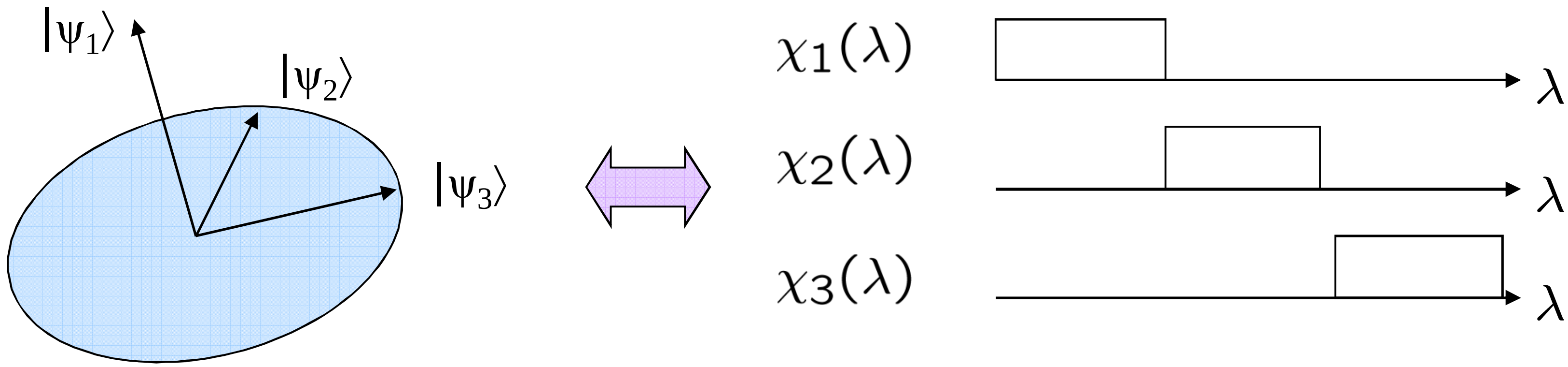
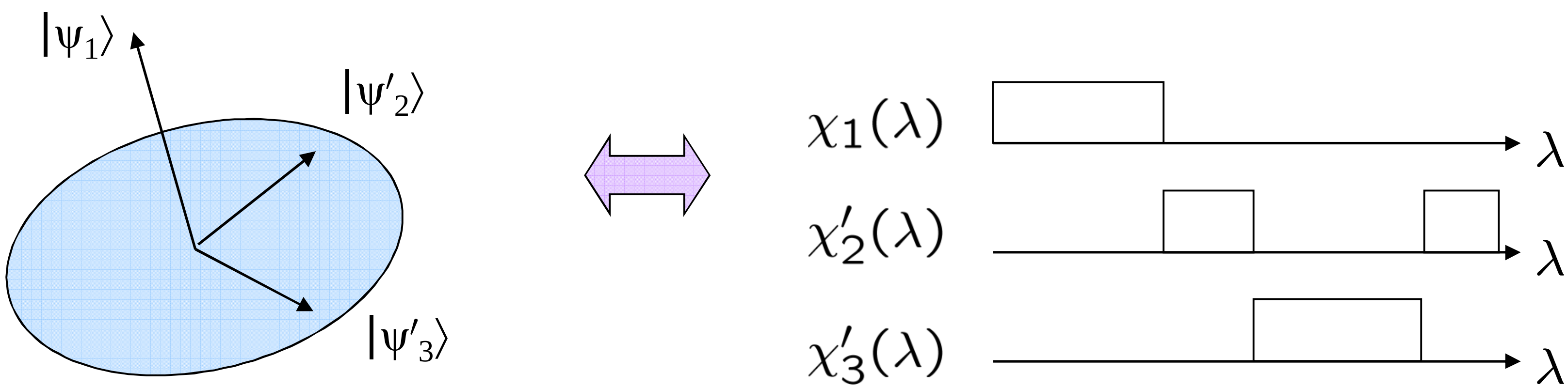
# The traditional notion of noncontextuality in quantum theory

# Traditional notion of noncontextuality

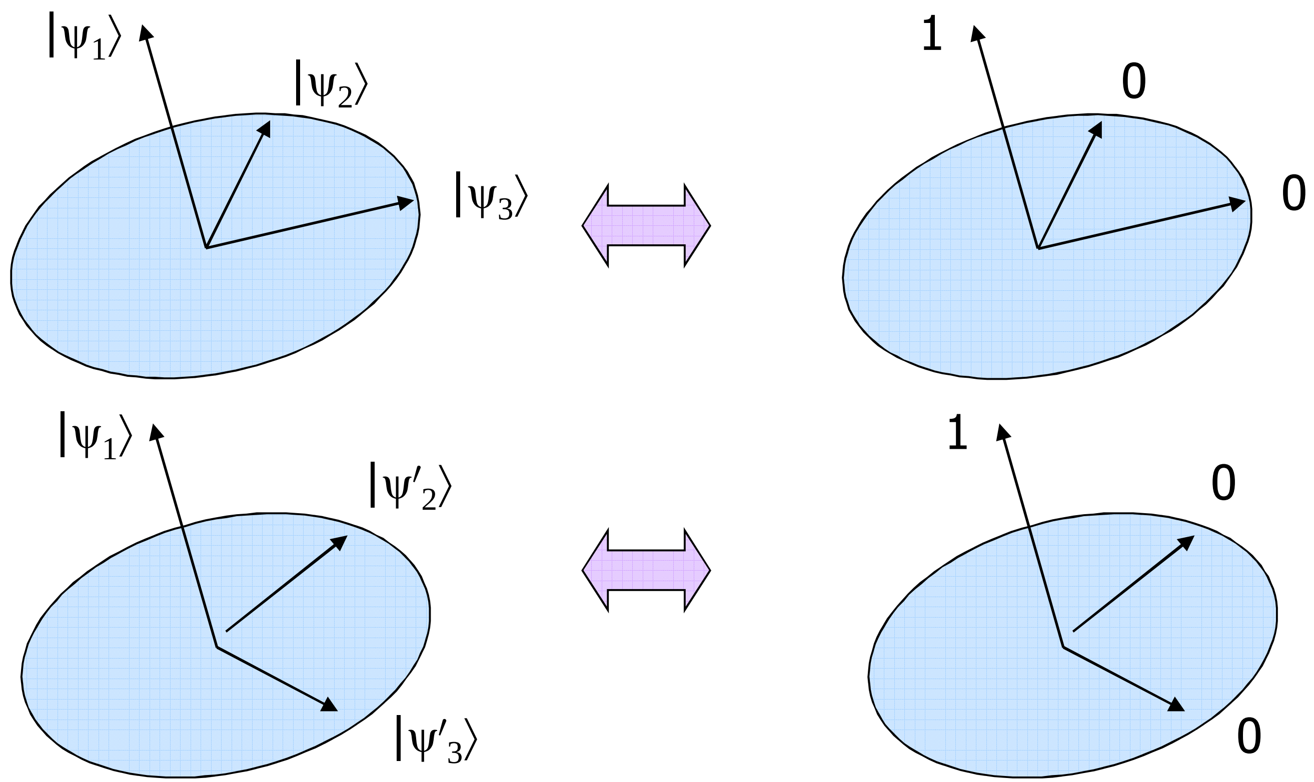
A given vector may appear in many different measurements




The traditional notion of noncontextuality:
Every vector is associated with the same $\chi(\lambda)$
regardless of how it is measured (i.e. the context)

The traditional notion of noncontextuality:
For every $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of the basis it is considered a part (i.e. the context).



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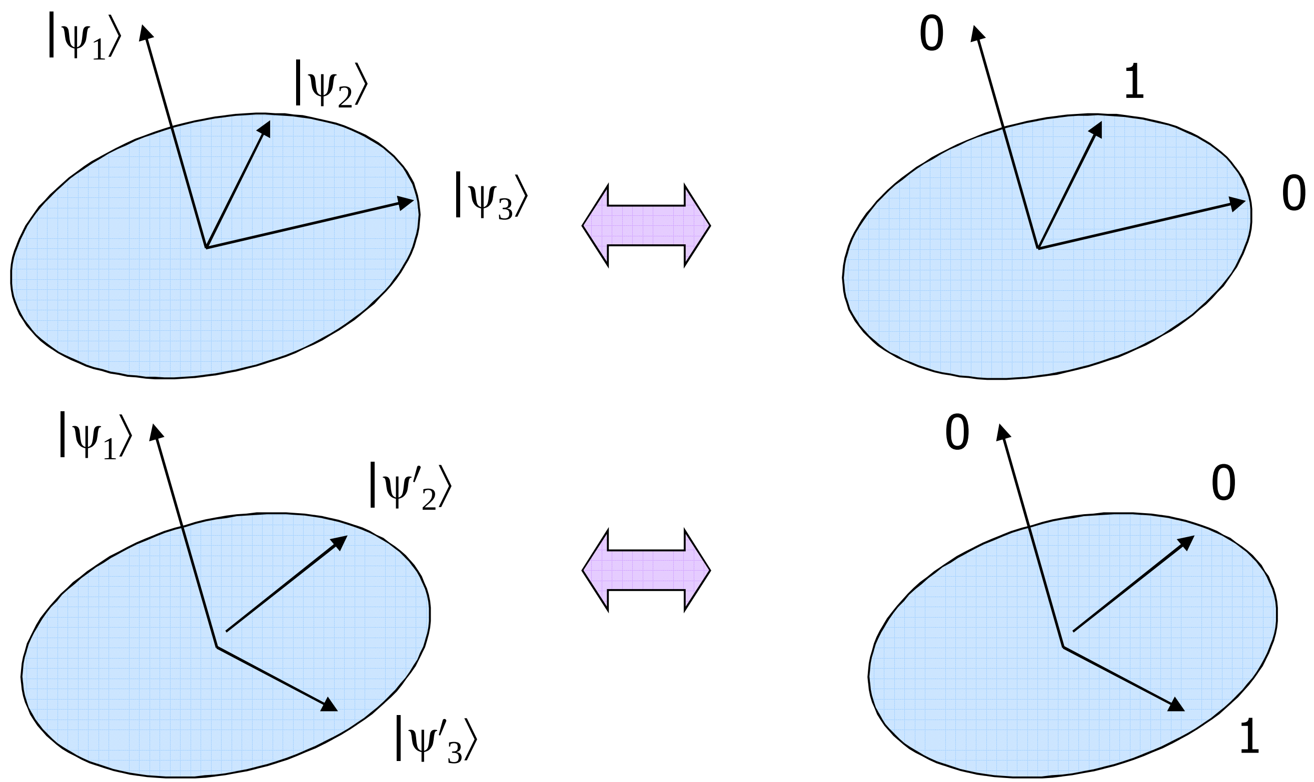
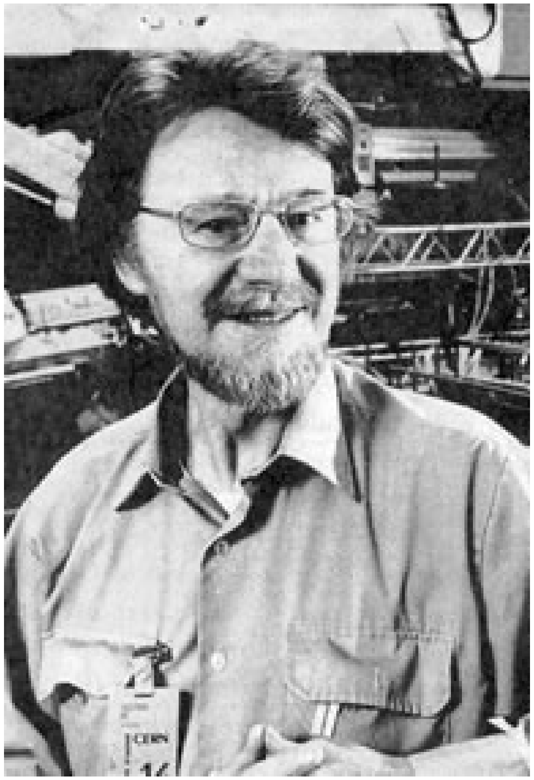




John S. Bell

Ernst Specker (with son) and
Simon Kochen

Bell-Kochen-Specker theorem: A noncontextual hidden variable model of quantum theory for Hilbert spaces of dimension 3 or greater is impossible.

## Example: The CEGA algebraic 18 ray proof in 4d:

Cabello, Estebaranz, Garcia-Alcaine, Phys. Lett. A 212, 183 (1996)

Each of the 18 rays appears twice in the following list

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,0,0,1 | 0,0,0,1 | 1,-1,1,-1 | 1,-1,1,-1 | 0,0,1,0 | 1,-1,-1,1 | 1,1,-1,1 | 1,1,-1,1 | 1,1,1,-1 |
| 0,0,1,0 | 0,1,0,0 | 1,-1,-1,1 | 1,1,1,1 | 0,1,0,0 | 1,1,1,1 | 1,1,1,-1 | -1,1,1,1 | -1,1,1,1 |
| 1,1,0,0 | 1,0,1,0 | 1,1,0,0 | 1,0,-1,0 | 1,0,0,1 | 1,0,0,-1 | 1,-1,0,0 | 1,0,1,0 | 1,0,0,1 |
| 1,-1,0,0 | 1,0,-1,0 | 0,0,1,1 | 0,1,0,-1 | 1,0,0,-1 | 0,1,-1,0 | 0,0,1,1 | 0,1,0,-1 | 0,1,-1,0 |

In each of the 9 columns, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

But each ray appears twice and so there must be an even number of rays assigned 1

CONTRADICTION!

## Example: The CEGA algebraic 18 ray proof in 4d:

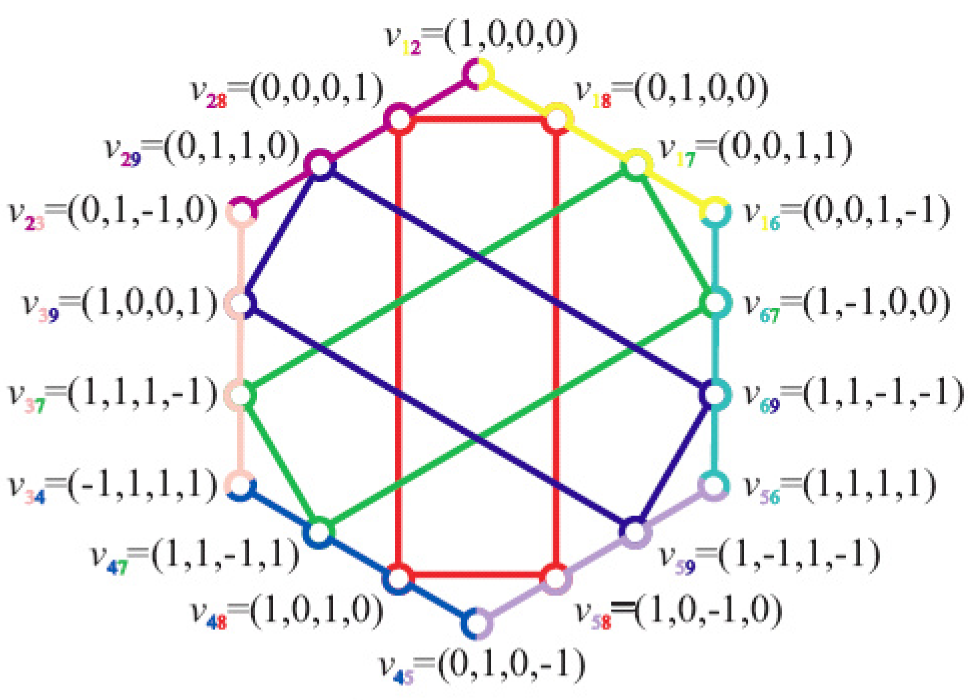
Cabello, Estebaranz, Garcia-Alcaine, Phys. Lett. A 212, 183 (1996)

Each of the 18 rays appears twice in the following list

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,0,0,1 | 0,0,0,1 | 1,-1,1,-1 | 1,-1,1,-1 | 0,0,1,0 | 1,-1,-1,1 | 1,1,-1,1 | 1,1,-1,1 | 1,1,1,-1 |
| 0,0,1,0 | 0,1,0,0 | 1,-1,-1,1 | 1,1,1,1 | 0,1,0,0 | 1,1,1,1 | 1,1,1,-1 | -1,1,1,1 | -1,1,1,1 |
| 1,1,0,0 | 1,0,1,0 | 1,1,0,0 | 1,0,-1,0 | 1,0,0,1 | 1,0,0,-1 | 1,-1,0,0 | 1,0,1,0 | 1,0,0,1 |
| 1,-1,0,0 | 1,0,-1,0 | 0,0,1,1 | 0,1,0,-1 | 1,0,0,-1 | 0,1,-1,0 | 0,0,1,1 | 0,1,0,-1 | 0,1,-1,0 |

In each of the 9 columns, c
Therefore, 9 rays must be a

But each ray appears twice
of rays assigned 1

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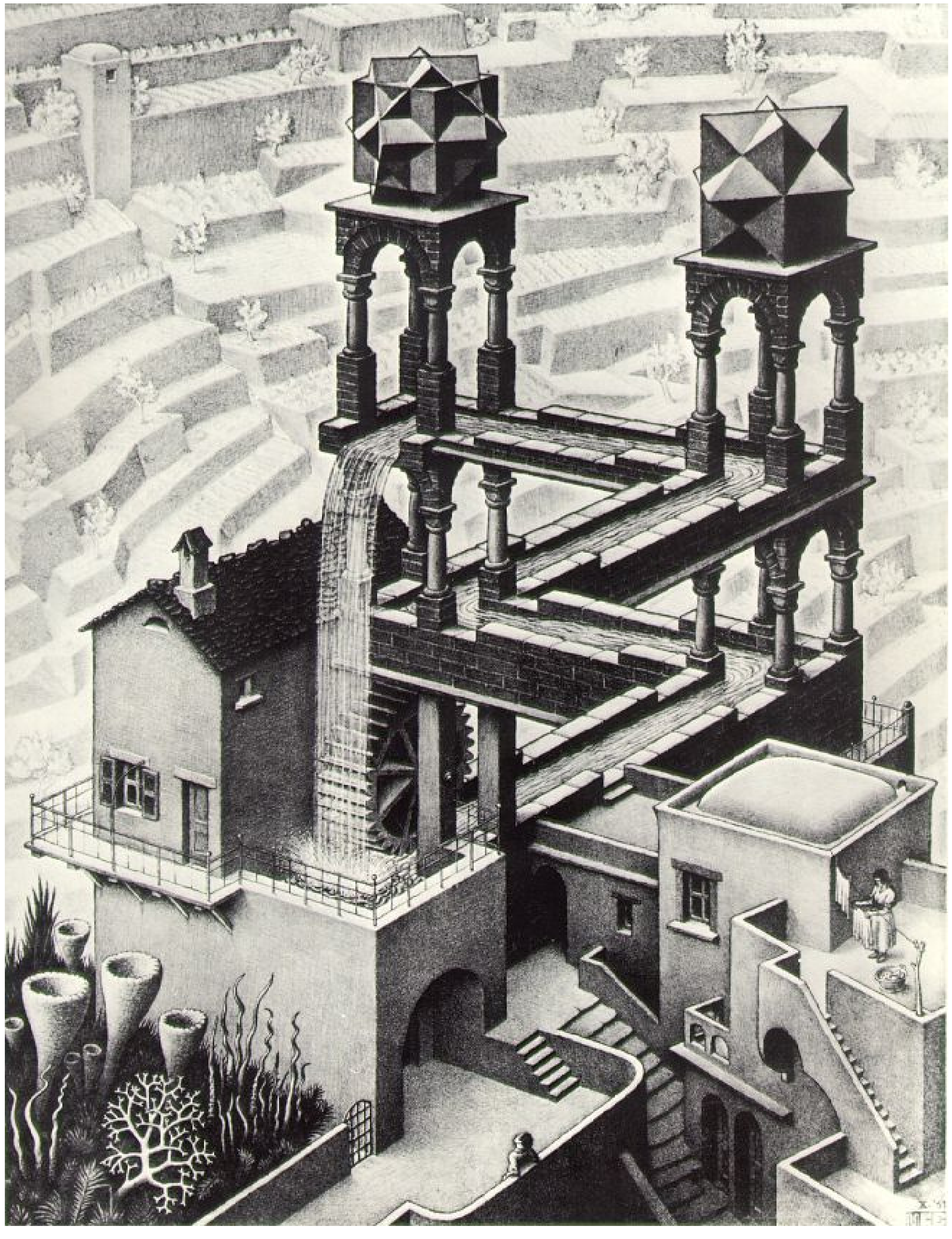
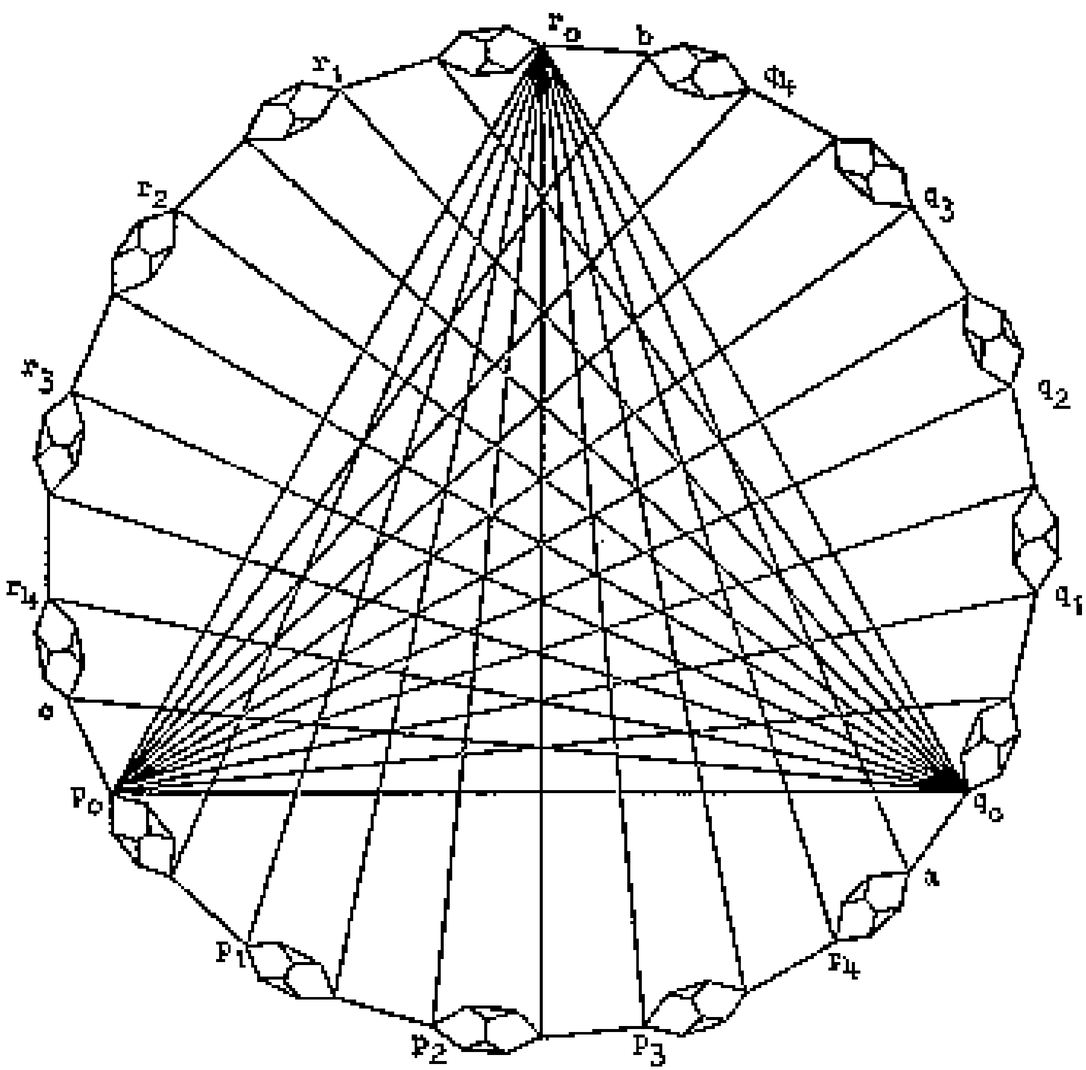
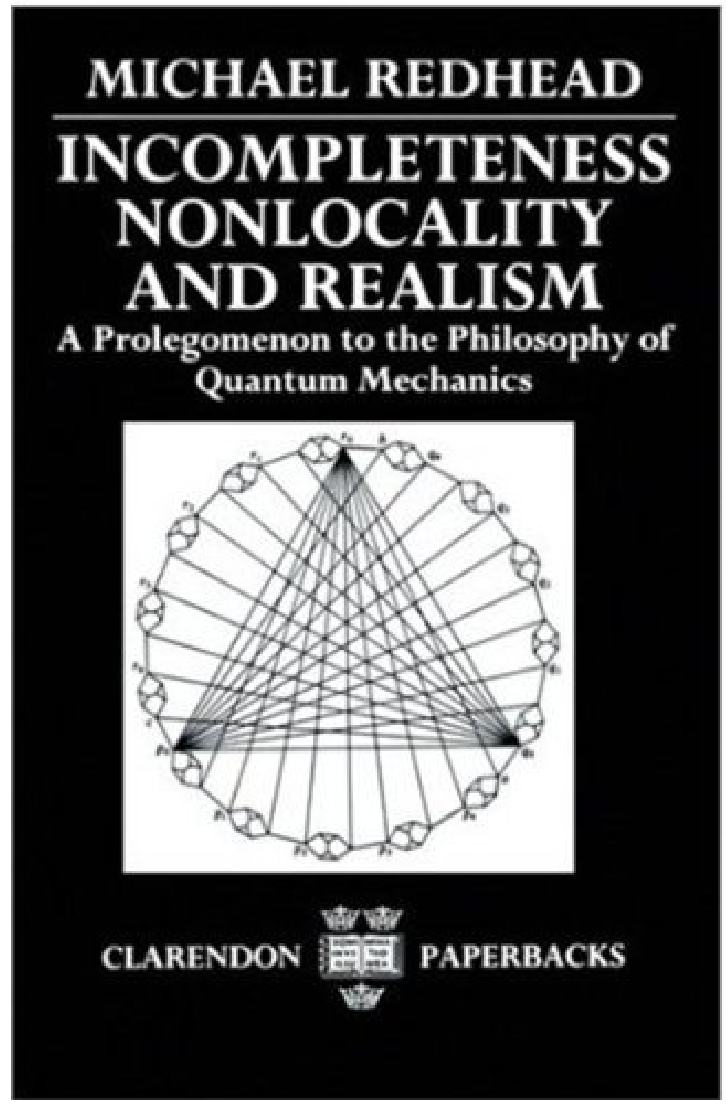
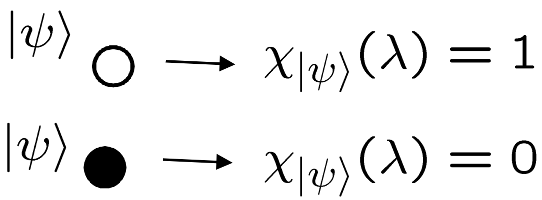


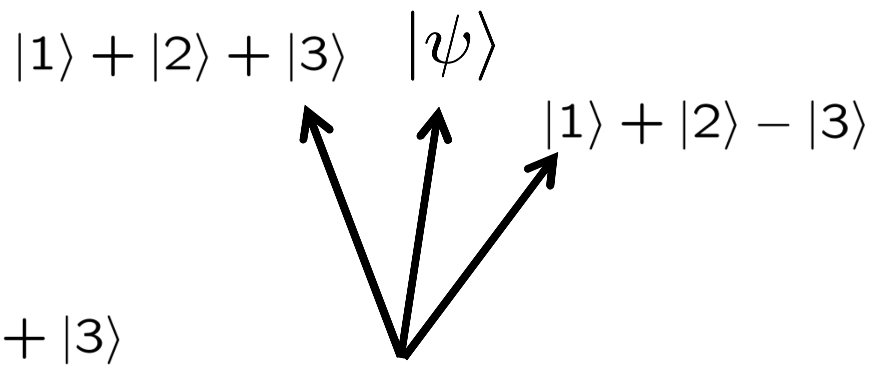
## Example: Kochen and Specker's original algebraic 117 ray proof in 3d

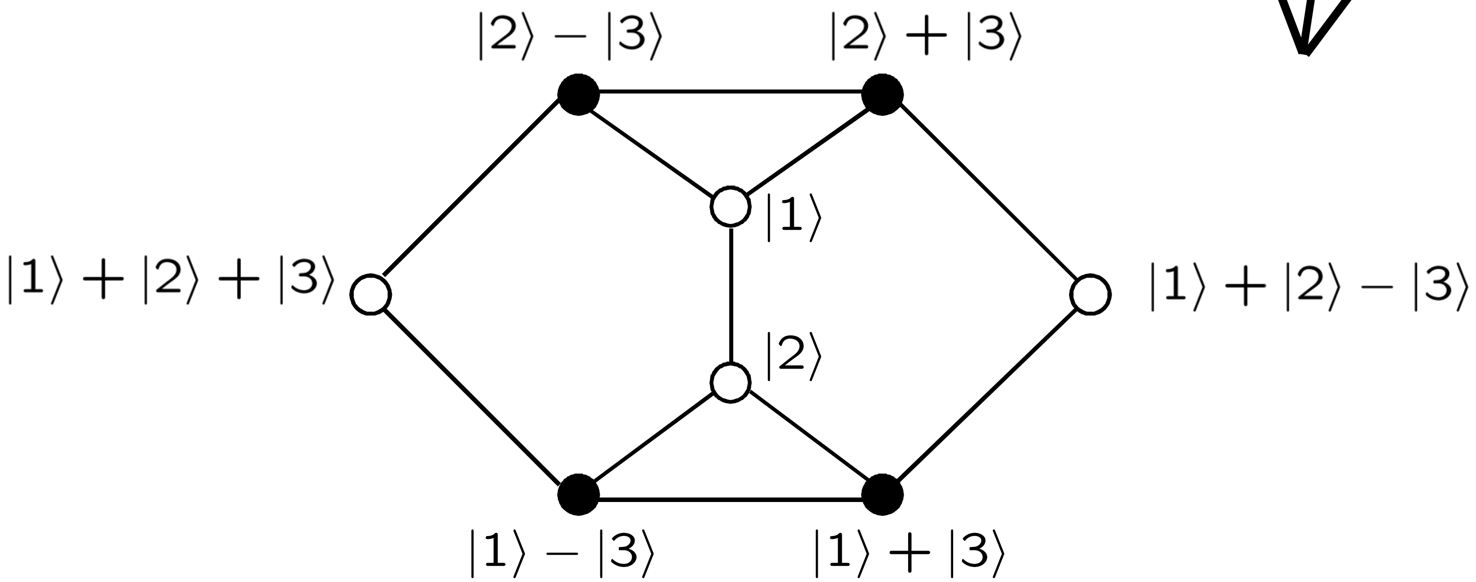


# Example: Clifton's state-specific 8 ray proof in 3d

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The traditional notion of noncontextuality:
For every $\lambda$, every projector P is assigned a value 0 or 1 regardless of how it is measured (i.e. the context)

$v(P) = 0$ or $1$ for all P

Every measurment has *some* outcome

$v(I) = 1$

Coarse-graining of a measurement implies a coarse-graining of the value (because it is just post-processing)

$v(\sum_k P_k) = \sum_k v(P_k)$

## Example: Bell's proof in 3d based on Gleason's theorem

Consider a function on projectors $P \mapsto \omega(P)$, satisfying:

1) $0 \leq \omega(P) \leq 1$ for all $P$
2) $\omega(I) = 1$
3) $\omega(\sum_k P_k) = \sum_k \omega(P_k)$

**Gleason's theorem**: For $\dim(\mathcal{H}) \geq 3$,

$$\omega(P) = \mathrm{Tr}(\rho P)$$

where $\rho$ is a density operator ($\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$).

But there is no $\rho$ such that $\omega(P)$=0 or 1 for all $P$
(Any given $\rho$ can only achieve a 0-1 valuation on a single basis)

CONTRADICTION

The traditional notion of noncontextuality:

For Hermitian operators A, B, C satisfying

$$[A, B] = 0 \qquad [A, C] = 0 \qquad [B, C] \neq 0$$

the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Measure A = measure projectors onto eigenspaces of A, $\{P_a\}$

Measure A with B
= measure projectors onto joint eigenspaces of A and B, $\{P_{ab}\}$
then coarse-grain over B outcome $P_a = \sum_b P_{ab}$

Measure A with C
= measure projectors onto joint eigenspaces of A and C, $\{P_{ac}\}$
Then coarse-grain over C outcome $P_a = \sum_b P_{ac}$

$v(P_a)$ is independent of context

Therefore $v(A)$ is independent of context

Functional relationships among commuting Hermitian operators must be respected by their values

If $f(L, M, N, ...) = 0$

then $f(v(L), v(M), v(N), ...) = 0$

Proof: the possible sets of eigenvalues one can simultaneously assign to L, M, N,… are specified by their joint eigenstates. By acting the first equation on each of the joint eigenstates, we get the second.

## Example: Mermin's magic square proof in 4d

| | | | |
|---|---|---|---|
| $X_1$ | $X_2$ | $X_1 X_2$ | $I$ |
| $Y_2$ | $Y_1$ | $Y_1 Y_2$ | $I$ |
| $X_1 Y_2$ | $Y_1 X_2$ | $Z_1 Z_2$ | $I$ |
| $I$ | $I$ | ¡ $I$ | |

$$X_1 \ X_2 \ X_1 X_2 = I$$
$$Y_1 \ Y_2 \ Y_1 Y_2 = I$$
$$X_1 Y_2 \ Y_1 X_2 \ Z_1 Z_2 = I$$
$$X_1 \ Y_2 \ X_1 Y_2 = I$$
$$Y_1 \ X_2 \ Y_1 X_2 = I$$
$$X_1 X_2 \ Y_1 Y_2 \ Z_1 Z_2 = -I$$

$$v(X_1)\ v(X_2)\ v(X_1 X_2) = 1$$
$$v(Y_1)\ v(Y_2)\ v(Y_1 Y_2) = 1$$
$$v(X_1 Y_2)\ v(Y_1 X_2)\ v(Z_1 Z_2) = 1$$
$$v(X_1)\ v(Y_2)\ v(X_1 Y_2) = 1$$
$$v(Y_1)\ v(X_2)\ v(Y_1 X_2) = 1$$
$$v(X_1 X_2)\ v(Y_1 Y_2)\ v(Z_1 Z_2) = -1$$

Product of LHSs = +1

Product of RHSs = -1

CONTRADICTION

# Aside: Local determinism is an instance of traditional noncontextuality where the context is remote

$S_a^A$ - $I^B$ is either measured with $I^A$ - $S_b^B$

or with $I^A$ - $S_{b'}^B$

Recall traditional noncontextuality:

For Hermitian operators A, B, C satisfying

$[A, B] = 0 \quad [A, C] = 0 \quad [B, C] \neq 0$

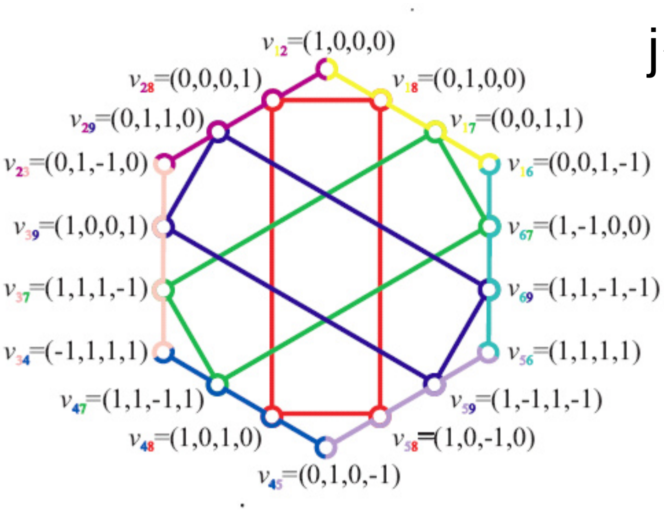
the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Therefore $v(S_a^A)$ is the same for the two contexts

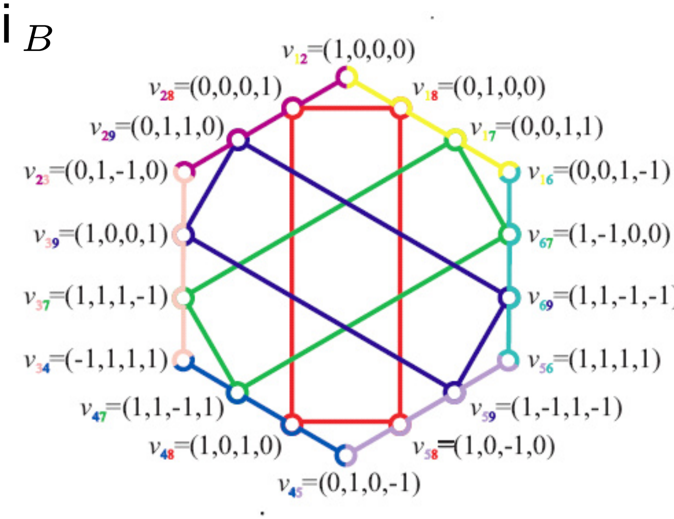
This is local determinism

Every proof of the impossibility of a locally deterministic model is a proof of the impossibility of a traditional noncontextual model

# Aside: Traditional noncontextuality can sometimes be justified by local causality




$$|\psi\rangle_{AB} = \frac{1}{2}\sum_{i=1}^{4} |i\rangle_A |i\rangle_B$$




Perfect correlation when same mmt is made on both wings
+ local causality
→ Traditional noncontextual hidden variable model for mmts on one wing

CONTRADICTION!

# The generalized notion of noncontextuality

Problems with the traditional definition of noncontextuality:

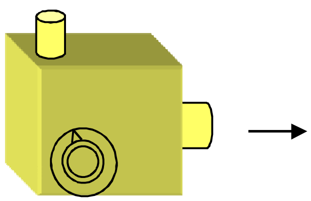
- applies only to sharp measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory

A better notion of noncontextuality would determine

- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model

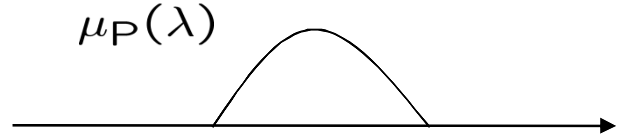
# A realist model of an operational theory

Preparation
P



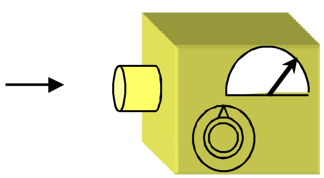
$\int \mu_{\mathrm{P}}(\lambda) d\lambda = 1$

$\mu_{\mathrm{P}}(\lambda)$



$\lambda$

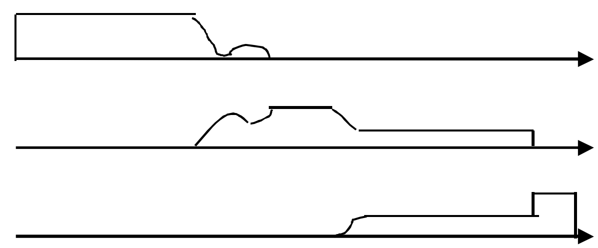
Measurement
M



$0 \leq \xi_{\mathrm{M},k} \leq 1$

$\sum_k \xi_{\mathrm{M},k}(\lambda) = 1$ for all $\lambda$

$\xi_{\mathrm{M},1}(\lambda)$, $\xi_{\mathrm{M},2}(\lambda)$, $\xi_{\mathrm{M},3}(\lambda)$



$\lambda$

$$p(k|\mathrm{P},\mathrm{M}) = \int d\lambda \, \xi_{\mathrm{M},k}(\lambda) \, \mu_{\mathrm{P}}(\lambda)$$

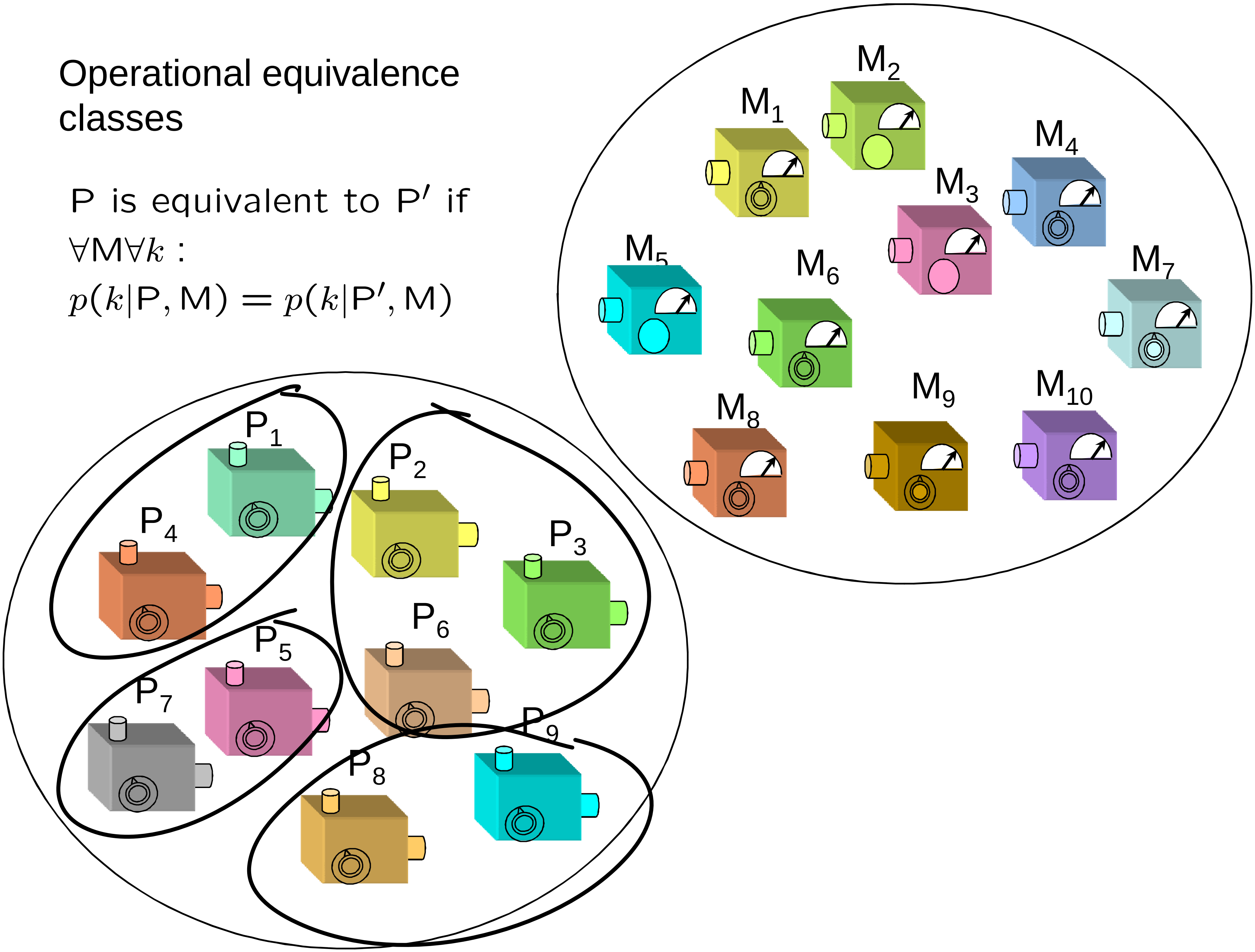
## Generalized definition of noncontextuality:

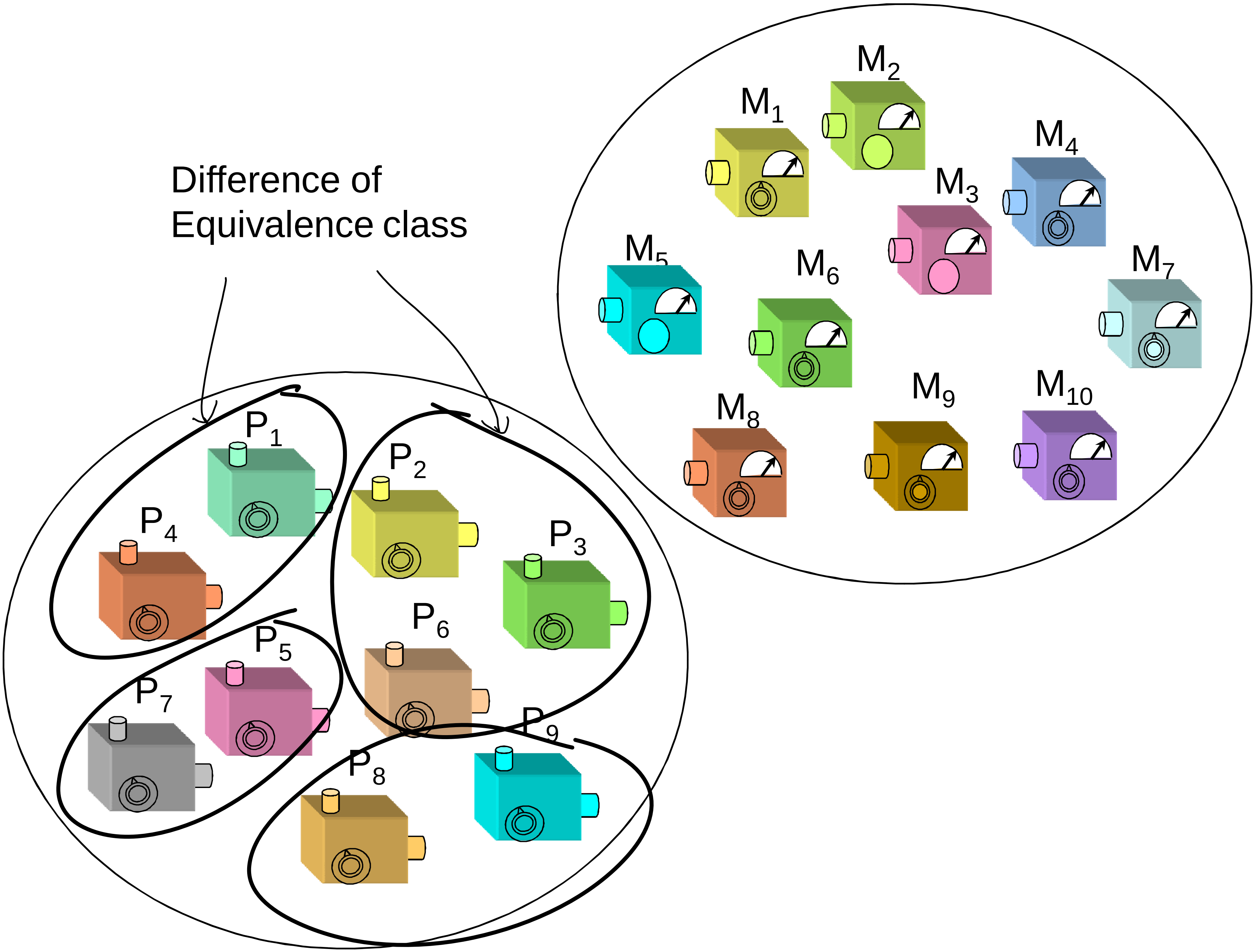
A realist model of an operational theory is noncontextual if

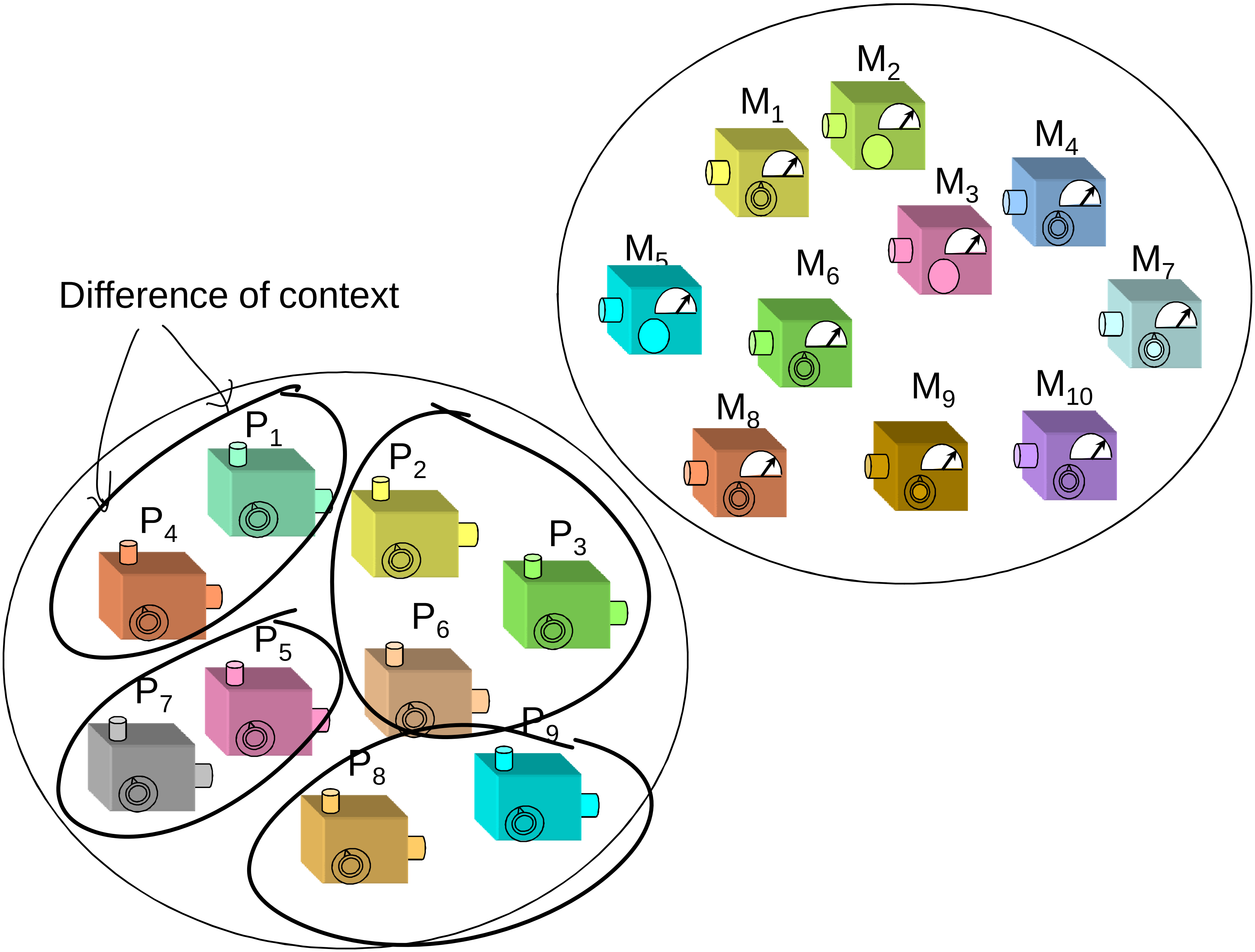
Operational equivalence of two experimental procedures → Equivalent representations in the realist model

Operational equivalence classes

P is equivalent to P′ if
∀M∀$k$ :
$p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$

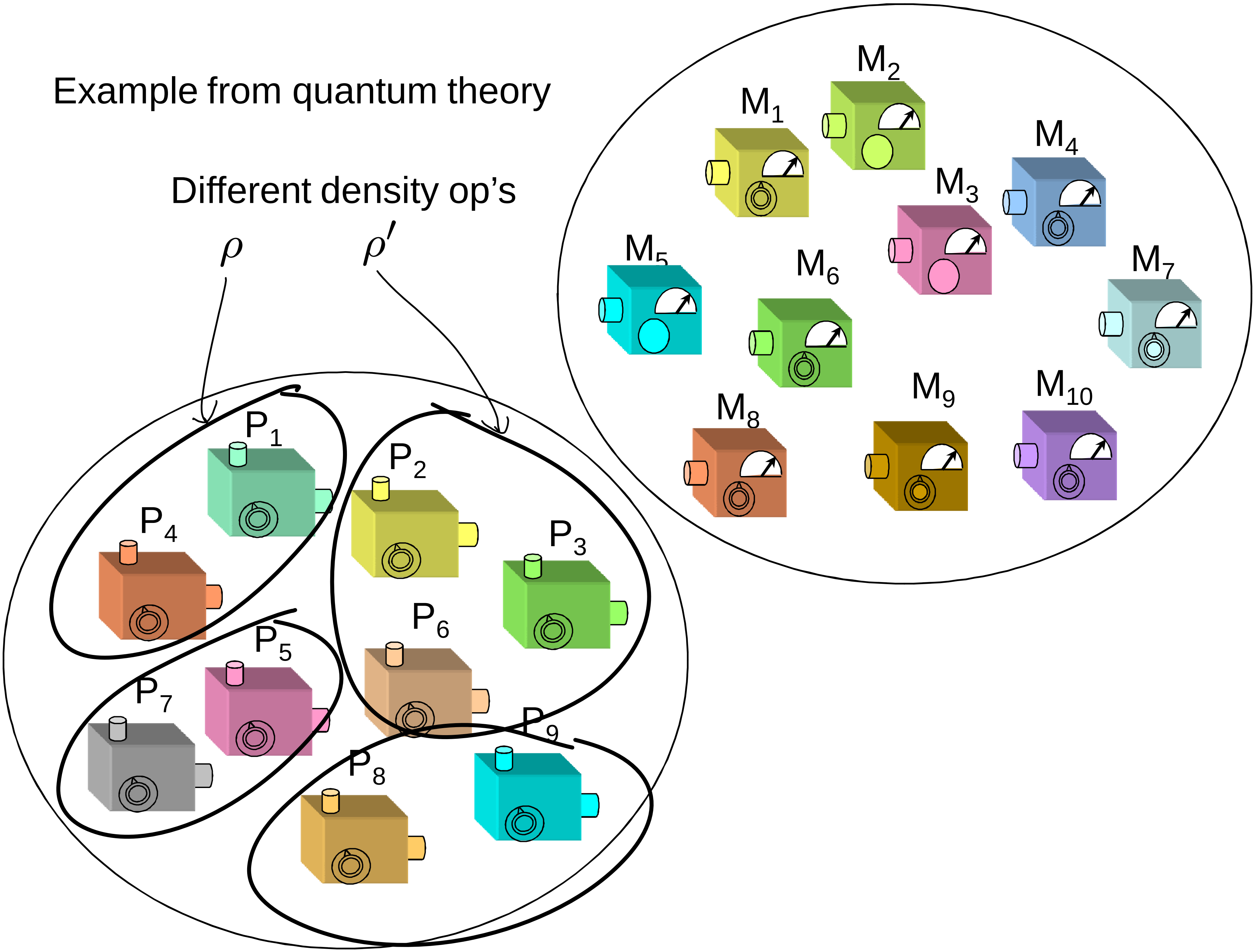



Difference of
Equivalence class
M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9



M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
Difference of context
P1
P2
P3
P4
P5
P6
P7
P8
P9

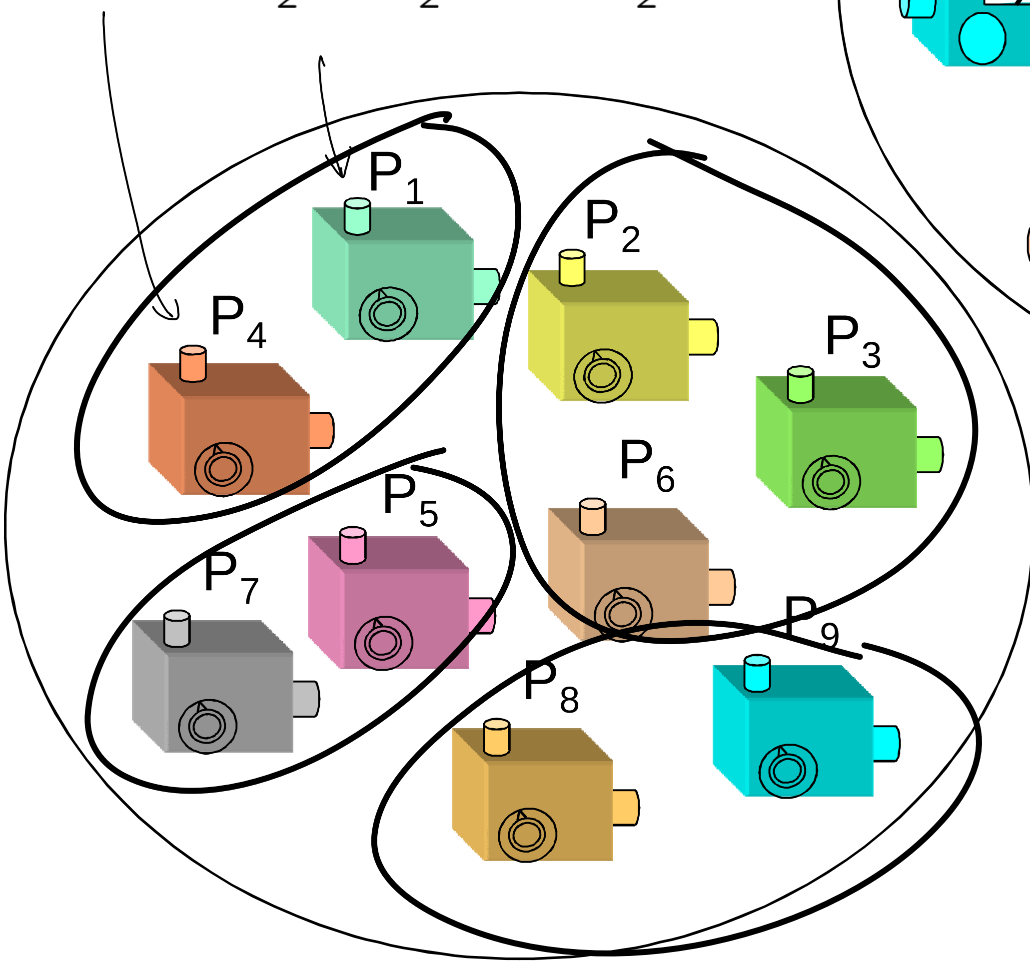
# Example from quantum theory

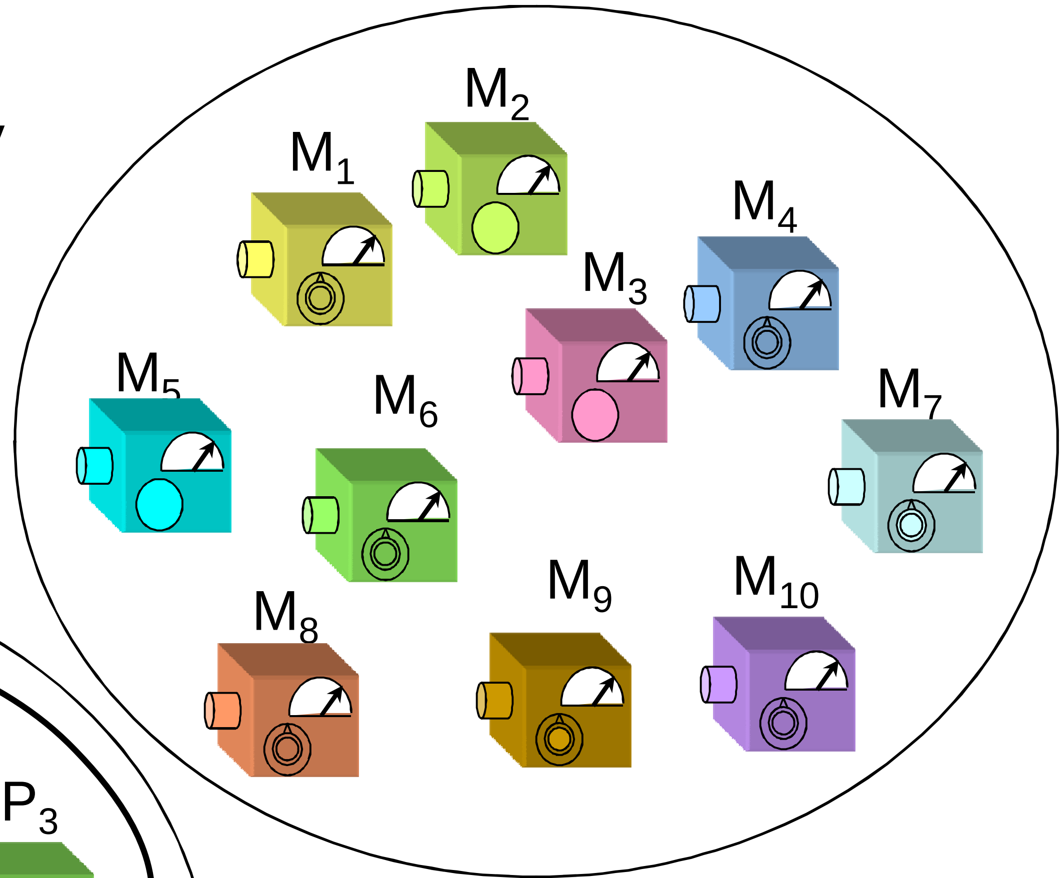


# Example from quantum theory

$$\frac{1}{2}I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\frac{1}{2}I = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

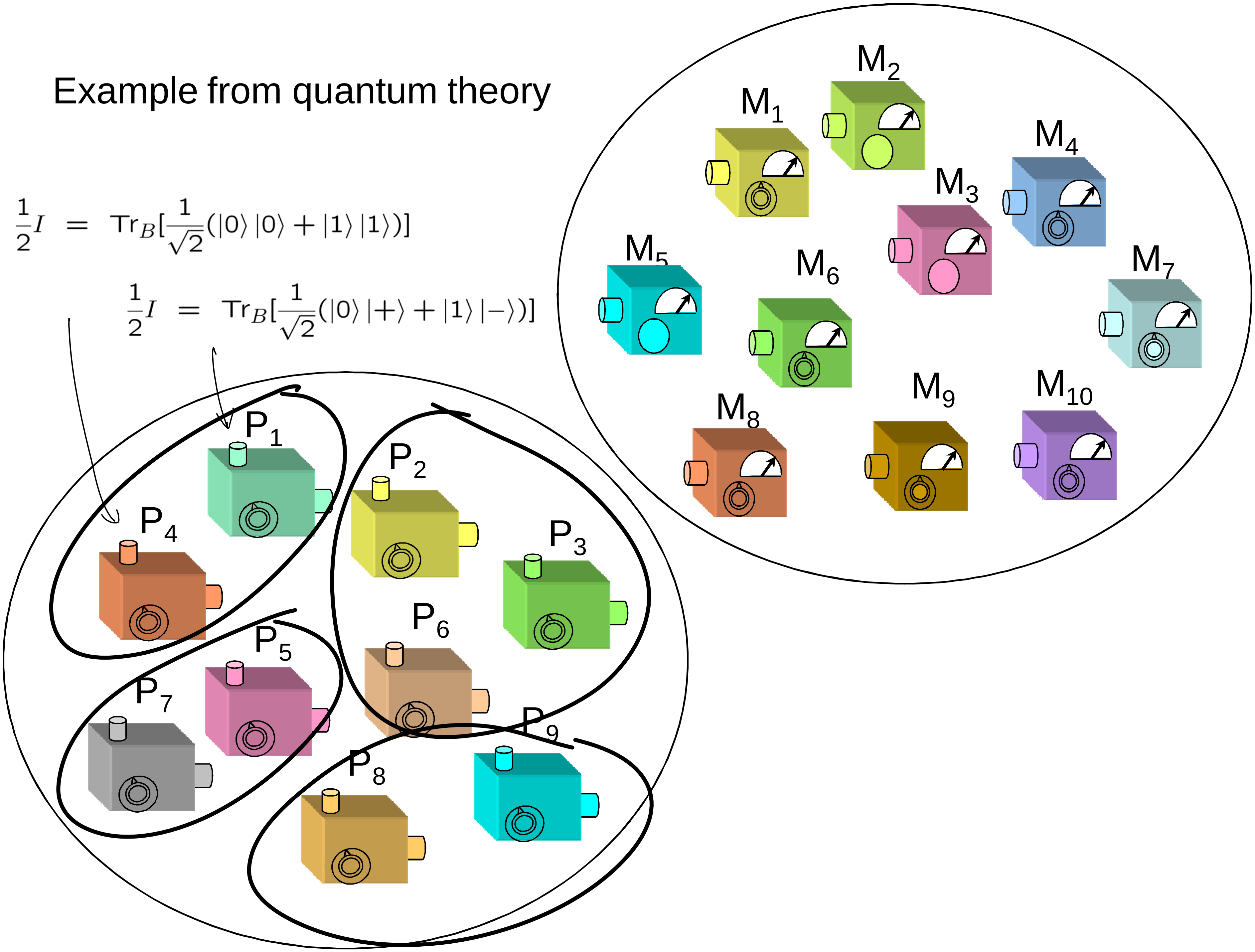


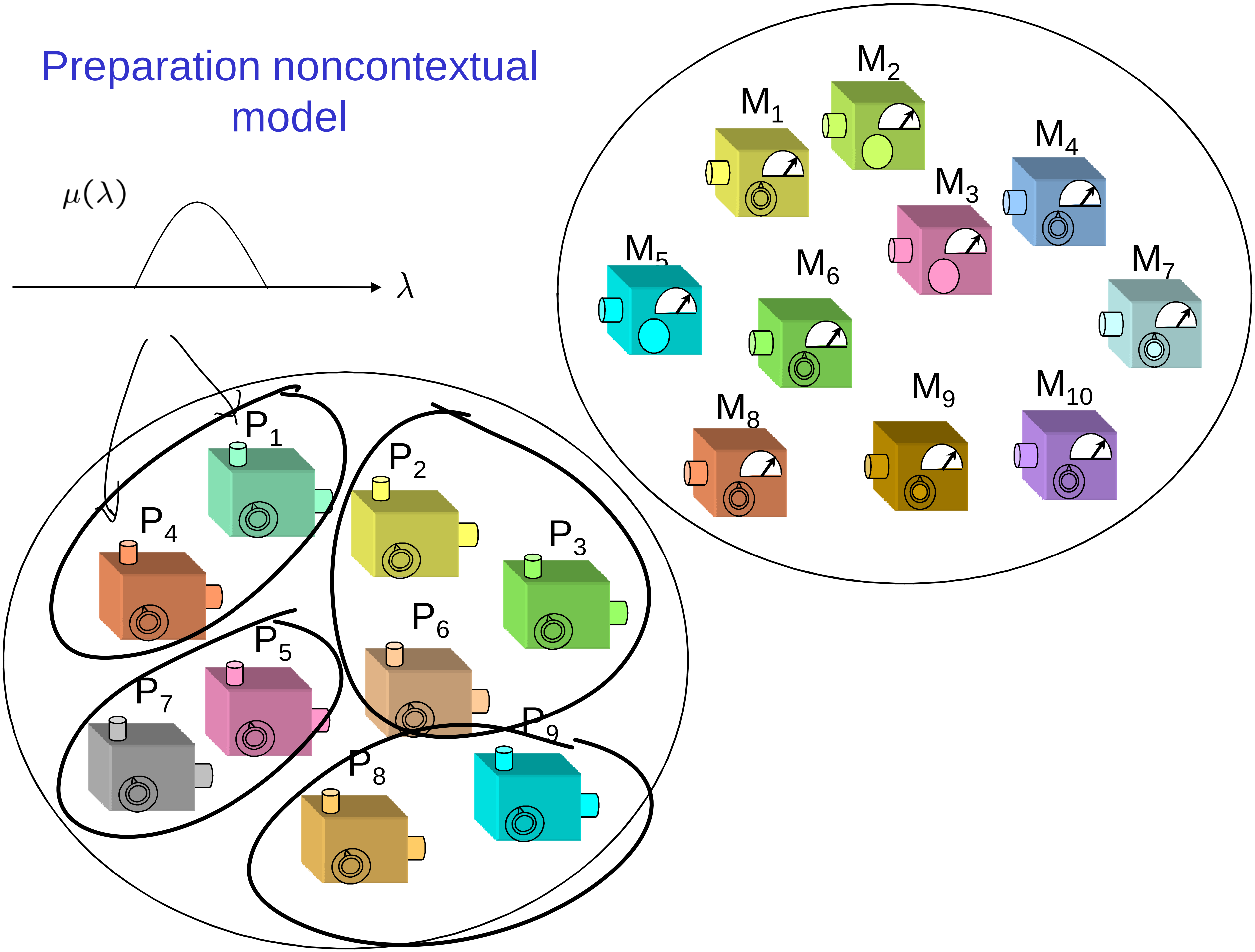
Example from quantum theory

$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)]$$

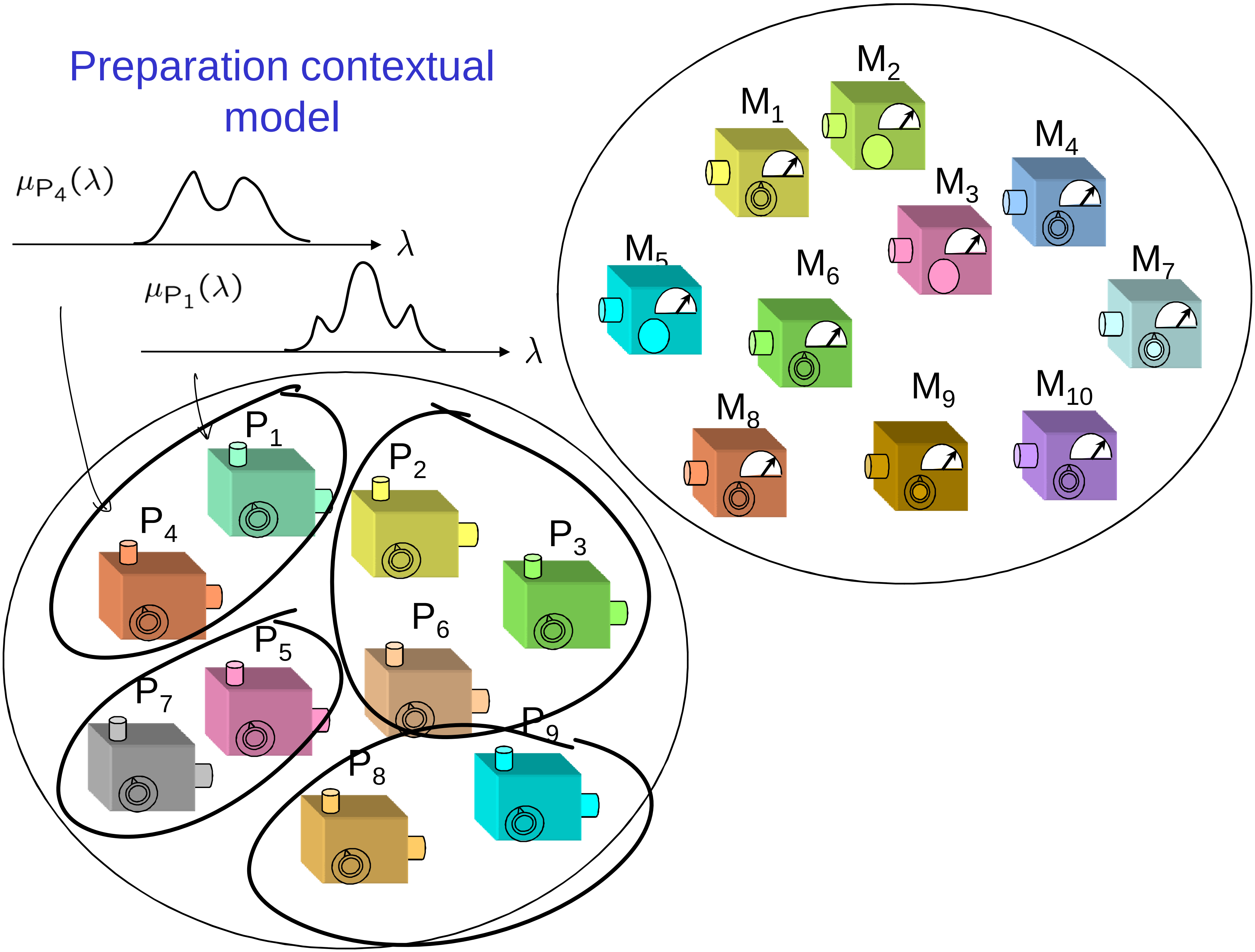
$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)]$$



# Preparation noncontextual model



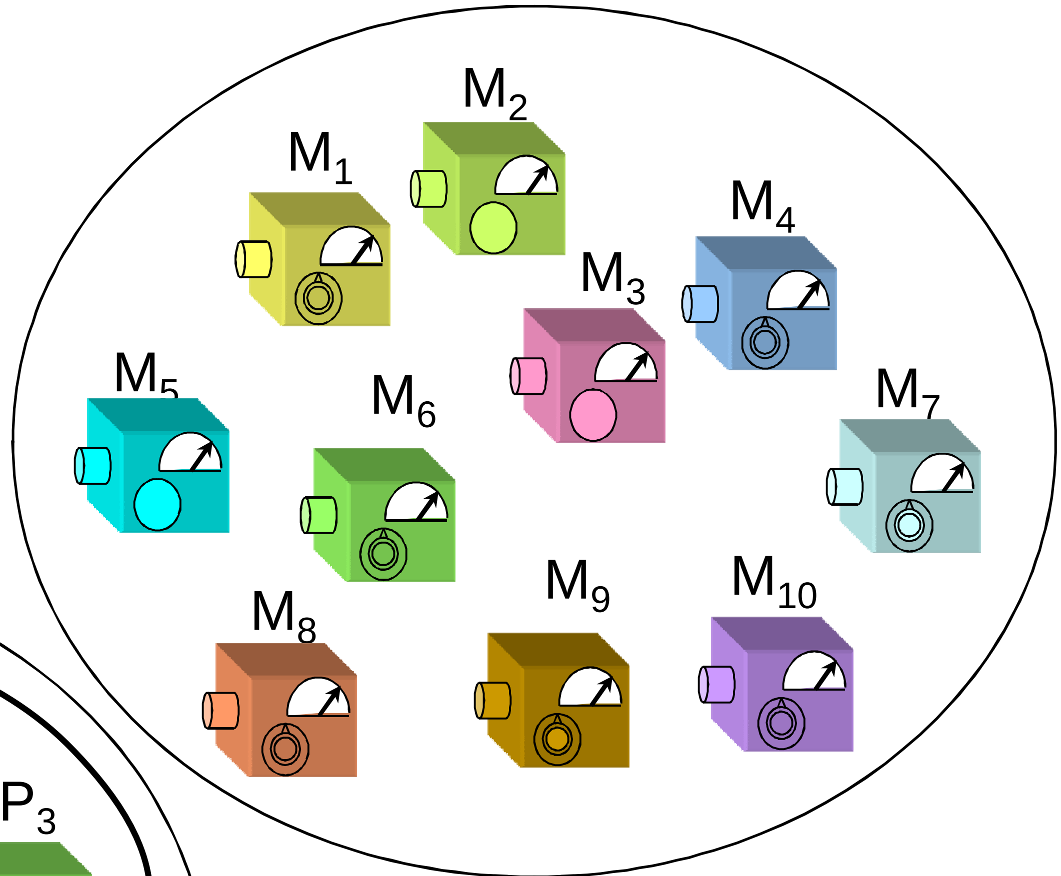
# Preparation contextual model

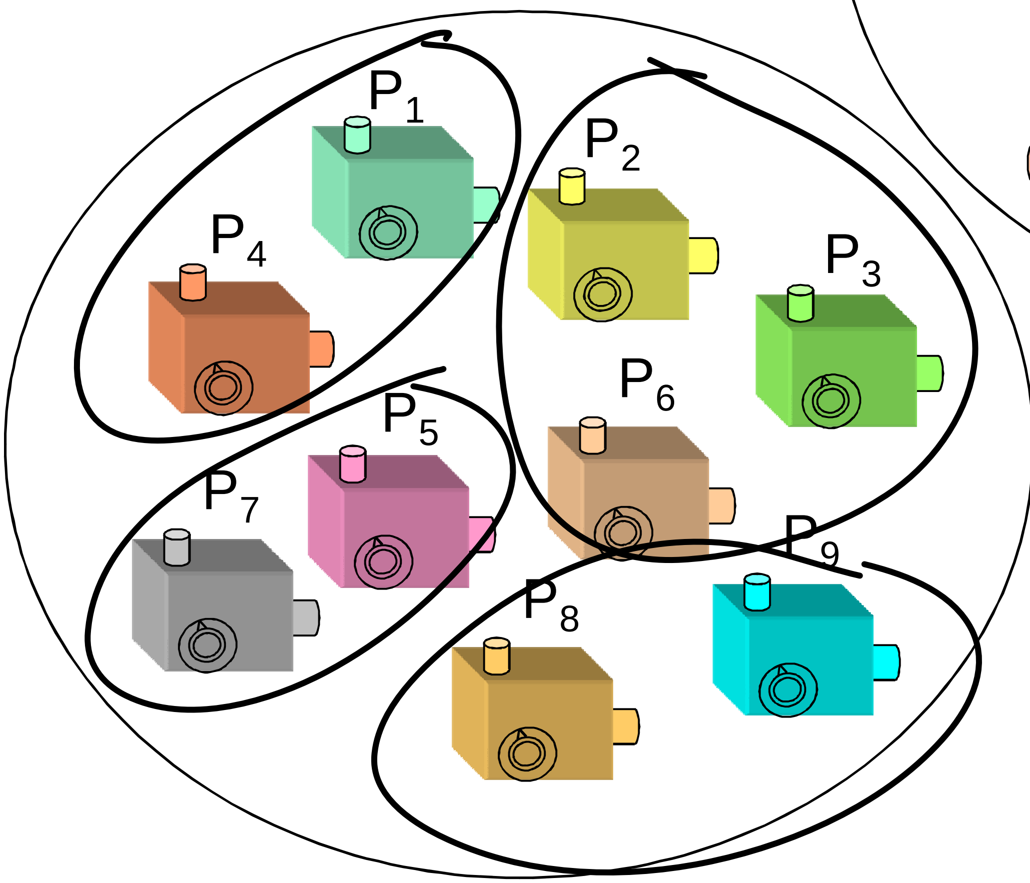


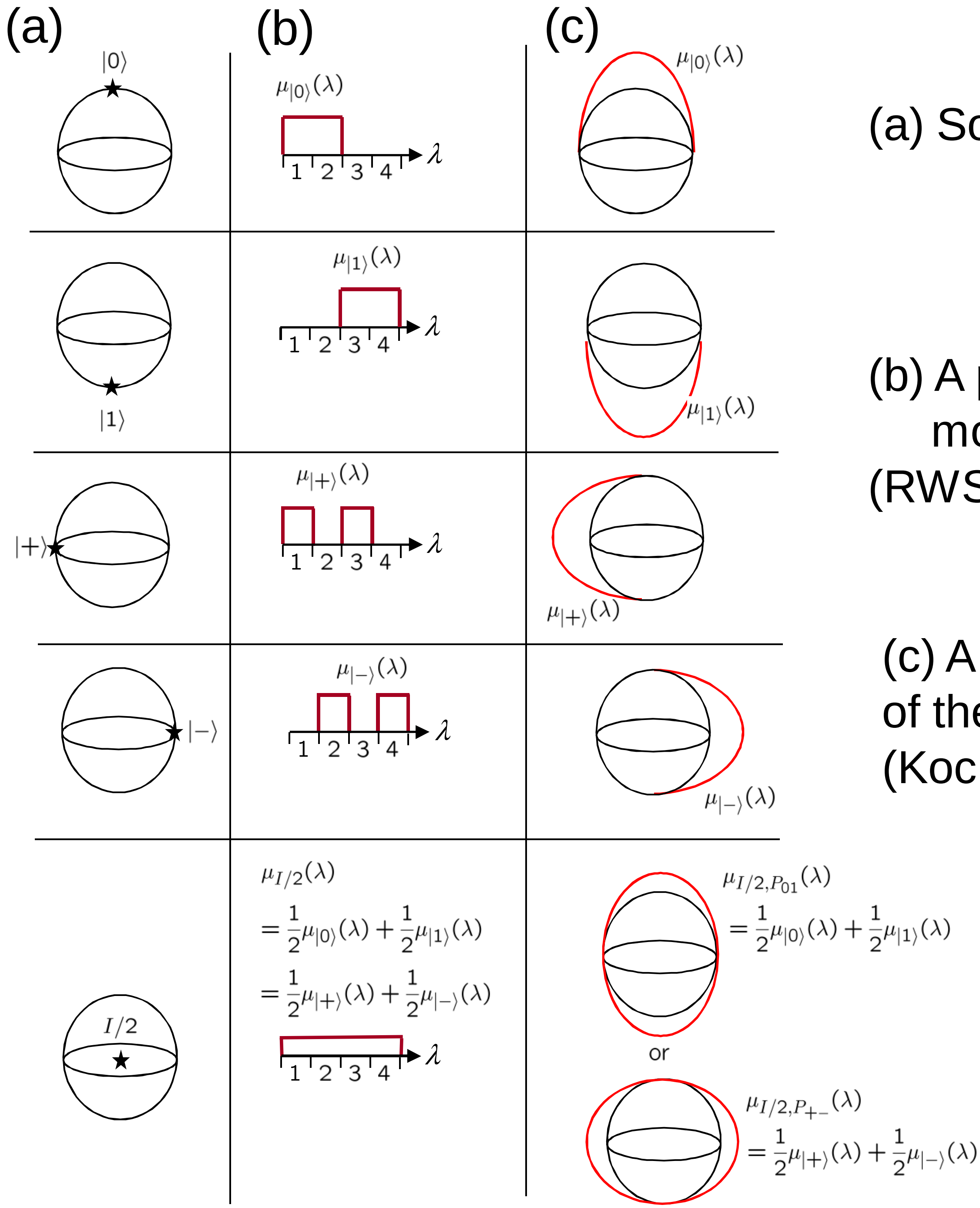
Definition of preparation noncontextual model:

$$\forall \mathsf{M} : p(k|\mathsf{P}, \mathsf{M}) = p(k|\mathsf{P}', \mathsf{M})$$

$$\longrightarrow \quad p(\lambda|\mathsf{P}) = p(\lambda|\mathsf{P}')$$



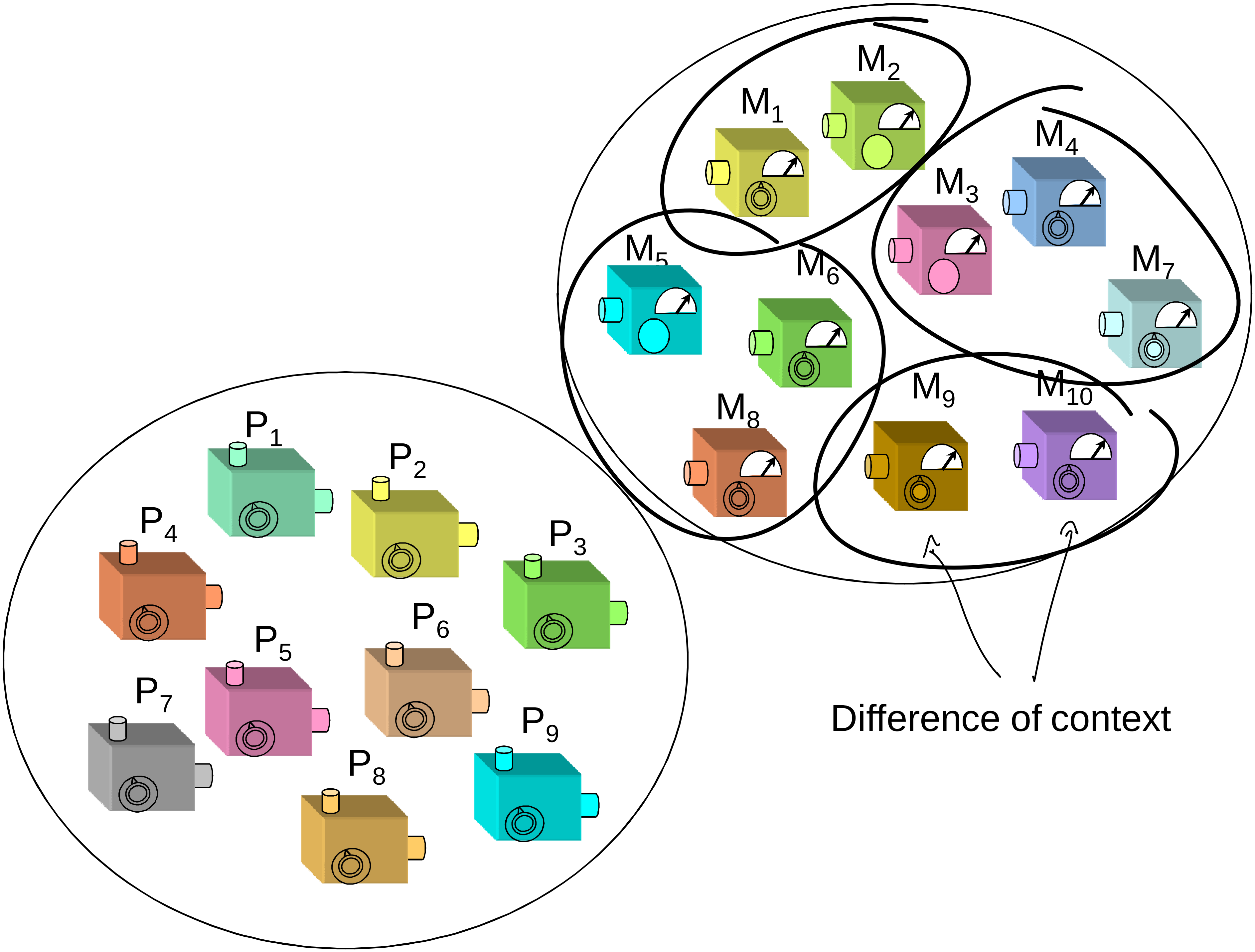





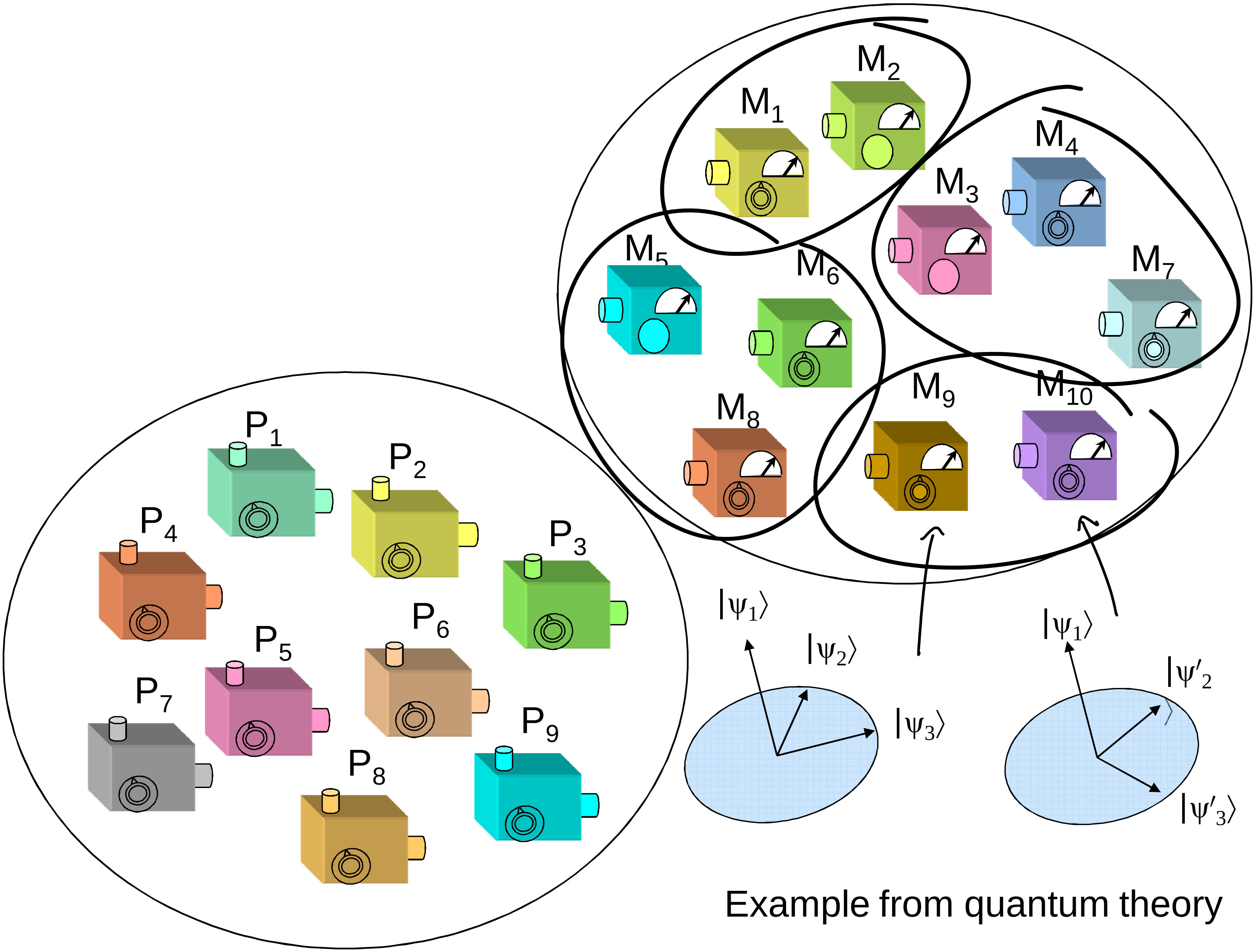

(a) Some states of a qubit

(b) A preparation noncontextual model of these (RWS, PRA 75, 032110, 2007)

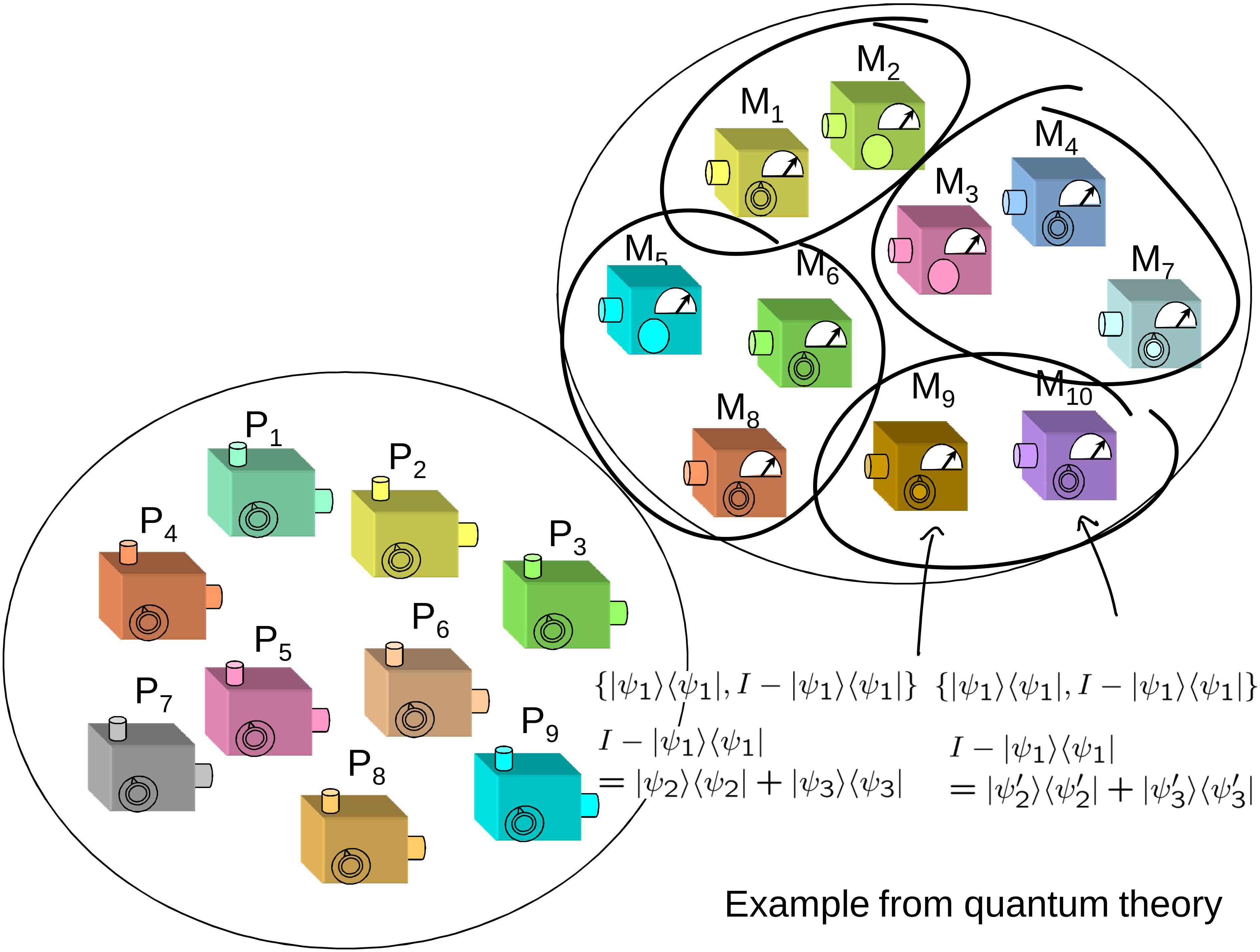
(c) A preparation contextual model of these (Kochen-Specker, 1967)



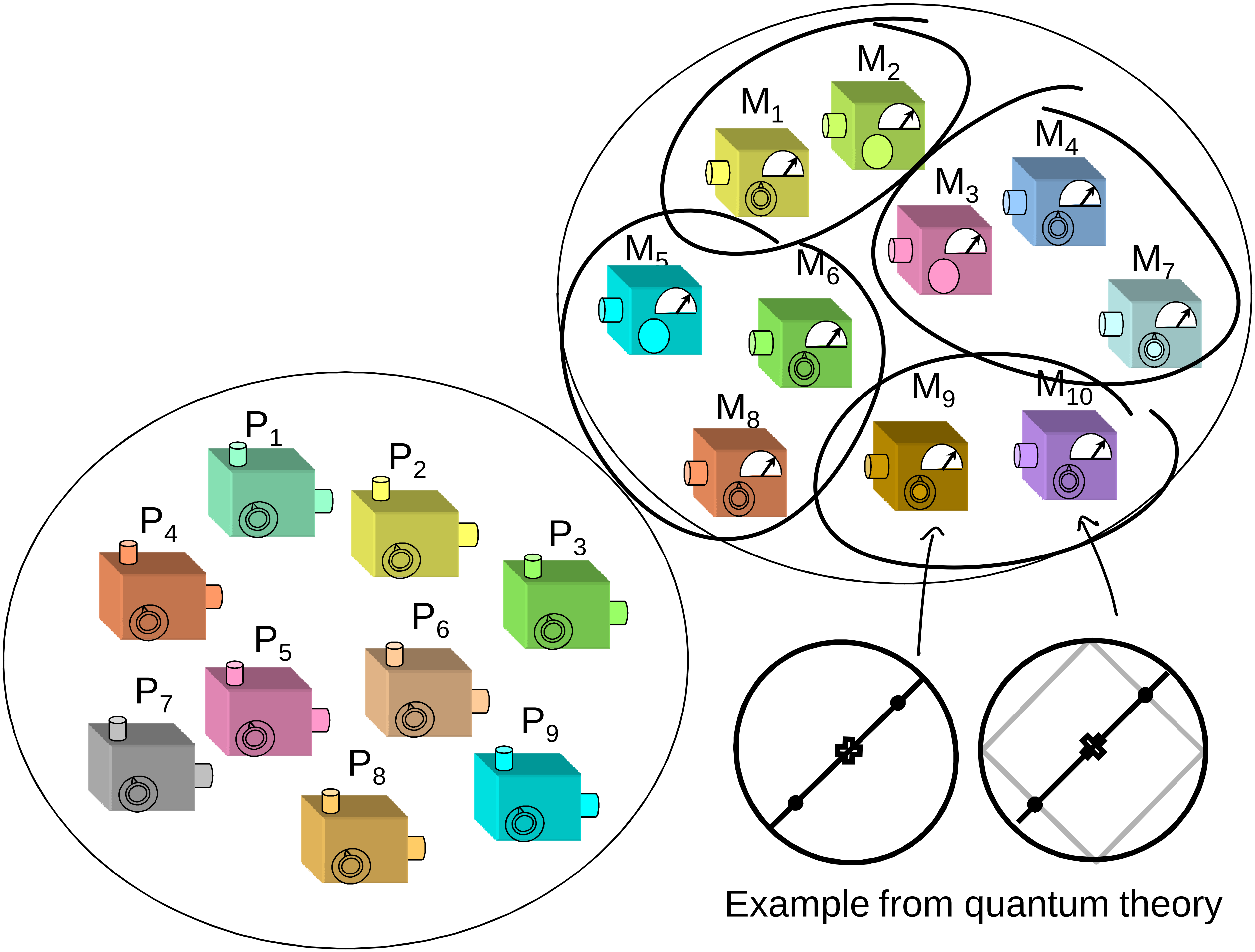
$M_1$
$M_2$
$M_3$
$M_4$
$M_5$
$M_6$
$M_7$
$M_8$
$M_9$
$M_{10}$
$P_1$
$P_2$
$P_3$
$P_4$
$P_5$
$P_6$
$P_7$
$P_8$
$P_9$
Difference of context



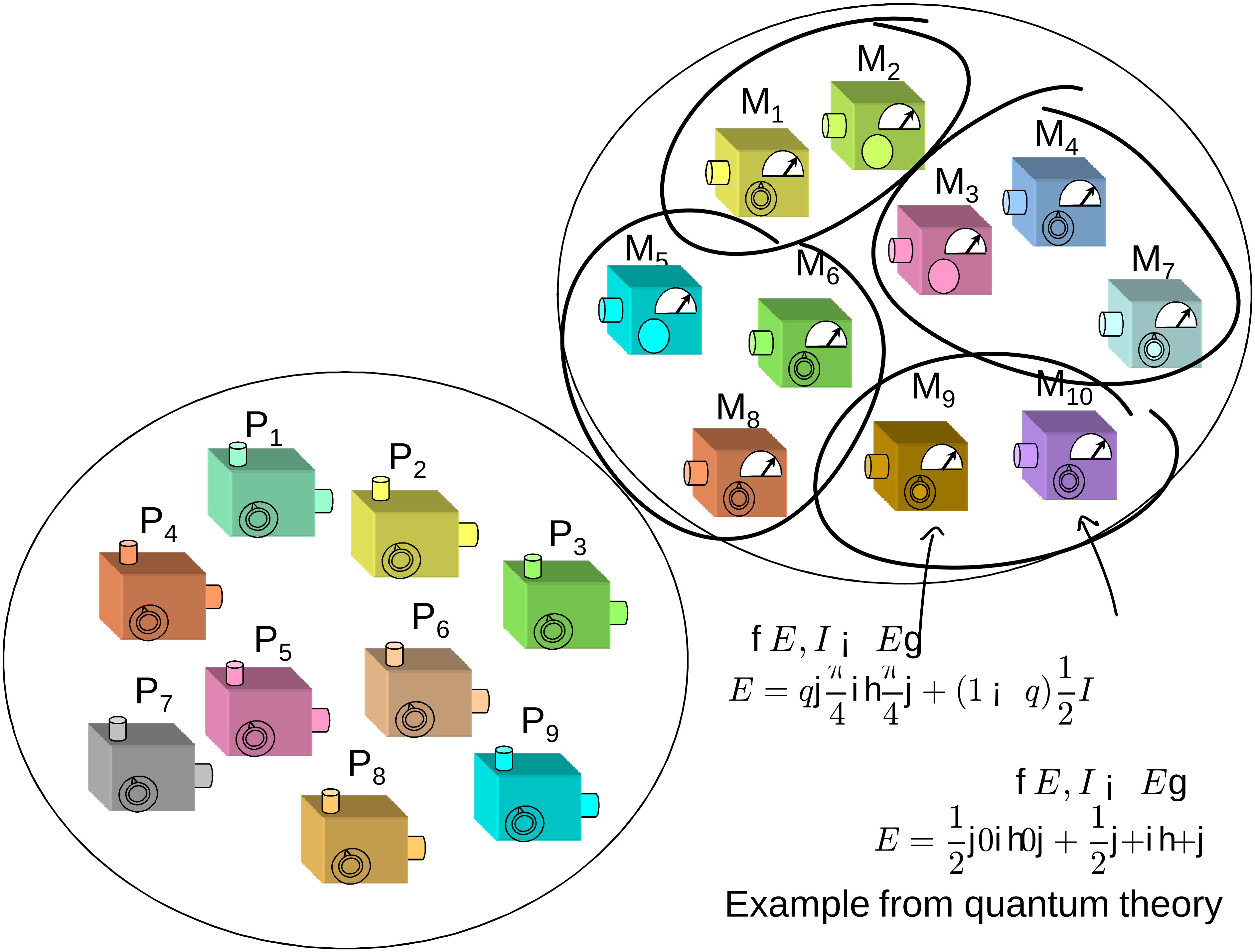

Example from quantum theory




Example from quantum theory



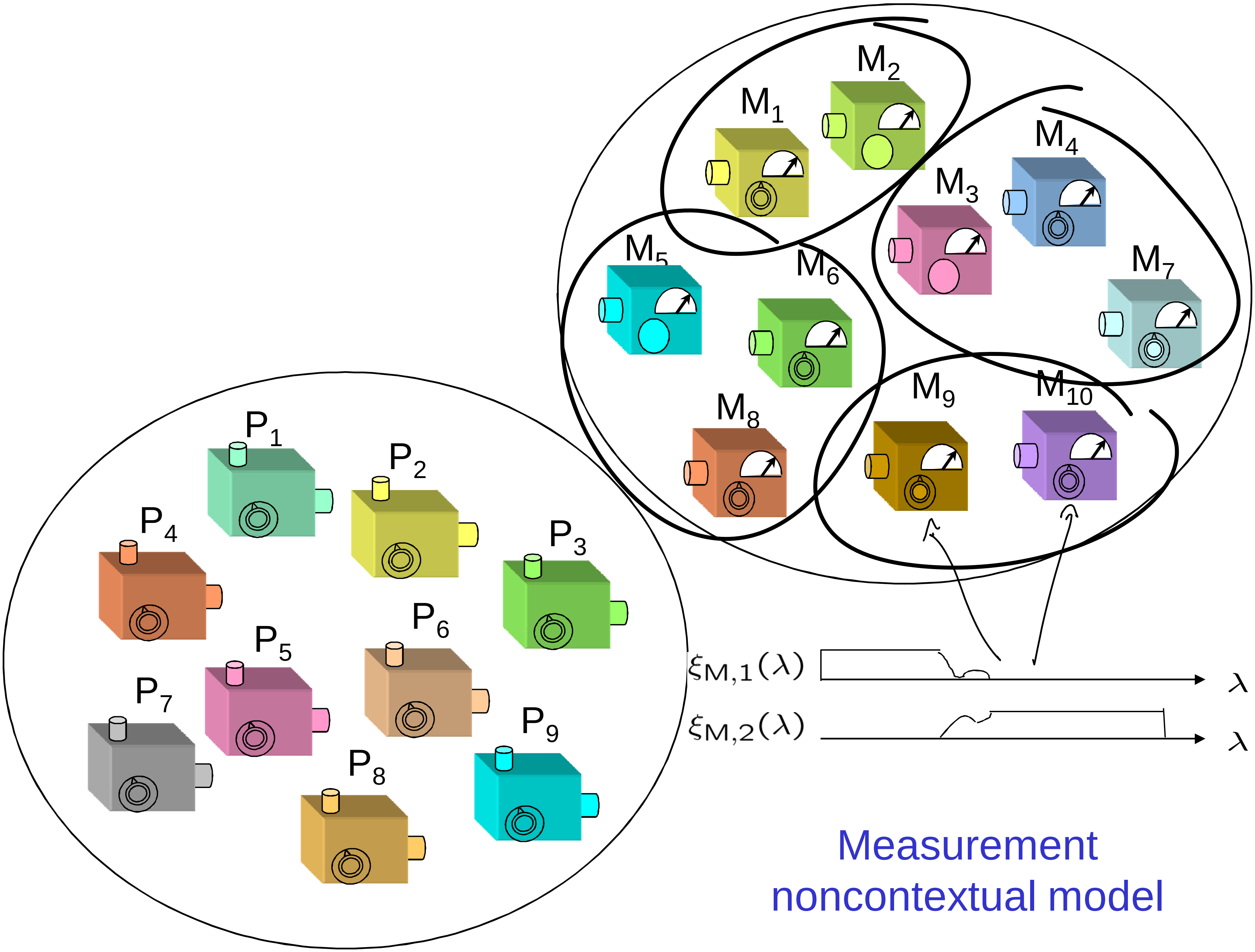
M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9
Example from quantum theory



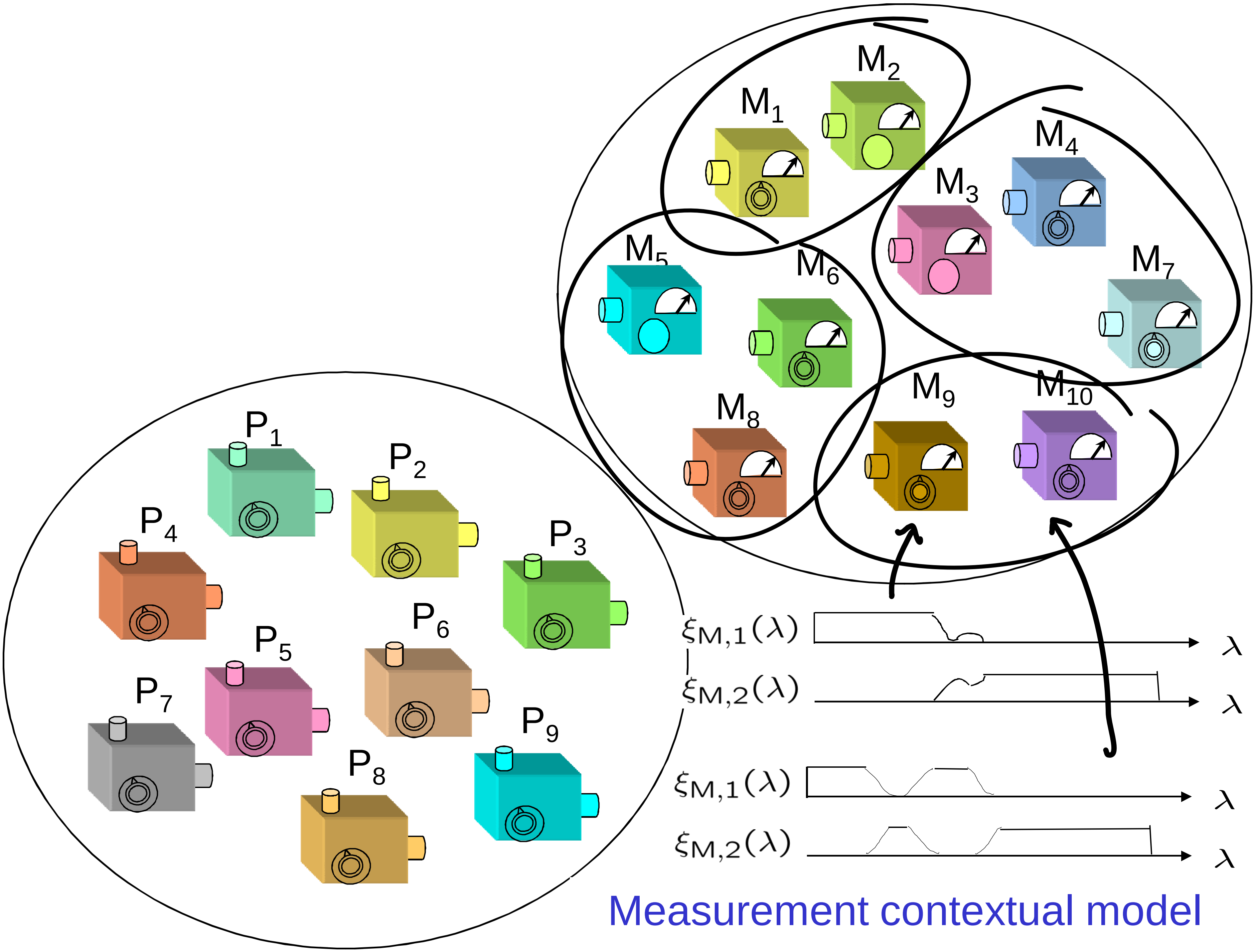

$$\{E, I - E\}$$
$$E = q|\tfrac{\pi}{4}\rangle\langle\tfrac{\pi}{4}| + (1 - q)\frac{1}{2}I$$

$$\{E, I - E\}$$
$$E = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

Example from quantum theory




Measurement
noncontextual model



Measurement contextual model

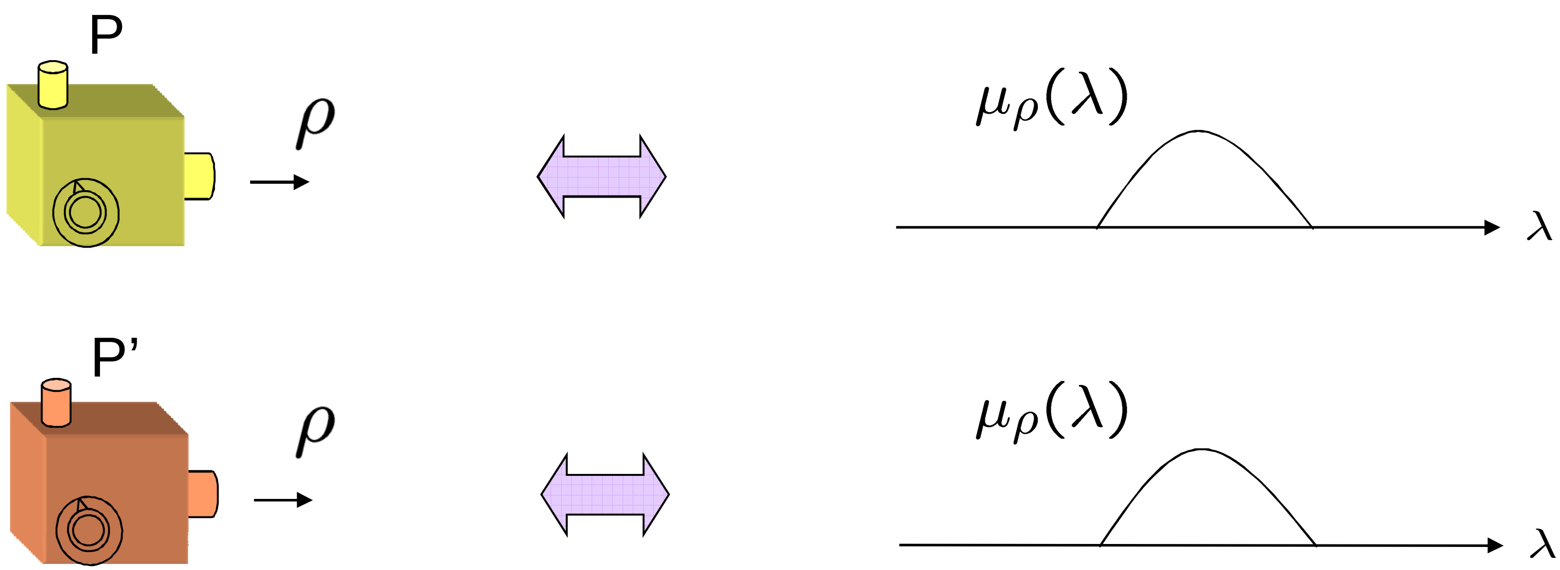
universal noncontextuality
= noncontextuality for preparations *and* measurements

# Generalized noncontextuality in quantum theory

# Defining noncontextuality in quantum theory

## Preparation Noncontextuality in QT

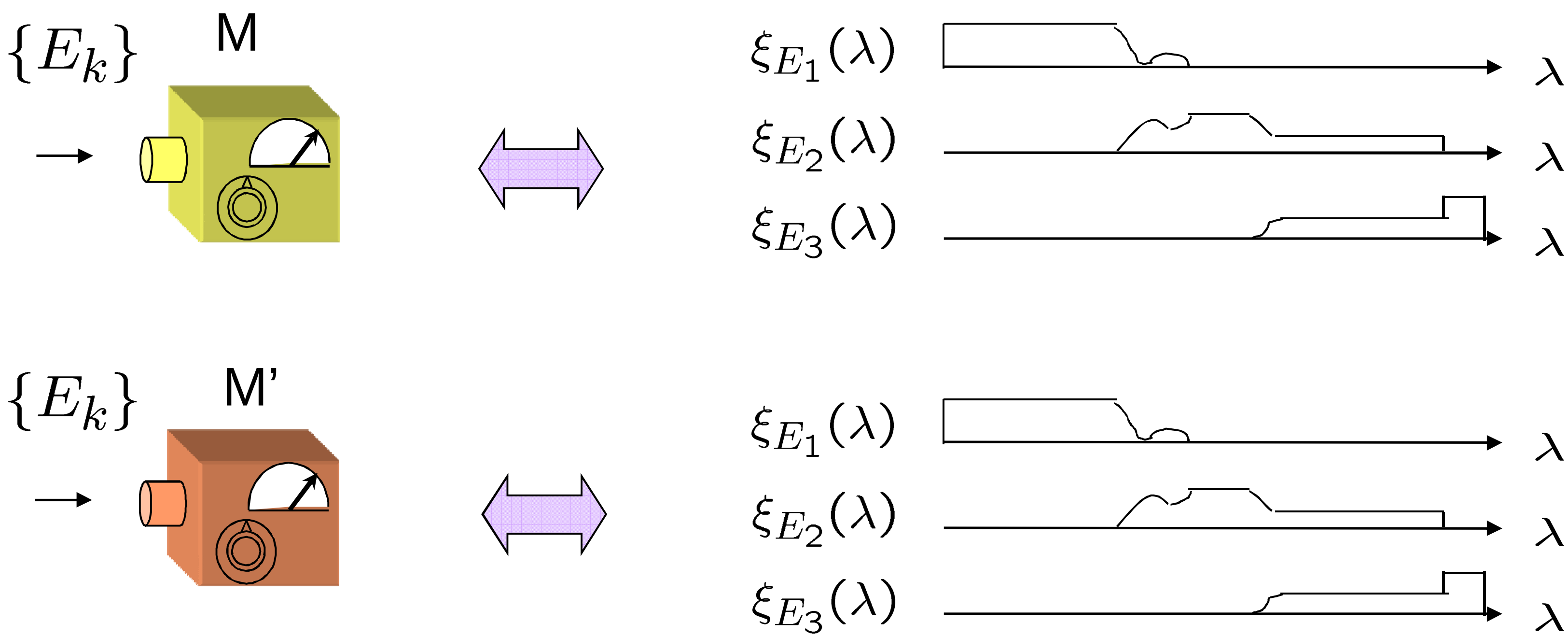
if P, P'$\rightarrow \rho$ then $\mu_{\mathrm{P}}(\lambda) = \mu_{\mathrm{P'}}(\lambda) = \mu_{\rho}(\lambda)$



# Defining noncontextuality in quantum theory

## Measurement Noncontextuality in QT

if M,M$' \to \{E_k\}$ then $\xi_{\mathrm{M},k}(\lambda) = \xi_{\mathrm{M}',k}(\lambda) = \xi_{E_k}(\lambda)$



# Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)

# Important features of realist models

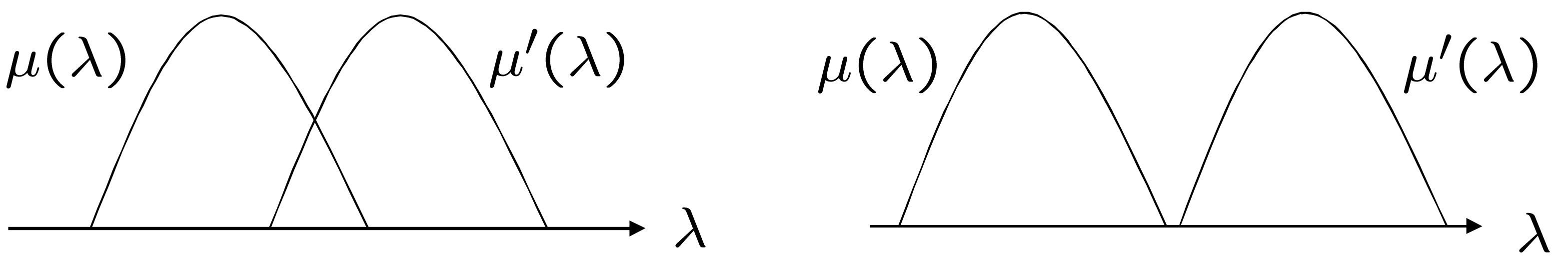
Let P $\leftrightarrow \mu(\lambda)$

P$' \leftrightarrow \mu'(\lambda)$

## Representing one-shot distinguishability:

If P and P$'$ are distinguishable with certainty

then $\mu(\lambda)\,\mu'(\lambda) = 0$




## Representing convex combination:

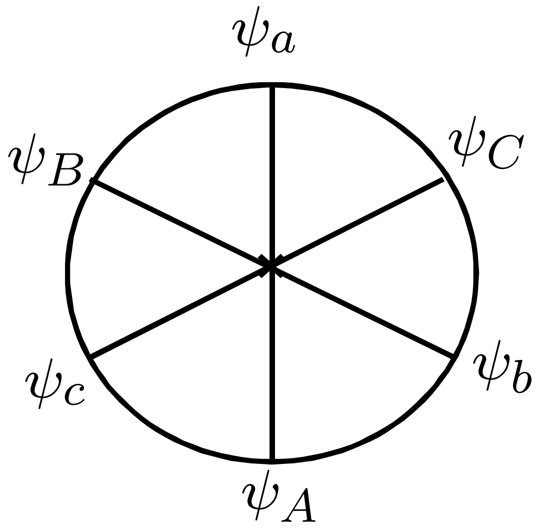
If P$''$ = P with prob. $p$ and P$'$ with prob. $1-p$

Then $\mu''(\lambda) = p\,\mu(\lambda) + (1-p)\,\mu'(\lambda)$

# Proof based on finite construction in 2d

$$\mathsf{P}_a \leftrightarrow \psi_a = (1,0)$$
$$\mathsf{P}_A \leftrightarrow \psi_A = (0,1)$$
$$\mathsf{P}_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2)$$
$$\mathsf{P}_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2)$$
$$\mathsf{P}_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2)$$
$$\mathsf{P}_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)$$

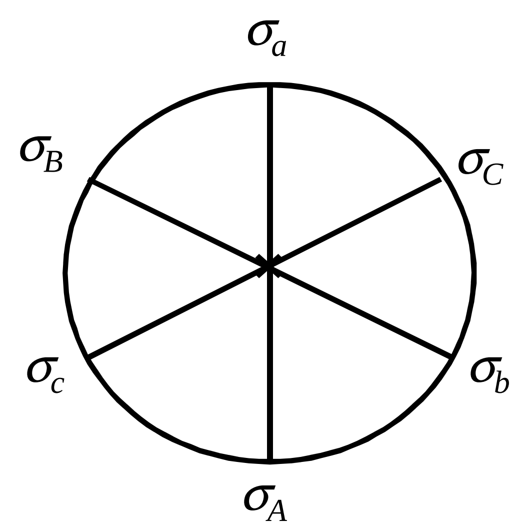
$$\psi_a \perp \psi_A$$
$$\psi_b \perp \psi_B$$
$$\psi_c \perp \psi_C$$



# Proof based on finite construction in 2d

$$
\begin{aligned}
\mathsf{P}_a &\leftrightarrow & \sigma_a &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathsf{P}_A &\leftrightarrow & \sigma_A &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathsf{P}_b &\leftrightarrow & \sigma_b &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
\mathsf{P}_B &\leftrightarrow & \sigma_B &= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \\
\mathsf{P}_c &\leftrightarrow & \sigma_c &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
\mathsf{P}_C &\leftrightarrow & \sigma_C &= \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_a \sigma_A &= 0 \\
\sigma_b \sigma_B &= 0 \\
\sigma_c \sigma_C &= 0
\end{aligned}
$$

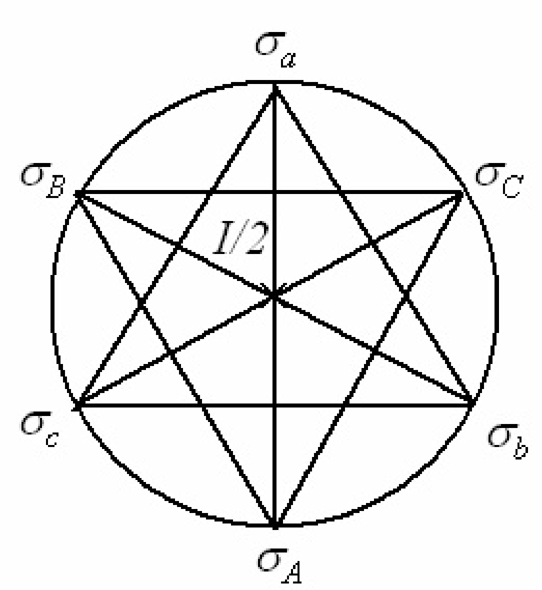



$\mathsf{P}_a$ and $\mathsf{P}_A$ are distinguishable with certainty
$\mathsf{P}_b$ and $\mathsf{P}_B$ are distinguishable with certainty
$\mathsf{P}_c$ and $\mathsf{P}_C$ are distinguishable with certainty

$$
\longrightarrow \quad
\begin{aligned}
\mu_a(\lambda)\,\mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\,\mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\,\mu_C(\lambda) &= 0
\end{aligned}
$$

$$\begin{aligned}
\mathsf{P}_{aA} &\equiv \mathsf{P}_a \text{ and } \mathsf{P}_A \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{bB} &\equiv \mathsf{P}_b \text{ and } \mathsf{P}_B \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{cC} &\equiv \mathsf{P}_c \text{ and } \mathsf{P}_C \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{abc} &\equiv \mathsf{P}_a,\ \mathsf{P}_b \text{ and } \mathsf{P}_c \text{ with prob. } 1/3 \text{ each} \\
\mathsf{P}_{ABC} &\equiv \mathsf{P}_A,\ \mathsf{P}_B \text{ and } \mathsf{P}_C \text{ with prob. } 1/3 \text{ each}
\end{aligned}$$

$$\longrightarrow \quad \begin{aligned}
\mu_{aA}(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
\mu_{bB}(\lambda) &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
\mu_{cC}(\lambda) &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
\mu_{abc}(\lambda) &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
\mu_{ABC}(\lambda) &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)
\end{aligned}$$




$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
&\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
\end{aligned}
$$

By preparation noncontextuality

$$
\begin{aligned}
&\mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
&= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
&\equiv \nu(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

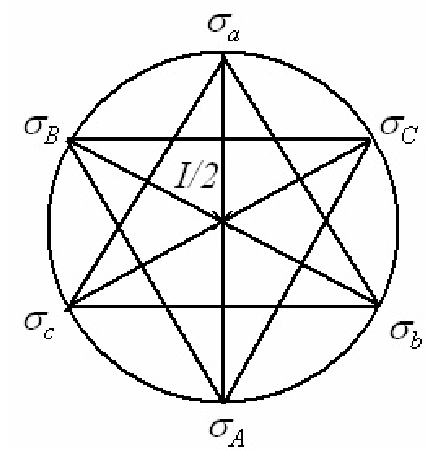
Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda)+\frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda)+\frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda)+\frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda)+\frac{1}{3}\mu_b(\lambda)+\frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda)+\frac{1}{3}\mu_B(\lambda)+\frac{1}{3}\mu_C(\lambda).\end{aligned}$$

i.e., paralleling the quantum structure:

$$\begin{aligned}\sigma_a\sigma_A &= 0\\ \sigma_b\sigma_B &= 0\\ \sigma_c\sigma_C &= 0\end{aligned}$$




$$\begin{aligned}I/2 &= \frac{1}{2}\sigma_a+\frac{1}{2}\sigma_A\\ &= \frac{1}{2}\sigma_b+\frac{1}{2}\sigma_B\\ &= \frac{1}{2}\sigma_c+\frac{1}{2}\sigma_C\\ &= \frac{1}{3}\sigma_a+\frac{1}{3}\sigma_b+\frac{1}{3}\sigma_c\\ &= \frac{1}{3}\sigma_A+\frac{1}{3}\sigma_B+\frac{1}{3}\sigma_C.\end{aligned}$$

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Consider $\lambda'$ such that $\nu(\lambda') \neq 0$

From decompositions (1)-(3)

$$\begin{aligned}\mu_a(\lambda') &= 0 \text{ or } 2\nu(\lambda')\\ \mu_b(\lambda') &= 0 \text{ or } 2\nu(\lambda')\\ \mu_c(\lambda') &= 0 \text{ or } 2\nu(\lambda')\end{aligned}$$

But then the RHS of decomposition (4) is

$$0, \tfrac{2}{3}\nu(\lambda'), \tfrac{4}{3}\nu(\lambda'), 2\nu(\lambda')$$
$$\neq \nu(\lambda')$$

CONTRADICTION

## Example: A “reverse” Gleason theorem for all dimensions

Consider a function on density operators $\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all $\rho$

2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$ and $\sum_k w_k = 1$.

**The "reverse" Gleason's theorem**:

$$f(\rho) = Tr\,(E\rho)$$

for some effect $E$ (i.e. $0 \leq E \leq I$).

Suppose $\rho \leftrightarrow \mu_\rho(\lambda)$ preparation noncontextuality

$\mu_\rho(\lambda) \geq 0$

$\mu_\rho(\lambda)$ is convex-linear in $\rho$

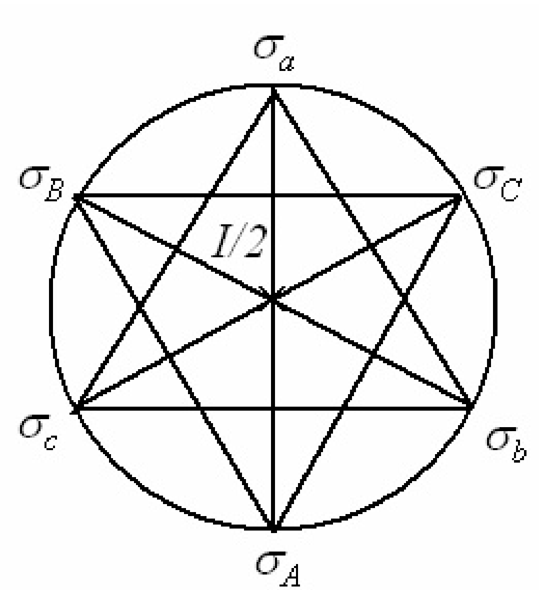
$\mu_\rho(\lambda) = \text{Tr}(\rho E_\lambda)$ for some effect $E_\lambda$

Recall: If $\rho_1 \rho_2 = 0$, then $\mu_{\rho_1}(\lambda)\mu_{\rho_2}(\lambda) = 0$

If one knew $\lambda$, one could retrodict with certainty which state was prepared from an orthogonal basis, for any basis. There is no effect such that finding it would allow one to achieve such a retrodiction.

CONTRADICTION

# Aside: justifying preparation noncontextuality by local causality




$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$

By preparation noncontextuality

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

PNC for *I/2* can be justified by local causality

But PNC for $\sigma_x$ cannot be justified by local causality

Also,

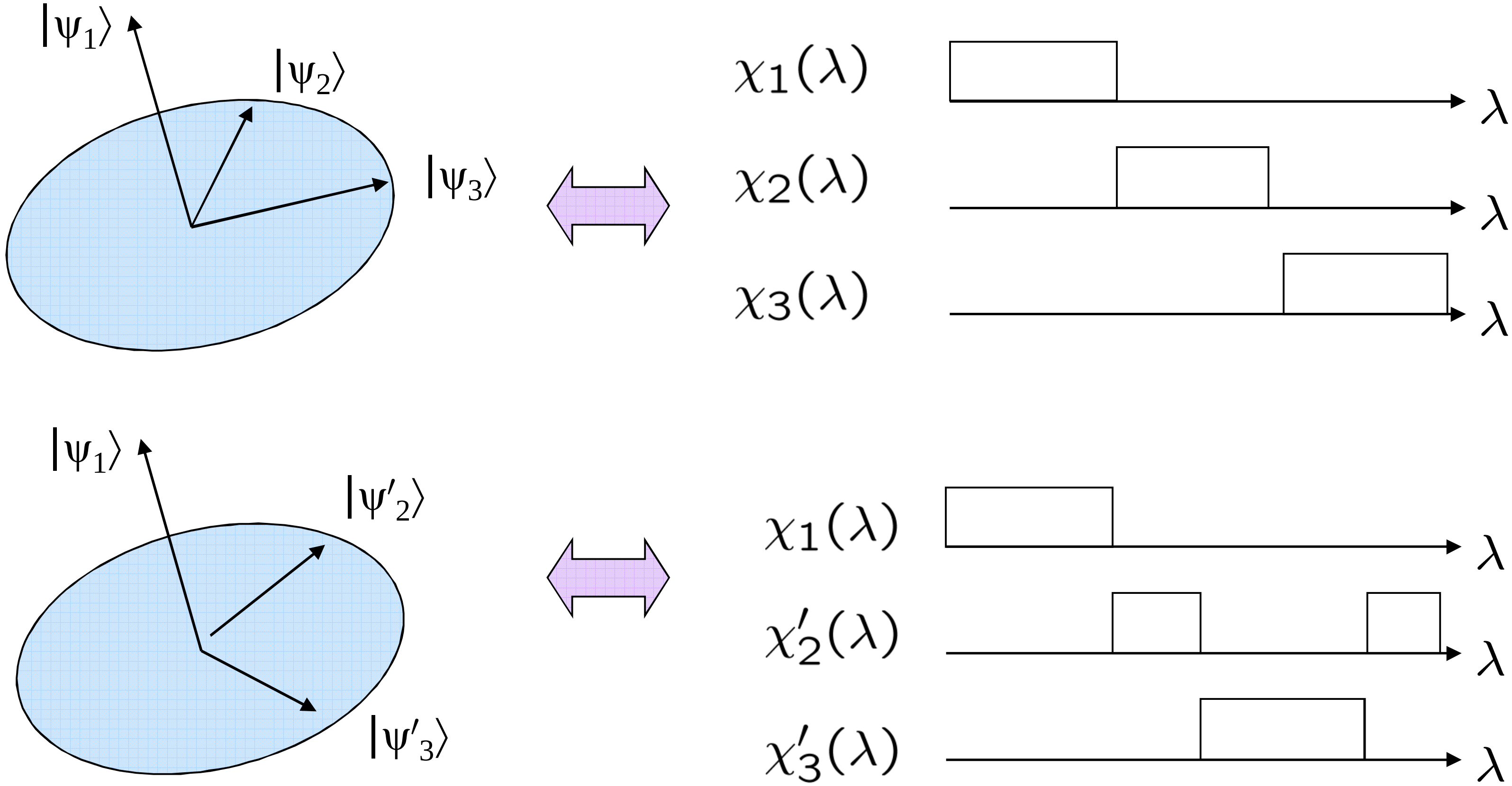
Any bipartite Bell-type proof of nonlocality $\rightarrow$ proof of preparation contextuality

(proof due to Jon Barrett)

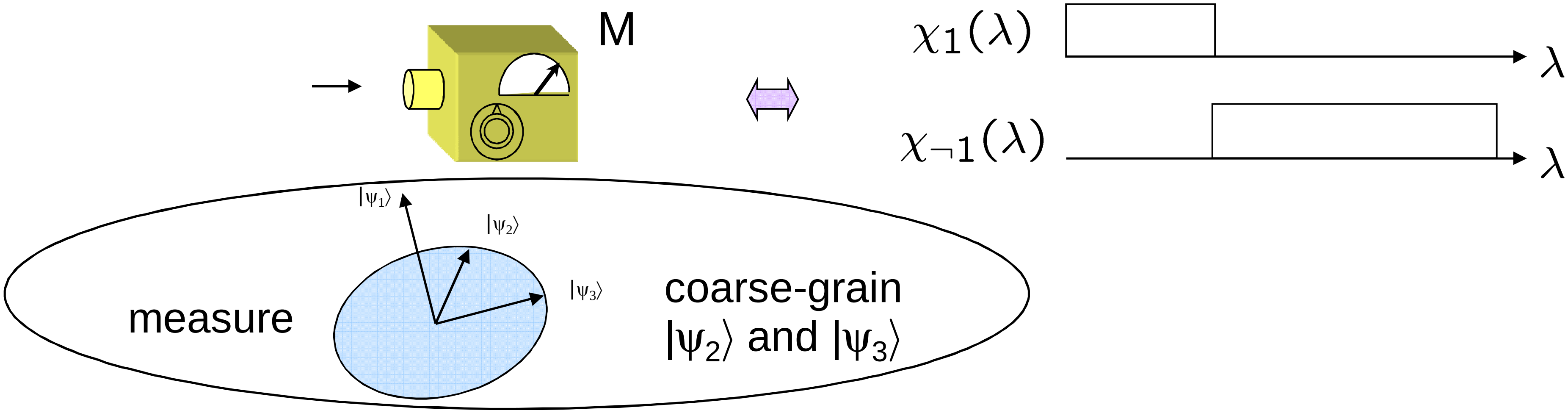
# Measurement contextuality

New definition versus traditional definition

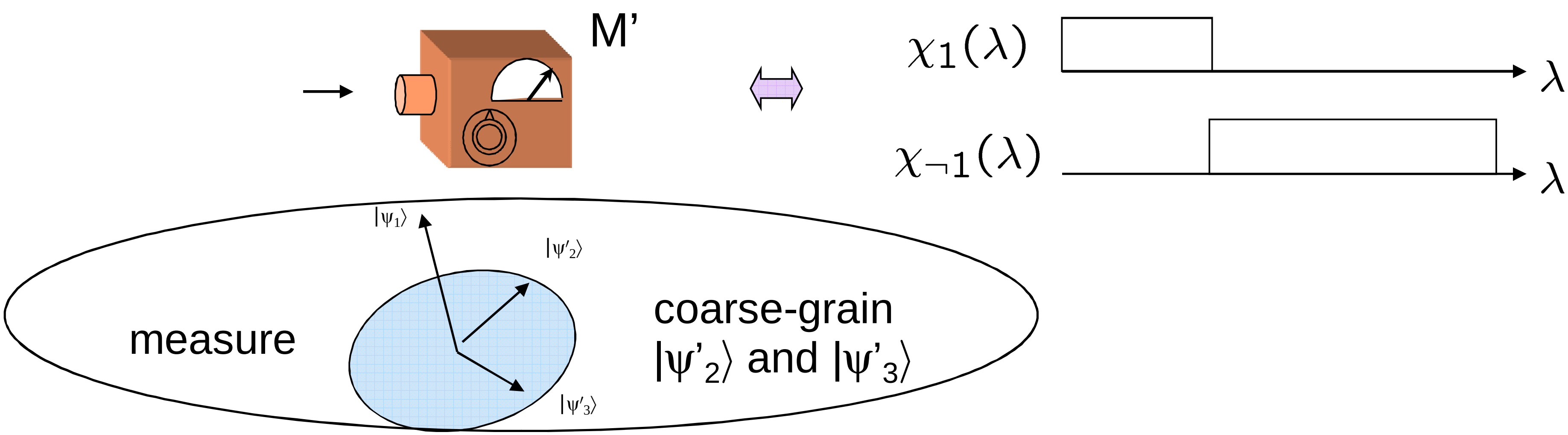
How to formulate the traditional notion of noncontextuality:



# This is equivalent to assuming:

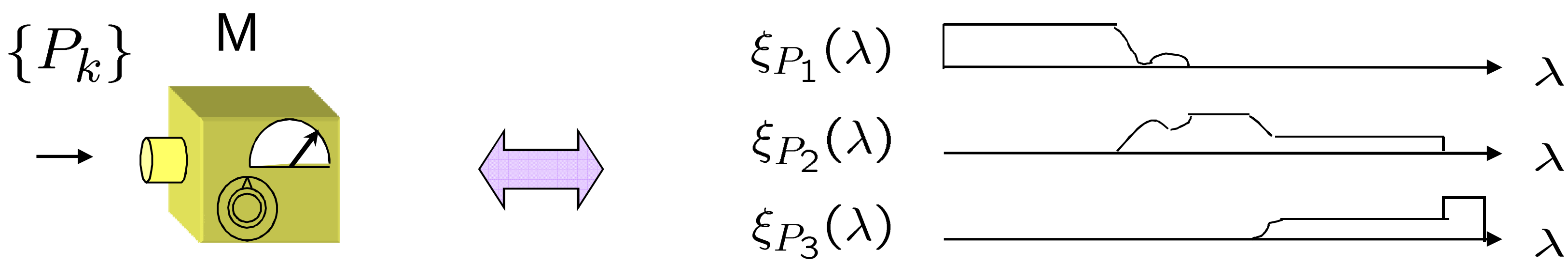



$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$




$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$

But recall that the most general representation was




Therefore:

traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and

outcome determinism for sharp measurements

So, the new definition of noncontextuality is not simply a generalization of the traditional notion

For sharp measurements, it is a revision of the traditional notion

Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

---

Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and $\lambda$)?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and $\lambda$)?

Noncontextuality and determinism are separate issues traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and

outcome determinism for sharp measurements

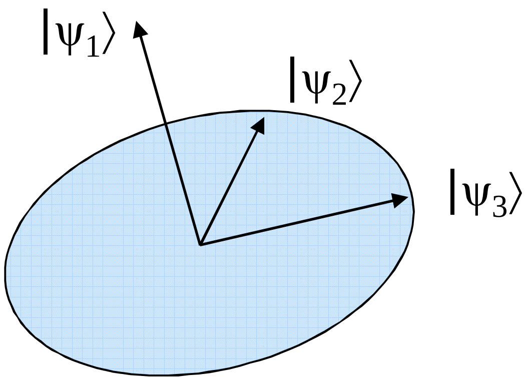
No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

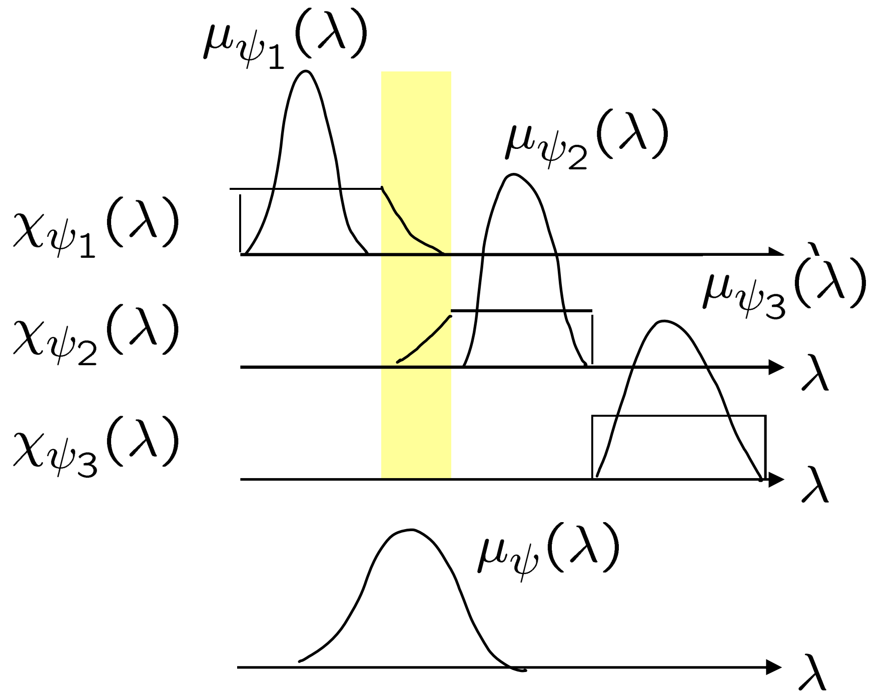
In face of contradiction, could give up ODSM

However, one can prove that

preparation noncontextuality → outcome determinism for sharp measurements

Proof




$$\mu_{I/3}(\lambda) = \tfrac{1}{3}\mu_{\psi_1}(\lambda) + \tfrac{1}{3}\mu_{\psi_2}(\lambda) + \tfrac{1}{3}\mu_{\psi_3}(\lambda)$$

$$\mu_{I/3}(\lambda) = p\mu_{\psi}(\lambda) + \ldots$$

We've established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality
and
preparation noncontextuality
→
measurement noncontextuality
and
outcome determinism for sharp measurements

We've established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality → Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion

… and there are many new proofs

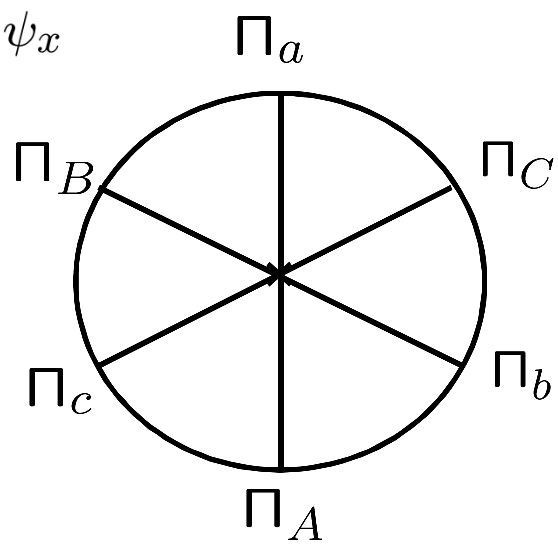
# Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)

# Proof of contextuality for unsharp measurements in 2d

$$\mathsf{M}_a \leftrightarrow \{\Pi_a, \Pi_A\}$$
$$\mathsf{M}_b \leftrightarrow \{\Pi_b, \Pi_B\}$$
$$\mathsf{M}_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

$\Pi_x$ projects onto $\psi_x$




$$\Pi_a + \Pi_A = I$$
$$\Pi_b + \Pi_B = I$$
$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$
$$\Pi_b \Pi_B = 0$$
$$\Pi_c \Pi_C = 0$$

$$\mathsf{M}_a \leftrightarrow \{\chi_a(\lambda), \chi_A(\lambda)\}$$
$$\mathsf{M}_b \leftrightarrow \{\chi_b(\lambda), \chi_B(\lambda)\}$$
$$\mathsf{M}_c \leftrightarrow \{\chi_{c(\lambda)}, \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$
$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$
$$\chi_c(\lambda) + \chi_C(\lambda) = 1$$

By outcome determinism for sharp measurements

$$\chi_a(\lambda)\chi_A(\lambda) = 0$$
$$\chi_b(\lambda)\chi_B(\lambda) = 0$$
$$\chi_c(\lambda)\chi_C(\lambda) = 0$$

Thus, $\{\chi_x(\lambda), \chi_X(\lambda)\}$
$= \{0,1\}$ or $\{1,0\}$ for every $\lambda$.

M $\equiv$ implement one of M$_a$, M$_b$ and M$_c$ with prob. 1/3 each, register only whether first or second outcome ocurred

$$\mathsf{M} \quad \leftrightarrow \quad \{\tfrac{1}{3}\Pi_a + \tfrac{1}{3}\Pi_b + \tfrac{1}{3}\Pi_c, \tfrac{1}{3}\Pi_A + \tfrac{1}{3}\Pi_B + \tfrac{1}{3}\Pi_C\} \; = \{\tfrac{1}{2}I, \tfrac{1}{2}I\}$$

$$\mathsf{M} \quad \leftrightarrow \quad \{\tfrac{1}{3}\chi_a(\lambda)+\tfrac{1}{3}\chi_b(\lambda)+\tfrac{1}{3}\chi_c(\lambda), \tfrac{1}{3}\chi_A(\lambda)+\tfrac{1}{3}\chi_B(\lambda)+\tfrac{1}{3}\chi_C(\lambda)\}$$

$\tilde{\mathsf{M}}$ $\equiv$ ignore the system, flip a fair coin

$$\tilde{\mathsf{M}} \quad \leftrightarrow \quad \{\tfrac{1}{2}I, \tfrac{1}{2}I\}$$

$$\tilde{\mathsf{M}} \quad \leftrightarrow \quad \{\tfrac{1}{2}, \tfrac{1}{2}\}$$

By the assumption of measurement noncontextuality

$$\mathsf{M} \simeq \tilde{\mathsf{M}} \quad \longrightarrow \quad \{\tfrac{1}{3}\chi_a + \tfrac{1}{3}\chi_b + \tfrac{1}{3}\chi_c, \tfrac{1}{3}\chi_A + \tfrac{1}{3}\chi_B + \tfrac{1}{3}\chi_C\} = \{\tfrac{1}{2}, \tfrac{1}{2}\}$$

But $\{0,1\}, \{\tfrac{1}{3}, \tfrac{2}{3}\}, \{1,0\}, \{\tfrac{2}{3}, \tfrac{1}{3}\} \neq \{\tfrac{1}{2}, \tfrac{1}{2}\}$

CONTRADICTION

## Example: A variant of Busch's generalization of Gleason to 2d

Busch, Phys. Rev. Lett. **91**, 120403 (2003)

Consider a function on effects $E \mapsto \omega(E)$, satisfying:

1) $0 \leq \omega(E) \leq 1$ for all $E$
2) $\omega(I) = 1$
3) $\omega(\sum_k w_k E_k) = \sum_k w_k \omega(E_k)$

**Generalized Gleason's theorem**: For all dim$(\mathcal{H})$,

$$\omega(E) = \text{Tr}(\rho E)$$

where $\rho$ is a density operator ($\rho \geq 0$, $\text{Tr}(\rho) = 1$).

$E \mapsto \xi_E(\lambda)$ Measurement noncontextuality

$\xi_E(\lambda)$ is convex linear in *E*

$\xi_E(\lambda)$ considered as a function of *E* satisfies the conditions of the generalized Gleason's theorem

$\xi_E(\lambda) = \mathrm{Tr}(\rho_\lambda E)$ for some density operator $\rho_\lambda$

By outcome determinism for sharp measurements
$\xi_P(\lambda)$ = 0 or 1 for all projectors *P*

But there is no $\rho$ such that $Tr(\rho P)$=0 or 1 for all *P*
(Any given $\rho$ can only achieve a 0-1 valuation on a single basis)

CONTRADICTION

# The mystery of contextuality

There is a tension between

1) the dependence of representation on certain details of the experimental procedure

and

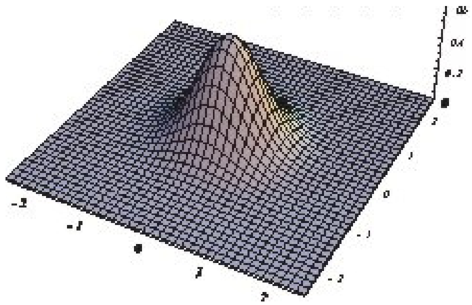
2) the independence of outcome statistics on those details of the experimental procedure

# Noncontextuality and the characterization of classicality

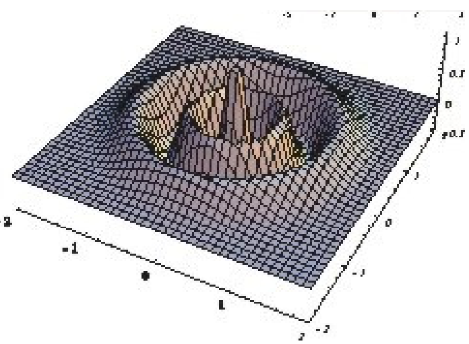
# Classicality as non-negativity

Continuous Wigner function for a harmonic oscillator

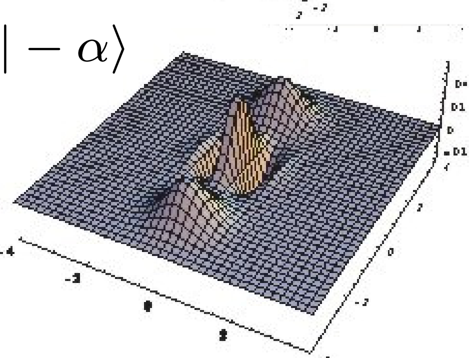
$|0\rangle$



$|4\rangle$



$|\alpha\rangle + |-\alpha\rangle$



From:
qis.ucalgary.ca/quantech/wiggalery.html

Common slogan:
A quantum state is nonclassical if it has a negative Wigner representation

Better to ask whether a quantum experiment admits of a classical explanation

Negativity is not necessary for nonclassicality: the nonclassicality could reveal itself in the negativity of the representation of the measurement rather than the state

Negativity is not sufficient for nonclassicality: When considering possibilities for a classical explanation, we need to look at representations other than that of Wigner

# Quasi-probability representations of QM:

**States**

$$\rho \leftrightarrow \mu_\rho(\lambda)$$

$$\mu_\rho : \Lambda \to \mathbb{R}$$

$$\int \mu_\rho(\lambda) d\lambda = 1$$

**Measurements**

$$\{E_k\} \leftrightarrow \{\xi_{E_k}(\lambda)\}$$

$$\xi_{E_k} : \Lambda \to \mathbb{R}$$

$$\sum_k \xi_{E_k}(\lambda) = 1$$

$$\text{Tr}[\rho E_k] = \int d\lambda\, \mu_\rho(\lambda)\, \xi_{E_k}(\lambda)$$

Examples:

- Wigner representation
- discrete Wigner representation (e.g. Wootters, quant-ph/0306135)
- Q representation of quantum optics
- P representation of quantum optics
- Hardy-type formulation of QM using fiducial measurements
- Hardy-type formulation of QM using fiducial preparations
- …

See Ferrie and Emerson, J. Phys. A 41 352001 (2008)

# Quasi-probability representations of QM:

States

$$\rho \leftrightarrow \mu_\rho(\lambda)$$

$$\mu_\rho : \Lambda \rightarrow \mathbb{R}$$

$$\int \mu_\rho(\lambda) d\lambda = 1$$

Measurements

$$\{E_k\} \leftrightarrow \{\xi_{E_k}(\lambda)\}$$

$$\xi_{E_k} : \Lambda \rightarrow \mathbb{R}$$

$$\sum_k \xi_{E_k}(\lambda) = 1$$

$$\mathrm{Tr}[\rho E_k] = \int d\lambda\, \mu_\rho(\lambda)\, \xi_{E_k}(\lambda)$$

This provides a classical explanation if and only if

$$\mu_\rho(\lambda) \geq 0$$

for all ρ

$$\xi_{E_k}(\lambda) \geq 0$$

for all $\{E_k\}$

## Classicality from nonnegativity, take II:

A quantum experiment is nonclassical if it fails to admit a quasi-probability representation that is nonnegative for all states and measurements

# Quasi-probability representations of QM:

**States**

$$\rho \leftrightarrow \mu_\rho(\lambda)$$

$$\mu_\rho : \Lambda \rightarrow \mathbb{R}$$

$$\int \mu_\rho(\lambda) d\lambda = 1$$

**Measurements**

$$\{E_k\} \leftrightarrow \{\xi_{E_k}(\lambda)\}$$

$$\xi_{E_k} : \Lambda \rightarrow \mathbb{R}$$

$$\sum_k \xi_{E_k}(\lambda) = 1$$

$$\mathrm{Tr}[\rho E_k] = \int d\lambda\, \mu_\rho(\lambda)\, \xi_{E_k}(\lambda)$$

This provides a classical explanation if and only if

$$\mu_\rho(\lambda) \geq 0$$

for all ρ

$$\xi_{E_k}(\lambda) \geq 0$$

for all $\{E_k\}$

Nonnegative quasi-probability representation of QM = Noncontextual ontological model of QM

## Equivalent notions of classicality

# Noncontextuality inequalities and applications of contextuality

# Quantum Spellcraft

*Based on noncontextuality-inequality violation*

Parity-oblivious multiplexing

RS, Buzacott, Keehn, Toner, Pryde, PRL 102, 010401 (2009)

Computational advantages?

Raussendorf, arXiv:0907.5449

Anders and Browne, Phys. Rev. Lett. 102, 050502 (2009)

Secure key distribution?

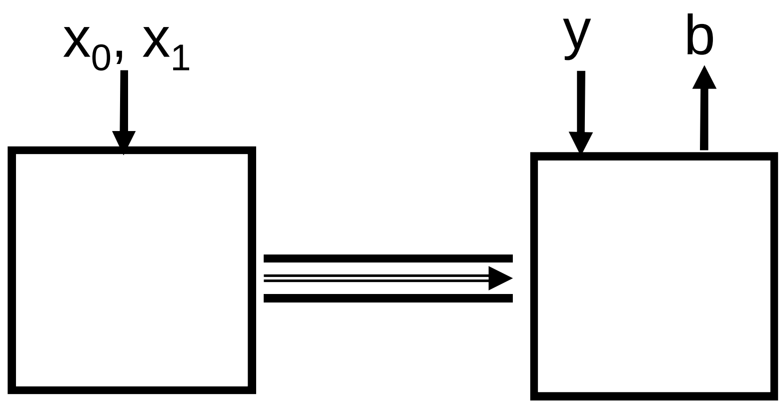
$Horodecki^4$, Pawlowski, Bourennane, arXiv:1002.2410

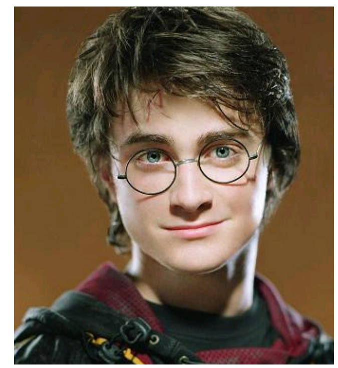
Why isn’t the world *more* contextual?

# The game of parity-oblivious multiplexing

Victor

Alice



Bob

Alice and Bob win if

$b = x_y$

The catch: no information about parity ($x_0 \oplus x_1$) can be conveyed!

Theorem: For all theories admitting a preparation noncontextual model

$$p(b=x_y) \leq 3/4$$

A “noncontextuality inequality”

RWS, Buzacott, Keehn, Toner, Pryde, PRL 102, 010402 (2009)