Graph Theory in Quantum Information

Lecture 2

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$\mathbb{C}$: d^2

$\mathbb{R}$: $\binom{d+1}{2}$

$$\langle P - tI, Q - tI \rangle$$

$$= \langle P, Q \rangle - t\langle P, I \rangle - t\langle I, Q \rangle + t^2 \langle I, I \rangle$$

$$= \alpha^2 - 2t + dt^2$$

$$t = \frac{1}{d}\left[1 \pm \sqrt{1 - d\alpha^2}\right]$$

$d\alpha^2 < 1$

$$I = \sum_{i=1}^{m} c_i P_i$$

$$P_1 = \sum_{i=1}^{m} c_i P_i P_1$$

Take trace:

$$1 = c_1\alpha^2 + \dots + c_m\alpha^2 + (1-\alpha^2)c_1$$

$$\therefore 1 = (c_1 + \dots + c_m)\alpha^2 + (1-\alpha^2)c_1$$

$$1 = (c_1 + \dots + c_m)\alpha^2 + (1-\alpha^2)c_2$$

$$\vdots$$

$$\mathrm{tr}(P_i P_1) = \alpha^2$$

$$\Rightarrow c_1 = c_2 = \dots = c_m$$

$$I = c \sum P_i$$

$$d = c \sum \mathrm{tr}(P_i) = mc$$

$$\Rightarrow c = \frac{d}{m}$$

equiangular lines

$\mathbb{C}$ · d^2

$\mathbb{R}$ · $\binom{d+1}{2}$

If we have d^2 equiangular lines in $\mathbb{C}^d$,

then
$$\sum_{i=1}^{m} P_i = \frac{m}{d} I$$

tight frames

d^2 e/a lines

tight frames

What if we have a tight frame?

Suppose $P_1, \dots, P_m$ are projections onto a set of equiangular lines.

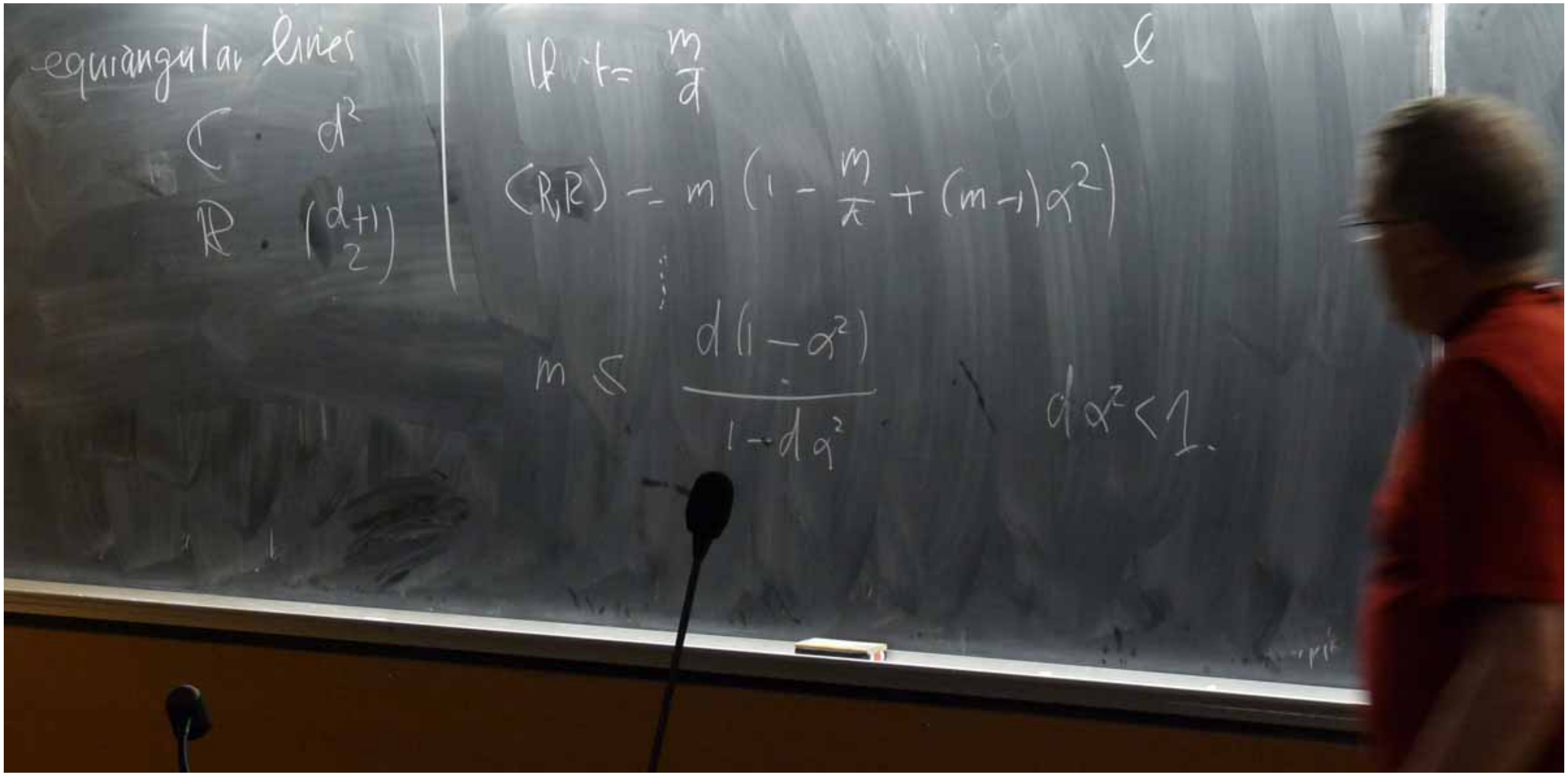
$$R = tI - \sum_{i=1}^{m} P_i$$

$$0 \le \langle R, R \rangle = \langle tI - \sum P_i, tI - \sum P_i \rangle$$

$$= t^2 \langle I, I \rangle - 2t \langle I, \sum P_i \rangle + \langle I, \sum_i P_i^2 \rangle + \langle I, \sum_{i \neq j} P_i P_j \rangle$$

$$= \underbrace{dt^2 - 2mt + m + m(m-1)\alpha^2}_{\text{quadratic in } t}$$

$$t = \frac{m}{d}$$


equiangular lines
ℂ · d²
ℝ · (d+1 choose 2)
If t = m/d
⟨P,P⟩ = m (1 − m/t + (m−1)α²)
m ≤ d(1−α²) / (1−dα²)
dα² < 1.

$\mathbb{R}$ $m \leq \binom{d+1}{2}$

If equality holds, it can be shown that $d+2$ is the square of an odd integer.

ex. $d=7$ $m=28$ — E_7, E_8

$d=23$ 276 — Leech lattice

$\mathbb{C}$ $m \leq d^2$

Choose an initial vector f in $\mathbb{C}^d$, $\|f\| = 1$.

$G = \langle P, D \rangle$ $\quad Pe_i = e_{i+1} \bmod d$ $\qquad$ Grassl.

$De_i = \theta^{i-1} e_i \quad \theta^d = 1, \theta \neq 1$ $\qquad$ Weyl-Heisenberg

I

Mub's

$U_1, \dots, U_m$

$U_i^* U_j$ — all entries have abs. value $\frac{1}{\sqrt{d}}$

flat

generalized Hadamard matrix

$MU_1, \dots, MU_m$

$M^* M = I$

$$\begin{pmatrix} I_d & K & \cdots & K \\ K & \ddots & & \vdots \\ \vdots & & \ddots & K \\ K & \cdots & K & I_d \end{pmatrix} \qquad K = \alpha^2 \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix}}_{d \times d}$$

$$m \leq d+1 \qquad \mathbb{C}.$$

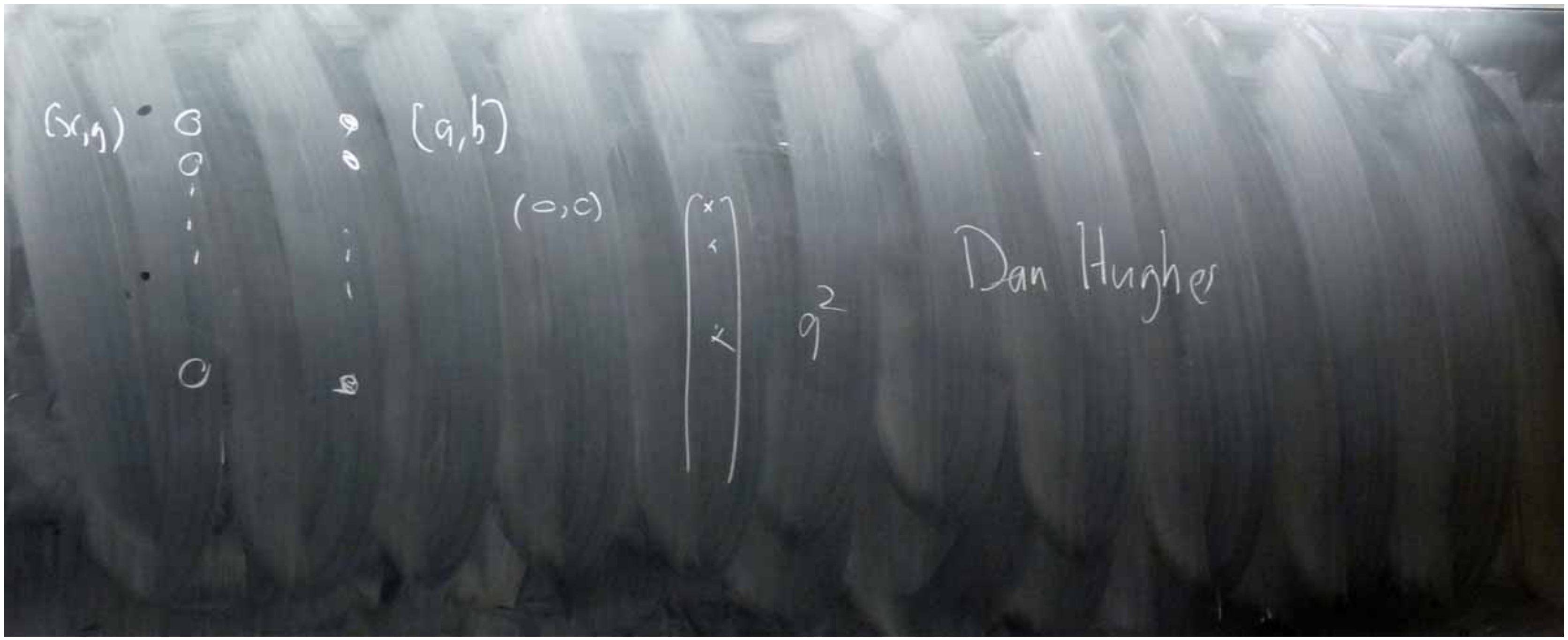
$$\left\lfloor \frac{d}{2} \right\rfloor + 1 \qquad \mathbb{R}.$$

Affine plane graphs

$q^2 \times q^2$

$$2q^2 \; \underset{2q^2}{\left(\begin{array}{c|c} O & B \\ \hline B^T & O \end{array}\right)}$$

Adjacency matrix of our affine plane graph.


(x,y)
(a,b)
(0,c)
Dan Hughes