Introduction to Adiabatic Quantum Computation

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Topics:

1. **Introduction to AQC**
2. **Effect of noise on AQC**
3. **Quantum annealing and quantum phase transition**
4. **Quantum annealing with superconducting qubits**

Quantum Evolution

Schrödinger equation:

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

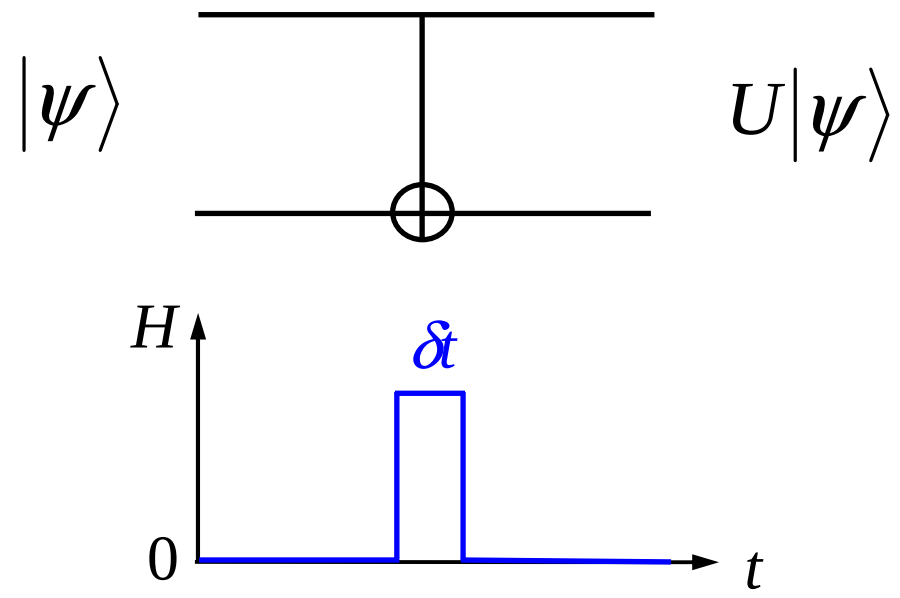
Planck constant

Hamiltonian

Time evolution operator: $|\psi(t)\rangle = U(t)|\psi(0)\rangle$

For time-independent H: $U(t) = e^{-iHt/\hbar}$

Quantum Gates



Hamiltonian is applied only at the time of gate operation

Eigenstates and Eigenvalues:

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle$$

Eigenstate Eigenvalue

Question:

What happens if the system starts in an eigenstate?

$$\left|\psi(0)\right\rangle = \left|\psi_n\right\rangle$$

$$\Rightarrow \qquad \left|\psi_n(t)\right\rangle = e^{-iHt/\hbar}\left|\psi_n\right\rangle = e^{-iE_nt/\hbar}\left|\psi_n\right\rangle$$

Answer: Nothing!

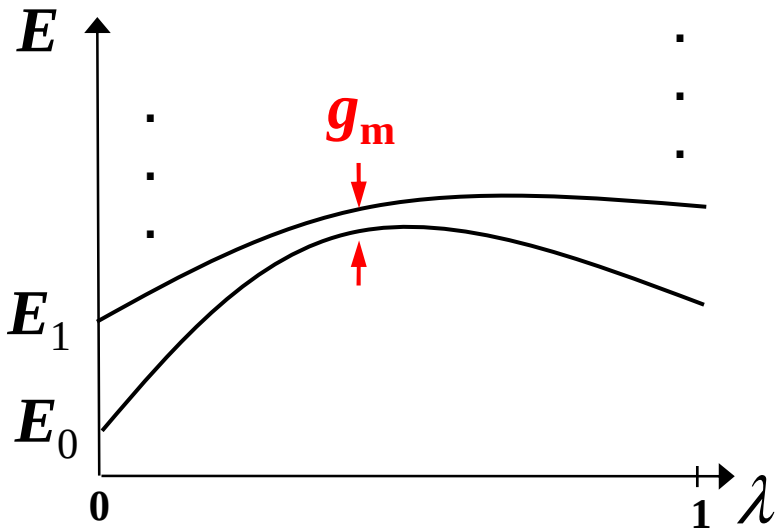
The system will stay in the same eigenstate

Adiabatic Quantum Computation (AQC)

E. Farhi et al., Science 292, 472 (2001)

System Hamiltonian:

$$H = (1-\lambda)\, H_B + \lambda\, H_P$$



- **Prepare the system in the ground state of** H_B
- **Slowly vary** $\lambda(t)$ **from 0 to 1**
- **Read out the final state which is ground state of** H_P

Computation Time

Energy-time uncertainty relation:

$$\delta t \cdot \delta E > h$$

Planck constant

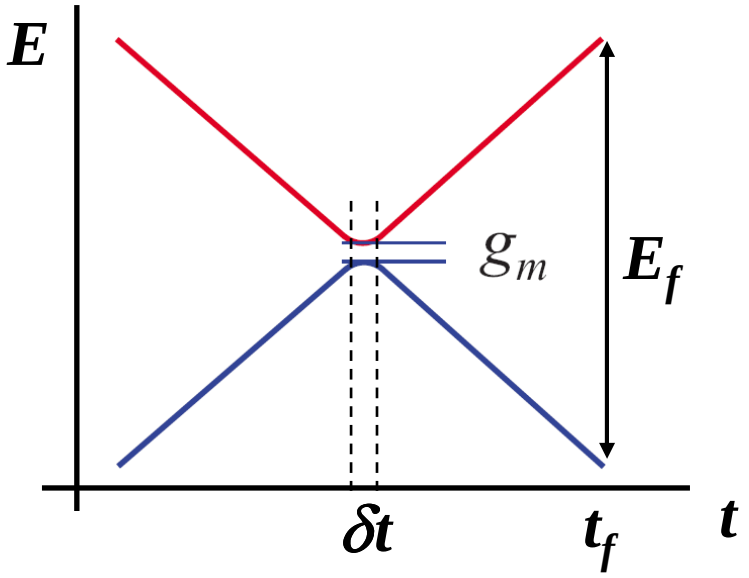
Uncertainty of energy (superposition in energy basis)

Time spent on the energy level

Computation Time

Energy-time uncertainty relation:

$$\delta t \cdot \delta E > h$$



$$\delta E < g_m \quad \Rightarrow \quad \delta t > \frac{h}{g_m}$$

Minimum gap

Uniform time sweep:

$$\frac{\delta t}{t_f} = \frac{g_m}{E_f} \quad \Rightarrow \quad t_f > \frac{hE_f}{g_m^2}$$

Non-uniform time sweep:

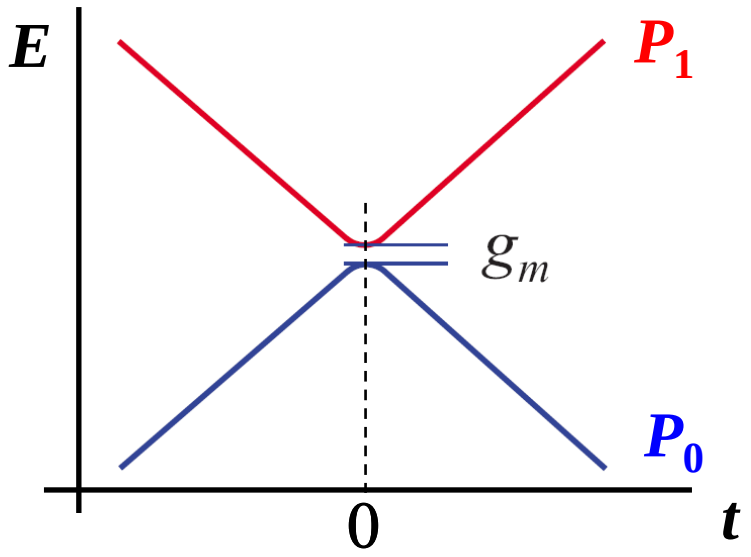
$$t_f \sim \delta t \quad \Rightarrow \quad t_f > \frac{h}{g_m}$$

Optimal

Landau-Zener Problem

2-State Hamiltonian:

$$H = -\frac{1}{2}\left(g_m \sigma_x + \nu t \sigma_z\right)$$



$$P_1(t=-\infty)=0 \quad \Rightarrow \quad P_1(t=+\infty)=e^{-\frac{\pi^2 g_m^2}{h\nu}}$$

Exact solution

$$\nu = \frac{E_f}{t_f} \quad \Rightarrow \quad P_1(t_f) \approx e^{-\frac{\pi^2 g_m^2}{hE_f} t_f} << 1 \quad \Rightarrow \quad \boxed{t_f >> \frac{hE_f}{g_m^2}}$$

Adiabatic Theorem

$$t_f >> \frac{\max_{\lambda\in[0,1]}\left|\langle 0|dH/d\lambda|1\rangle\right|}{g_m^2}$$

Controversy about adiabatic theorem:

K.-P. Marzlin and B. C. Sanders, PRL 93,160408 (2004)

M.S. Sarandy, L.-A. Wu, D.A. Lidar, Quant. Info. Proc. 3, 331 (2004)

D. M. Tong, K. Singh, L. C. Kwek, and C. H. Oh, PRL 95, 110407 (2005)

Z. Wu and H. Yang, PRA 72, 012114 (2005)

S. Duki, H. Mathur, and O. Narayan, PRL 97,128901 (2006)

M.H.S. Amin, PRL 102, 220401 (2009)

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Adiabatic Theorem

$$t_f >> \frac{\max_{\lambda \in [0,1]} \left| \langle 0 | dH / d\lambda | 1 \rangle \right|}{g_m^2}$$

Rigorous versions of adiabatic theorem:

A. Ambainis, O. Regev, arXiv:quant-ph/0411152

S. Jansen, M.-B. Ruskai, and R. Seiler, J. Math. Phys. 48, 102111 (2007)

D.A. Lidar, A.T. Rezakhani, and A. Hamma, J. Math. Phys 50, 102106 (2009)

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Two Types of Computation

1. Universal Adiabatic Quantum Computation

D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev, SIAM J. Comput. 37, 166 (2007)

A. Mizel, D.A. Lidar, M. Mitchell, Phys. Rev. Lett. 99, 070502 (2007)

2. Adiabatic Quantum Optimization

E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda, Science 292, 472 (2001)

Or Quantum Annealing

A. B. Finnila, M. A. Gomez, C. Sebenik, C. Stenson, and J. D. Doll, Chemical Physics Letters 219, 343 (1994)

Universal AQC

Seth Lloyd, arXiv:0805.2757

Consider a quantum algorithm with n gates represented by unitary operators: $U_1, U_2, \ldots, U_n$

After *l* gates: $|\psi_l\rangle = U_l \ldots U_2 U_1 |\psi_0\rangle$

Therefore $|\psi_{l+1}\rangle = U_{l+1}|\psi_l\rangle$ **and** $|\psi_{l-1}\rangle = U_l^{-1}|\psi_l\rangle$

Solution: $|\psi_{n-1}\rangle = U_{n-1} \ldots U_2 U_1 |\psi_0\rangle$

Reset: $|\psi_n\rangle = U_n |\psi_{n-1}\rangle = |\psi_0\rangle$

Write down a Hamiltonian with $|\psi_{n-1}\rangle$ being in its ground state

Pointer or Clock Qubits

Define orthogonal states: $\left|l\right\rangle, \quad l = 0, \ldots, n$

They can be represented by n additional qubits:

$$\left|0\right\rangle = 0...0001$$
$$\left|1\right\rangle = 0...0010$$
$$\left|2\right\rangle = 0...0100$$
$$\ldots$$
$$\left|n-1\right\rangle = 1...0000$$
$$\left|n\right\rangle = \left|0\right\rangle$$

Hamiltonian

$$H_P = -\sum_{l=0}^{n-1}\left[U_{l+1}\otimes\left|l+1\right\rangle\left\langle l\right| + U_{l+1}^{-1}\otimes\left|l\right\rangle\left\langle l+1\right|\right]$$

Define states $$\left|\Psi_k\right\rangle = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{i2\pi kl/n}\left|\psi_l\right\rangle\otimes\left|l\right\rangle$$

$$H_P\left|\Psi_k\right\rangle = -\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{i2\pi kl/n}\left|\psi_{l+1}\right\rangle\otimes\left|l+1\right\rangle - \frac{1}{\sqrt{n}}\sum_{l=1}^{n} e^{i2\pi kl/n}\left|\psi_{l-1}\right\rangle\otimes\left|l-1\right\rangle$$

$$= -\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left(e^{i2\pi k(l-1)/n} + e^{i2\pi k(l+1)/n}\right)\left|\psi_l\right\rangle\otimes\left|l\right\rangle$$

$$= -\left(e^{i2\pi k/n} + e^{-i2\pi k/n}\right)\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{i2\pi kl/n}\left|\psi_l\right\rangle\otimes\left|l\right\rangle$$

$$= -2\cos\left(2\pi k/n\right)\left|\Psi_k\right\rangle$$

Eigenstates of H

Ground State

$$\Rightarrow \quad H_P \left|\Psi_k\right\rangle = E_k \left|\Psi_k\right\rangle$$

Eigenstates: $$\left|\Psi_k\right\rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} e^{i 2\pi k l / n} \left|\psi_l\right\rangle \otimes \left|l\right\rangle$$

Eigenvalues: $$E_k = -2\cos(2\pi\, k / n)$$

Ground state: $$k = 0 \quad \Rightarrow \quad E_0 = -2$$

First excited state: $$k = 1 \quad \Rightarrow \quad E_1 = -2\cos\frac{2\pi}{n}$$

Energy gap: $$E_1 - E_0 = 2\left(1 - \cos\frac{2\pi}{n}\right) \approx \frac{2\pi^2}{n^2}$$

Adiabatic Evolution

$$H = (1-\lambda)H_B + \lambda H_P$$

Initial Hamiltonian:

$$H_B = -\left|l=0\right\rangle\left\langle l=0\right| + \eta\left(1-\left|\psi_0\right\rangle\left\langle\psi_0\right|\right)\otimes\left|l=0\right\rangle\left\langle l=0\right|$$

Final Hamiltonian:

$$H_P = -\sum_{l=0}^{n-1}\left[U_{l+1}\otimes\left|l+1\right\rangle\left\langle l\right| + U_{l+1}^{-1}\otimes\left|l\right\rangle\left\langle l+1\right|\right]$$

Minimum gap happens at $\lambda = 1$, therefore $g_m \approx \dfrac{2\pi^2}{n^2}$

Computation time ~ n^4 **n = # of gates**

Universality

AQC is computationally equivalent to gate model quantum computation

They can be mapped to each other with polynomial overhead

Practicality

$$H_P = -\sum_{l=0}^{n-1}\left[U_{l+1}\otimes\left|l+1\right\rangle\left\langle l\right| + U_{l+1}^{-1}\otimes\left|l\right\rangle\left\langle l+1\right|\right]$$

2-qubit interaction **+** **2-qubit interaction**

= **4-qubit interaction** **Not practical!**

Using perturbation it can be reduce to 2-qubit interactions

J. Kempe, A. Kitaev, and O. Regev, SIAM J. Computing 35(5), 1070 (2006)

Only XZ coupling or XX + ZZ couplings is enough

J.D. Biamonte and P.J. Love, Phys. Rev. A 78, 012352 (2008)

H_p is not diagonal in computation basis

Adiabatic Quantum Optimization

Hamiltonian:

$$H = (1-\lambda)\, H_B + \lambda\, H_P$$

Off-diagonal (in computation basis)

Diagonal (in computation basis)

The final state is a classical state that minimizes the energy of H_p

Adiabatic Grover Search

Unstructured search problem:

Find state m in a data base containing $N = 2^n$ entries with no structure

Classical search algorithm: $t_f = O(N)$

Grover search algorithm $t_f = O(\sqrt{N})$

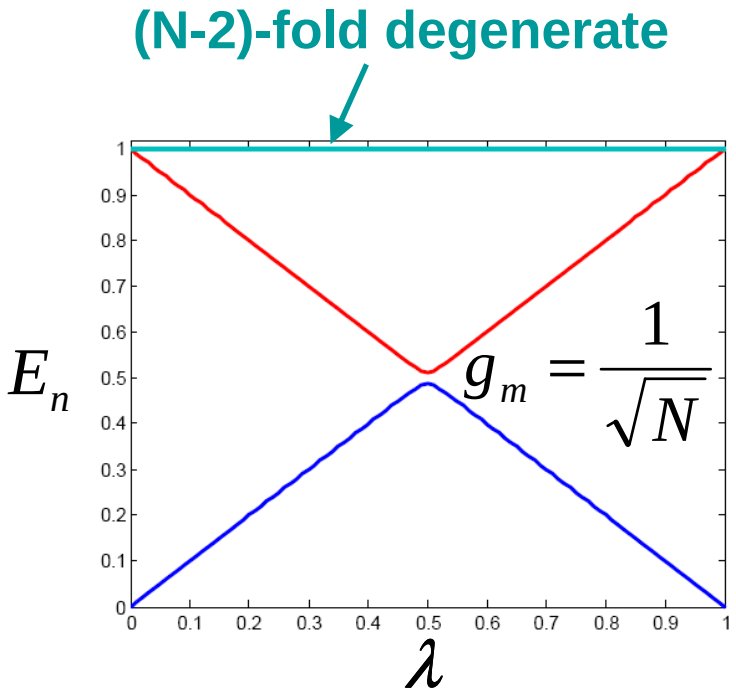
Adiabatic Grover Search

*J. Roland and N. Cerf, PRA **65**, 042308 (2002)*

$$H = (1-\lambda)\, H_B + \lambda\, H_P$$

$$H_B = 1 - |+\rangle\langle +| \qquad H_P = 1 - |m\rangle\langle m|$$

$$|+\rangle = \frac{1}{\sqrt{N}} \sum_{z=0}^{N-1} |z\rangle, \qquad N = 2^n$$



Linear time sweep: $t_f \sim \frac{1}{g_m^2} \sim N$ **Classical speed**

Optimal time sweep: $t_f \sim \frac{1}{g_m} \sim \sqrt{N}$ **Grover speed**

Not practical!

Ising Problem

Find a set of $s_i = \pm 1$ that minimizes

$$F(\vec{s}) = \sum_{i=1}^{n} h_i s_i + \sum_{i,j=1}^{n} J_{ij} s_i s_j, \qquad \vec{s} = [s_1, s_2, \ldots, s_n]$$

Ising problem is NP-Hard

Ising Hamiltonian

Find a set of $s_i = \pm 1$ that minimizes

$$F(\vec{s}) = \sum_{i=1}^{n} h_i s_i + \sum_{i,j=1}^{n} J_{ij} s_i s_j, \qquad \vec{s} = [s_1, s_2, \ldots, s_n]$$

Hamiltonian: $$H = (1-\lambda)\, H_B + \lambda\, H_P$$

$$H_P = \sum_{i=1}^{n} h_i \sigma_i^z + \sum_{i,j=1}^{n} J_{ij} \sigma_i^z \sigma_j^z$$

Diagonal (in σ_z basis)

$$H_B = -\Delta \sum_{i=1}^{n} \sigma_i^x$$

Off-diagonal (transverse)

Open Question

How does computation time scale with n?

G. Rose et al., arXiv:1006.4147