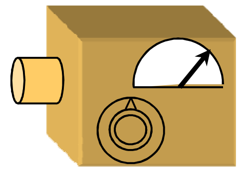
Then f is linear on opt'l states

$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$ $\quad \alpha_i \in \mathbb{R}$ and $\sum_i \alpha_i = 1$

Therefore $\exists \mathbf{r} : f(\mathbf{s}) = \mathbf{r} \cdot \mathbf{s}$

A convex operational theory





Preparation

P

Measurement

M

$\mathbf{s}_P \in S$

"operational states"

$\mathbf{r}_{M,k} \in R$

"operational effects"

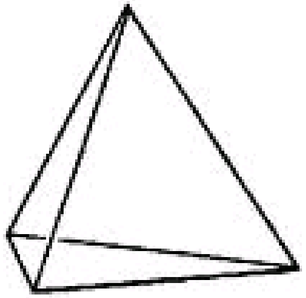
S = Convex set

R = Interval of positive cone

S and R characterize the operational theory!

$$Pr(k|\text{P},\text{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$$

Operational classical theory



$\mathbf{s}$ can be any probability distributions

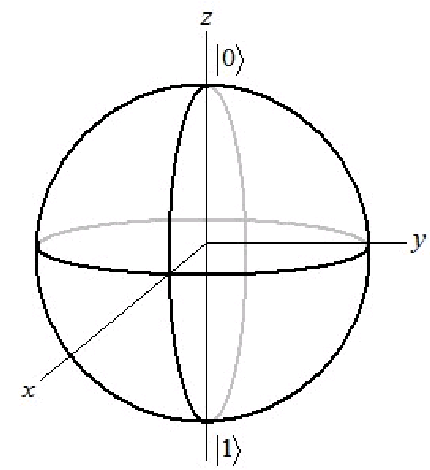
S = a simplex

$\mathbf{r}$ can be any vector of conditional probabilities

R = the unit hypercube

Operational quantum theory

Recall: The Hermitian operators on *H* of dimension *d* form a real Euclidean vector space of dimension d^2



The inner product is $(A, B) = \mathrm{Tr}(AB)$

$$Pr(k|\mathsf{P},\mathsf{M}) = (\rho, E_k) = \mathrm{Tr}(\rho E_k)$$

S = the convex set of positive trace-one operators

$Pr(k|\mathsf{P},\mathsf{M}) \geq 0$ for all P $\rightarrow$ $\langle\psi|E_k|\psi\rangle \geq 0 \;\; \forall|\psi\rangle \in H$

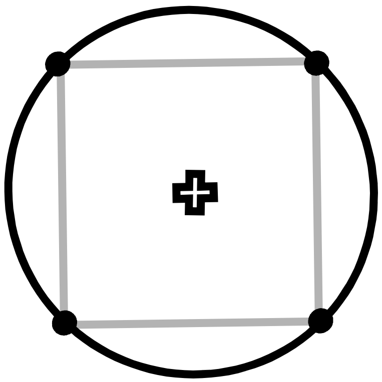
$\sum_k Pr(k|\mathsf{P},\mathsf{M}) = 1$ $\rightarrow$ $\sum_k E_k = I$

R = the set of all positive operators less than identity

An axiomatization must derive S and R

See e.g. L. Hardy, quant-ph/0101012, and J. Barrett, quant-ph/0508211

Is the operational interpretation satisfactory?



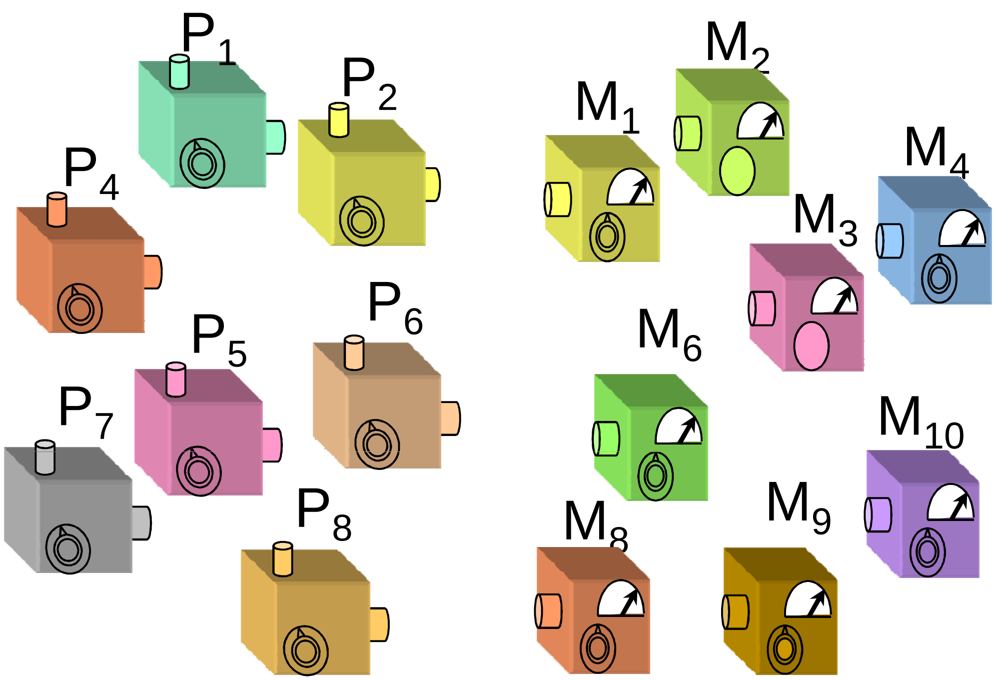
$\theta = \pi/4$

$$\left\{ \frac{1}{2}|\theta\rangle\langle\theta| , \frac{1}{2}|-\theta\rangle\langle-\theta| , \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta| , \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta| \right\}$$

Naimark's theorem: Every POVM can be implemented by coupling to an ancilla and implementing a projective measurement

Two approaches to axiomatization

<u>Operational approach</u>

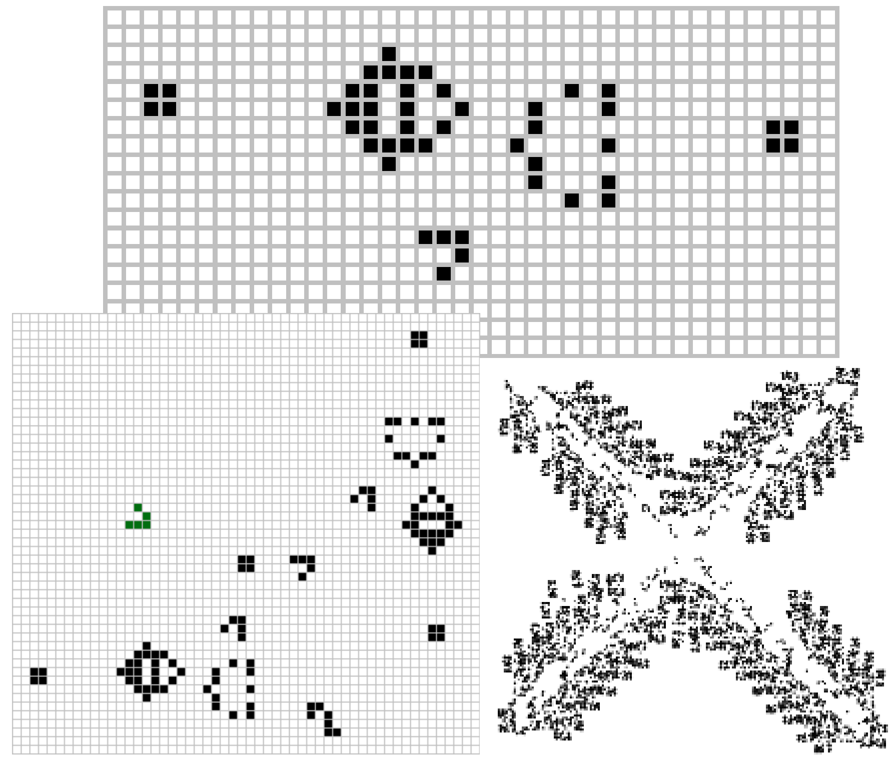


Preparations

Measurements

Axioms are constraints on
experimental statistics p(k|M,P)

<u>Ontological approach</u>



Axioms are constraints on the
ontology and its dynamics