Effect of Environment on Adiabatic Quantum Computation

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Some Common Statements

- Decoherence makes qubits classical bits
- Quantum information is lost after decoherence time
- Computation should be done within decoherence time

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Wrong!!!

Some apply to gate model computation

Open Quantum Sysytem

$$H = H_S + H_{\text{env}} + H_{\text{int}}$$

System

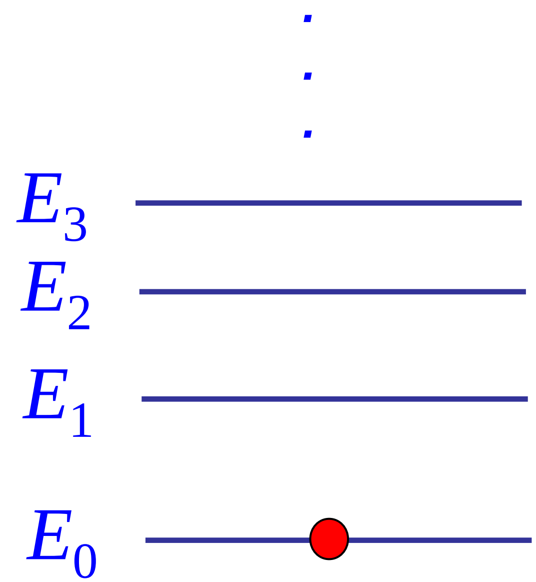
Interaction

Environment (Bath)

Open Quantum Sysytem

$$H = H_S + H_{\text{env}} + H_{\text{int}}$$

$$H_S|\psi_n\rangle = E_n|\psi_n\rangle$$



Question:

Assume weak coupling to environment and $E_n - E_0 >> T$, what happens if the system starts in the ground state?

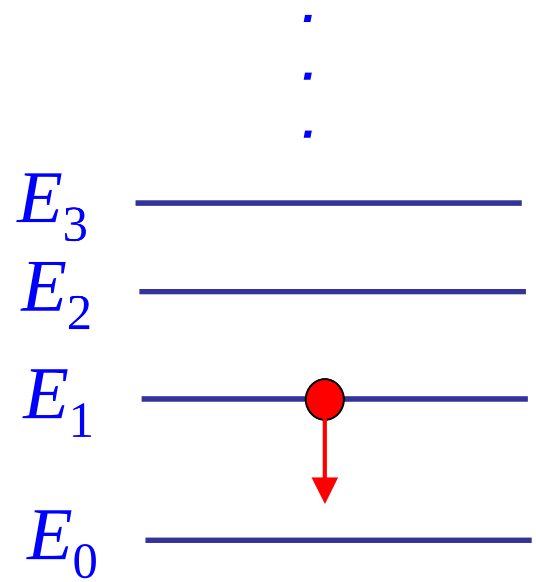
Answer:

System will remain in the ground state with highest probability.

Open Quantum Sysytem

$$H = H_S + H_{\text{env}} + H_{\text{int}}$$

$$H_S |\psi_n\rangle = E_n |\psi_n\rangle$$



Question:

Assume weak coupling to environment and $E_n - E_0 >> T$, what happens if the system starts in an excited state?

Answer:

System will relax to the ground state.

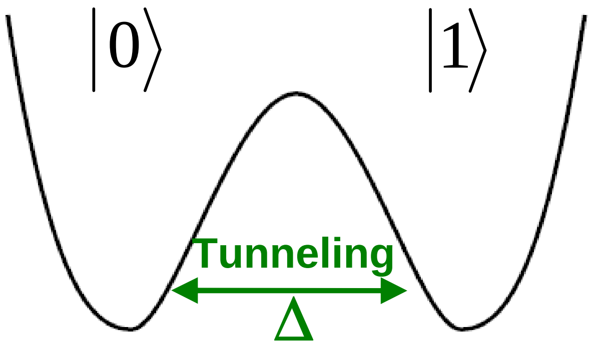
Quantum Coherence

Single qubit Hamiltonian: $H = -\tfrac{1}{2}\Delta\sigma_x$

Logical basis:

$$\sigma_z|0\rangle = -|0\rangle$$

$$\sigma_z|1\rangle = |1\rangle$$

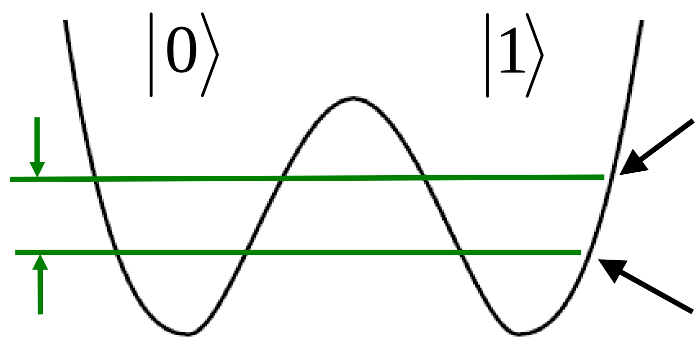


Quantum Coherence

Single qubit Hamiltonian: $H = -\tfrac{1}{2}\Delta\sigma_x$

Eigenstates and Eigenvalues: $\left|\pm\right\rangle = \dfrac{\left|0\right\rangle \pm \left|1\right\rangle}{\sqrt{2}}, \qquad E_{\pm} = \mp\dfrac{\Delta}{2}$

Energy gap = Δ



$$\left|-\right\rangle = \frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}$$

$$\left|+\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}$$

Quantum Coherence

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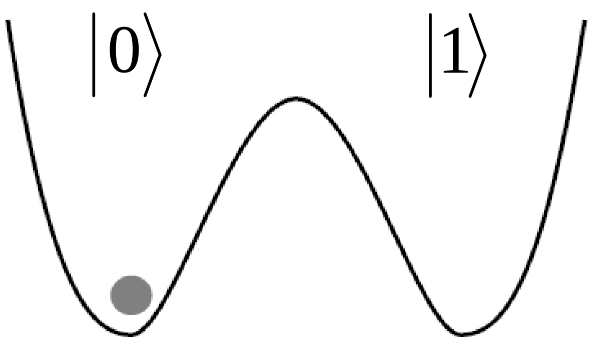
Initializing in state "0": $\left|\psi(t=0)\right\rangle = \left|0\right\rangle = \dfrac{\left|+\right\rangle + \left|-\right\rangle}{\sqrt{2}}$



Quantum Coherence

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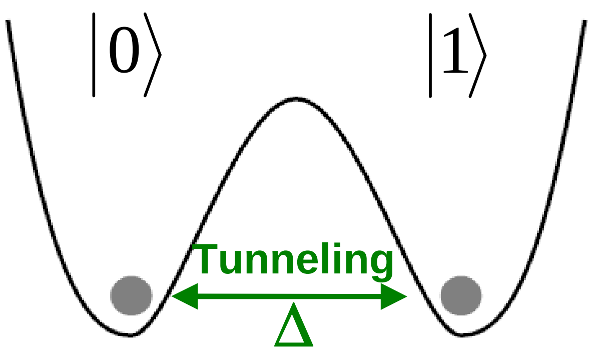
$$\left|\psi(t)\right\rangle = \frac{e^{i\Delta t/2\hbar}\left|+\right\rangle + e^{-i\Delta t/2\hbar}\left|-\right\rangle}{\sqrt{2}} = \cos\frac{\Delta t}{2\hbar}\left|0\right\rangle + i\sin\frac{\Delta t}{2\hbar}\left|1\right\rangle$$



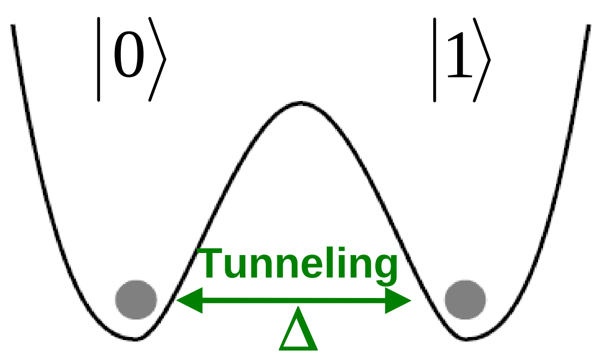
Quantum Coherence

Single qubit Hamiltonian: $H = -\tfrac{1}{2}\Delta\sigma_x$

Probability of finding the qubit in state "0":

$$P_0(t) = \left|\langle 0 | \psi(t) \rangle\right|^2 = \frac{1}{2}(1 + \cos \Delta t / \hbar)$$

Coherent Oscillations



What happens if there is an environment?

Open Quantum System

Single qubit Hamiltonian: $H = -\frac{1}{2}\Delta\sigma_x + H_{\text{int}} + H_{\text{env}}$

Eigenstates and Eigenvalues:

$$\left|\pm\right\rangle \approx \frac{\left|0\right\rangle \pm \left|1\right\rangle}{\sqrt{2}}, \qquad E_\pm = \mp\frac{\Delta}{2} + \delta E_\pm$$

Uncertainty in Energy

Energy-time uncertainty relation: $\delta t \cdot \delta E > h$

Relaxation time *T_1*

Dephasing Time

Wave function: $$\left|\psi(t)\right\rangle \approx \frac{\left|+\right\rangle + e^{-i(\Delta+\delta E)t/\hbar}\left|-\right\rangle}{\sqrt{2}}$$

Uncertainty in phase: $$\delta\varphi = \delta E \cdot t/\hbar$$

Phase information is lost within $$t \sim \hbar/\delta E$$

Dephasing time T_2

$$\boxed{\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_\varphi}}$$

Pure dephasing
due to classical noise

Density Matrix Approach

Pure state density matrix: $\rho = |\psi\rangle\langle\psi|$

All information can be extracted from the density matrix:

$$P_a = |\langle a|\psi\rangle|^2 = \langle a|\rho|a\rangle \qquad \langle\psi|A|\psi\rangle = \mathrm{Tr}[\rho A]$$

Time evolution of the density matrix:

Schrodinger equation → Liouville equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho]$$

Quantum mechanics can be formulated in terms of the density matrix

Pure State vs. Mixed State

Pure state density matrix: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\rho = |\psi\rangle\langle\psi| = |\alpha|^2|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

$$= \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

Nonzero off-diagonal elements represent quantum superposition

Mixed state density matrix:

$$\rho = P_0|0\rangle\langle 0| + P_1|1\rangle\langle 1| = \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix}$$

Classical probabilities

Open Quantum System

System　Bath　Interaction

Hamiltonian: $$H = H_S + H_B + H_{\mathrm{int}}$$

Total (system + bath) density matrix: $$\rho_{SB} = \left|\psi_{SB}\right\rangle\left\langle\psi_{SB}\right|$$

Reduced (system) density matrix: $$\rho_S = \mathrm{Tr}_B[\rho_{SB}]$$

$$\mathrm{Tr}_B[x] = \sum_n \left\langle \Psi_{Bn} \middle| x \middle| \Psi_{Bn} \right\rangle$$

Environment eigenfunctions

Entanglement with Environment

$$|\psi_S\rangle = \alpha|0\rangle + \beta|1\rangle \quad \rightarrow \quad |\psi_{SB}\rangle = \alpha|0\rangle \otimes |\psi_{B0}\rangle + \beta|1\rangle \otimes |\psi_{B1}\rangle$$

≈1 ≈0

$$\rho_S = \mathrm{Tr}_B\left[|\psi_{SB}\rangle\langle\psi_{SB}|\right] = \begin{pmatrix} |\alpha|^2 \mathrm{Tr}_B\left[|\psi_{B0}\rangle\langle\psi_{B0}|\right] & \alpha\beta^* \mathrm{Tr}_B\left[|\psi_{B0}\rangle\langle\psi_{B1}|\right] \\ \alpha^*\beta \mathrm{Tr}_B\left[|\psi_{B1}\rangle\langle\psi_{B0}|\right] & |\beta|^2 \mathrm{Tr}_B\left[|\psi_{B1}\rangle\langle\psi_{B1}|\right] \end{pmatrix}$$

≈0 ≈1

$$\boxed{\text{if } \langle\psi_{B0}|\psi_{B1}\rangle \approx 0}$$

$$\Rightarrow \qquad \rho_S = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

Decoherence

Entanglement with environment decoheres the qubit and generates mixed state

Coherent Oscillations

$$\left|\psi(t)\right\rangle = \frac{e^{i\Delta t/2}\left|+\right\rangle + e^{-i\Delta t/2}\left|-\right\rangle}{\sqrt{2}} = \cos\frac{\Delta t}{2}\left|0\right\rangle + i\sin\frac{\Delta t}{2}\left|1\right\rangle$$

Density matrix in computation basis:

$$\rho = \left|\psi\right\rangle\left\langle\psi\right| = \begin{pmatrix} \cos^2\frac{\Delta t}{2} & -\frac{i}{2}\sin\Delta t \\ \frac{i}{2}\sin\Delta t & \sin^2\frac{\Delta t}{2} \end{pmatrix}$$

Density matrix in energy basis: $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{i\Delta t} \\ \frac{1}{2}e^{-i\Delta t} & \frac{1}{2} \end{pmatrix}$

Diagonal part of the density matrix does not change

Off-diagonal part oscillates

Open System Oscillations

Density matrix in energy basis (weak coupling limit):

Relaxation (T_1) process

Dephasing (T_2) process

$$\rho = \begin{pmatrix} P_+^{eq} + (\frac{1}{2} - P_+^{eq})e^{-t/T_1} & \frac{1}{2} e^{i\Delta t} e^{-t/T_2} \\ \frac{1}{2} e^{-i\Delta t} e^{-t/T_2} & P_-^{eq} + (\frac{1}{2} - P_-^{eq})e^{-t/T_1} \end{pmatrix}$$

$$\rho \xrightarrow{t\to\infty} \begin{pmatrix} P_+^{eq} & 0 \\ 0 & P_-^{eq} \end{pmatrix} \qquad P_\pm^{eq} = \frac{e^{-E_\pm/T}}{e^{-E_+/T} + e^{-E_-/T}}$$

Equilibrium (Boltzmann) Distribution

Is this a completely classical state?

Equilibrium State

Density matrix in energy basis:

$$\rho = \begin{pmatrix} P_+^{eq} & 0 \\ 0 & P_-^{eq} \end{pmatrix}$$

Density matrix in computation basis ("0" , "1"):

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & P_+^{eq} - P_-^{eq} \\ P_+^{eq} - P_-^{eq} & 1 \end{pmatrix}$$

Signature of coherent mixture

ρ **is diagonal in logical basis only if**

$$P_+^{eq} = P_-^{eq} = \tfrac{1}{2} \quad \text{i.e.,} \quad T >> \Delta$$

Quantum Superposition

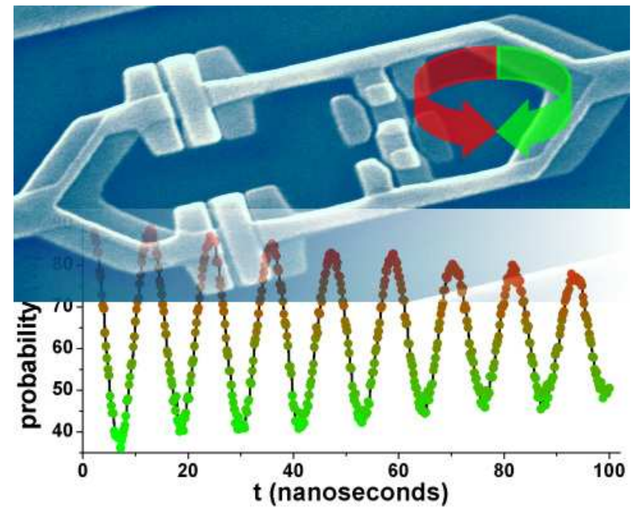
(coherent mixture)

can persist in equilibrium

Coherent Tunneling

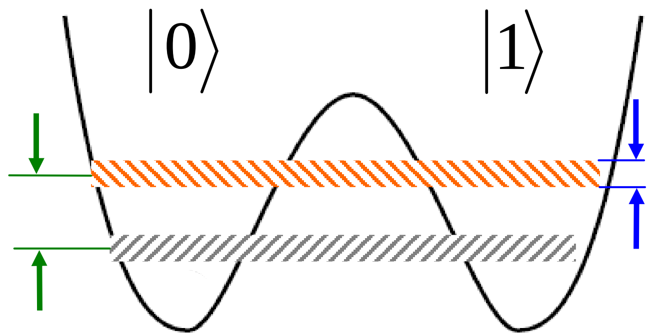
Coherent oscillations

$$P_0(t) = \langle 0 | \rho(t) | 0 \rangle = \tfrac{1}{2}(1 + e^{-\gamma t} \cos \Delta t),$$



Decoherence rate $\gamma = 1/T_2 < \Delta$

Energy gap = Δ



γ = broadening

Well-defined gap

Incoherent Tunneling

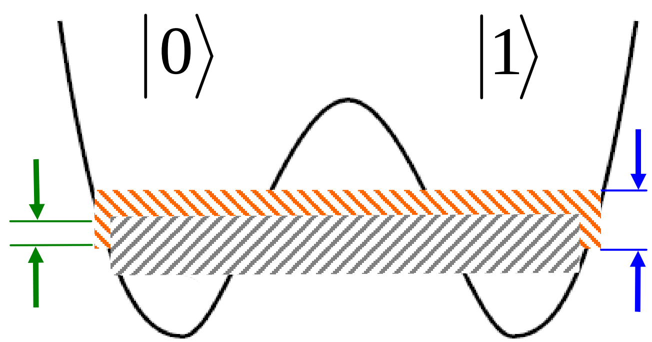
If decoherence rate $\gamma = 1/T_2 > \Delta$

$$P_0(t) = \tfrac{1}{2}(1 + e^{-\Gamma t}),$$

$$\boxed{\Gamma = \frac{\Delta^2}{\gamma}}$$ Incoherent tunneling rate

Incoherent tunneling is still a quantum effect

Energy gap = Δ



γ = broadening

$\boxed{\textbf{Gap not well-defined}}$

Two-Qubit Example

Hamiltonian: $$H = -\tfrac{1}{2}\Delta(\sigma_x^1 + \sigma_x^2) - \tfrac{1}{2}J\sigma_z^1\sigma_z^2, \qquad J >> \Delta$$

Ferromagnetic coupling

Lowest two energy eigenstates: $$|\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

Entangled states

Energy eigenvalues: $$E_\pm = \mp \frac{\Delta^2}{2J}$$

Two-Qubit Entanglement

Equilibrium density matrix ($J >> T, \Delta$):

$$\rho = P_+^{eq} \left| + \right\rangle \left\langle + \right| + P_-^{eq} \left| - \right\rangle \left\langle - \right|$$

Concurrence (entanglement measure):
W.K. Wootters, PRL 80, 2245 (1998)

$$C(\rho) = P_+^{eq} - P_-^{eq}$$

$C(\rho) = 0$, **(i.e., unentangled) only if**

$$P_+^{eq} = P_-^{eq} = \tfrac{1}{2} \quad \text{i.e.,} \quad T >> \Delta^2 / J$$

Quantum Entanglement
can persist in equilibrium

Summary:

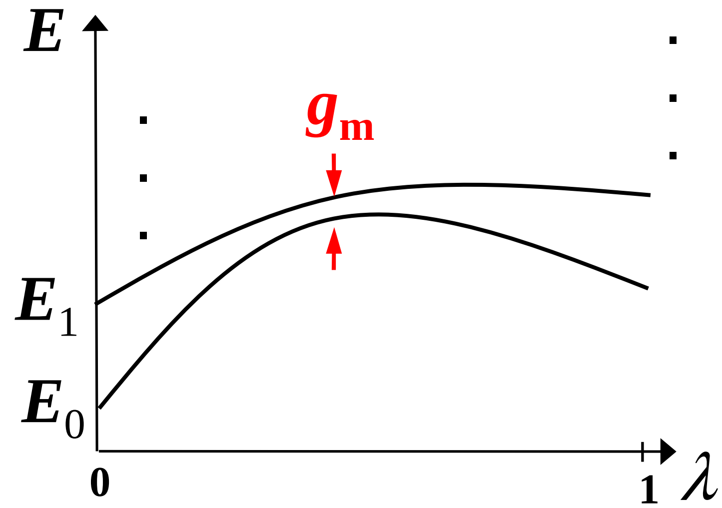
- **Classical limit is large T (compared to energy spacings) and not long t (compared to decoherence time)**

- **Without a Hamiltonian, the system will be classical after the decoherence time**

- **With a well-defined Hamiltonian (stronger than noise) system may stay quantum mechanical even in equilibrium as long as T *is* small**

Adiabatic Quantum Computation

If the excited state is not occupied its phase does not matter



Temperature — Energy broadening (decoherence rate)

If $g_m >> T, \gamma$, system will stay in the ground state throughout the computation

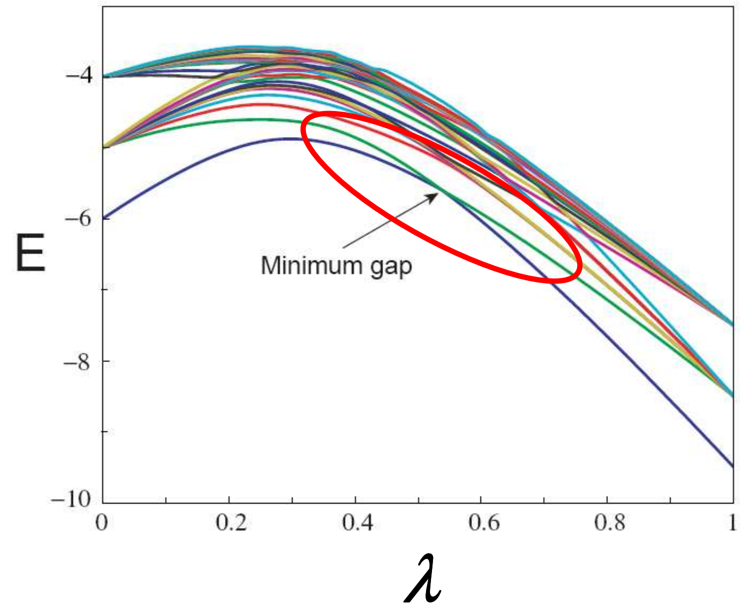
- **What if $g_m < T, \gamma$?**
- **What about the computation time?**

Small Gap Regime

System Hamiltonian:

$$H = (1-\lambda)H_B + \lambda H_P$$

Energy Spectrum

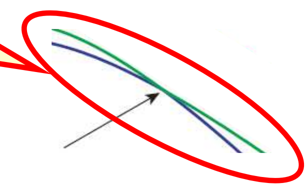


Small Gap Regime

System Hamiltonian:

$$H = (1-\lambda)H_B + \lambda H_P$$

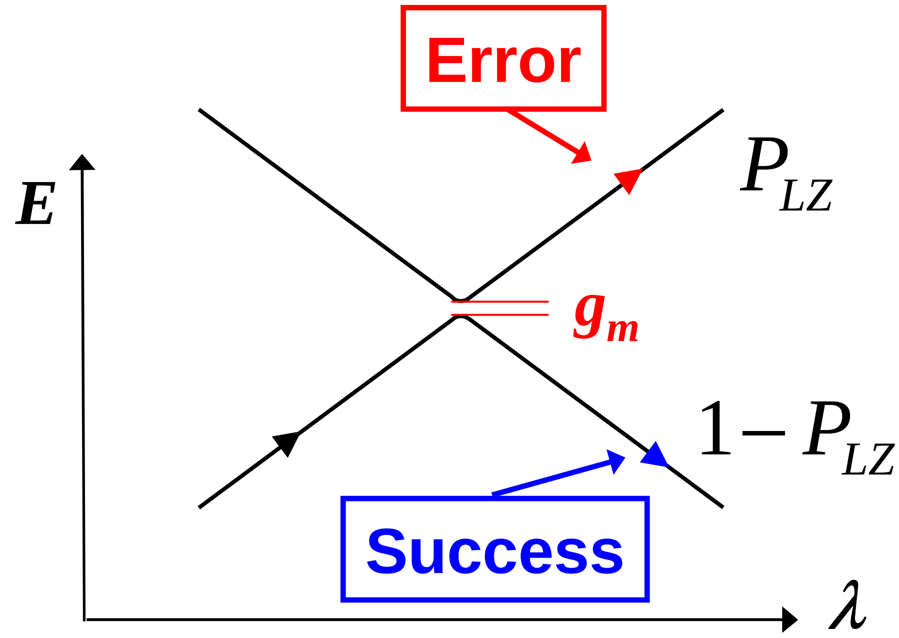
Effective two-state system



Landau-Zener physics approximately apply

Landau-Zener Transition

$$H = -\tfrac{1}{2}(\Delta\sigma_x + vt\sigma_z)$$



Landau-Zener formula:

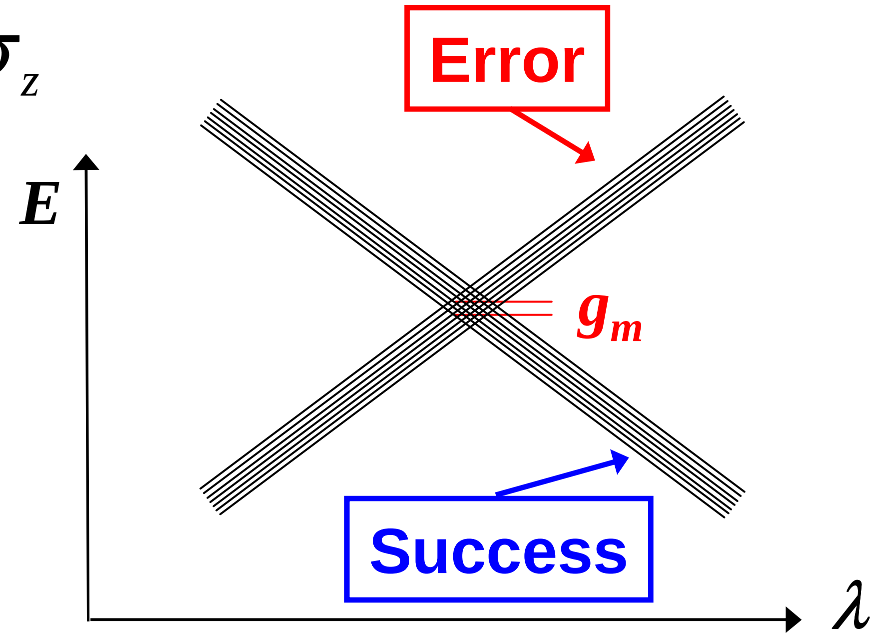
$$P_{LZ} = e^{-\pi g_m^2 / 2v}$$

$$v \sim \dot{\lambda} \sim 1/t_f$$

What happens if there is an environment?

Landau-Zener Transition

$$H = -\tfrac{1}{2}(\Delta\sigma_x + vt\sigma_z) - Q\sigma_z$$

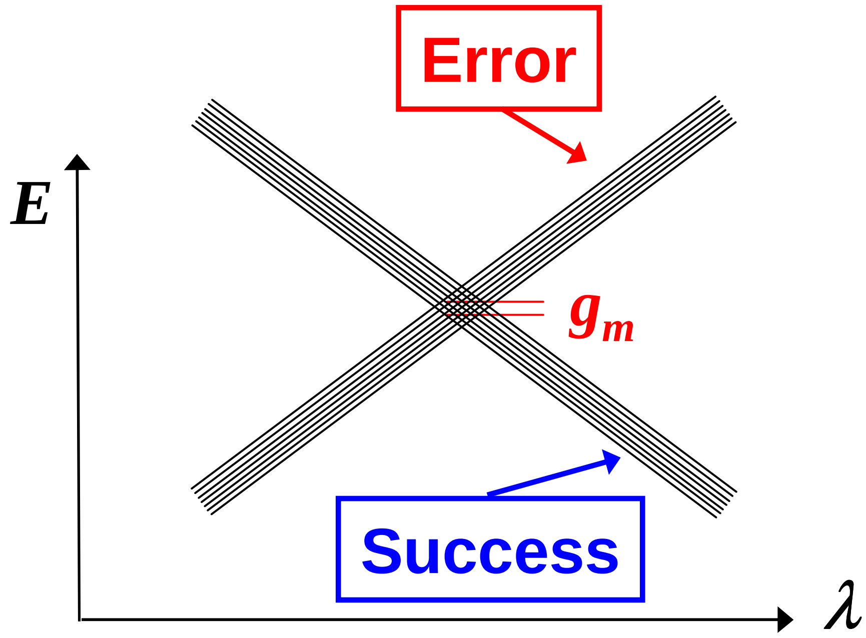


Anticrossing is replaced by many anticrossings

No well-defined gap!

Landau-Zener Probability at T=0

Landau-Zener probability is exactly the same as that for a closed system



Spin environment:

A.T.S. Wan, M.H.S. Amin, S.X. Wang, Int. J. Quant. Inf. 7, 725 (2009)

Harmonic oscillator environment:

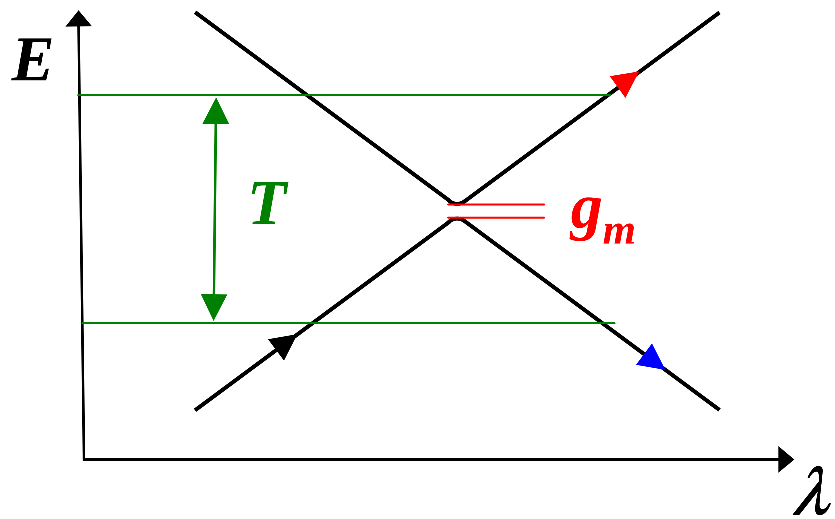
M. Wubs et al., PRL 97, 200404 (2006)

General environment:

K. Saito et al., PRB 75, 214308 (2007)

Landau-Zener Probability at finite T

$T >> g_m \Rightarrow P_0 \rightarrow 1/2$



There is always more chance of success than failure

What about computation time?

The computation time scale remains the same

M.H.S. Amin, P.J. Love. C.J.S. Truncik, PRL 100, 060503 (2008)

M.H.S Amin, D.V. Averin, J.A. Nesteroff, PRA 80, 022107 (2009)

Beyond 2-State Approximation

Density matrix formalism:

Hamiltonian:

$$H = \underbrace{H_S}_{\text{System}} + \underbrace{H_B}_{\text{Bath}} + \underbrace{H_{\text{int}}}_{\text{Interaction}}$$

Liouville equation:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)]$$

Find a differential equation that describes the evolution of the reduced density matrix.

Interaction Representation

$$\rho_I(t) \;=\; e^{i\int_0^t (H_S+H_B)dt'/\hbar}\rho(t)e^{-i\int_0^t (H_S+H_B)dt'/\hbar}$$

$$H_I(t) = e^{i\int_0^t (H_S+H_B)dt'/\hbar}H_{\mathrm{int}}(t)e^{-i\int_0^t (H_S+H_B)dt'/\hbar}$$

$$\Longrightarrow \qquad \dot{\rho}_I(t) = -\frac{i}{\hbar}[H_I(t), \rho_I(t)]$$

Integrating

$$\rho_I(t) = \rho_I(0) - \frac{i}{\hbar}\int_0^t d\tau [H_I(\tau), \rho_I(\tau)]$$

Interaction Representation

Substituting back into the integrand

$$\rho_I(t) = \rho_I(0) - \frac{i}{\hbar}\int_0^t d\tau[H_I(\tau),\rho_I(0)]$$
$$-\frac{1}{\hbar^2}\int_0^t d\tau\int_0^\tau d\tau'[H_I(\tau),[H_I(\tau'),\rho_I(\tau')]]$$

Differentiating + Tracing over the environment

$$\dot{\rho}_{SI}(t) = -\frac{1}{\hbar^2}\int_0^t d\tau \mathrm{Tr}_B[H_I(t),[H_I(\tau),\rho_{SI}(\tau)\rho_B(0)]]$$

Assumption: $\rho_I(t) = \rho_{SI}(t)\rho_B(0)$

Instantaneous Energy Basis

$$H_S(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

Define $\rho_{nm}(t) = \langle n(t)|\rho_S|m(t)\rangle = \mathrm{Tr}_B\{\langle n(t)|\rho|m(t)\rangle\}$

$$\dot{\rho}_{nm} = \mathrm{Tr}_B\{\langle n|\dot{\rho}|m\rangle + \langle\dot{n}|\rho|m\rangle + \langle n|\rho|\dot{m}\rangle\}$$

$$= -(i/\hbar)\mathrm{Tr}_B\{\langle n|[H,\rho]|m\rangle\} + \langle\dot{n}|\rho_S|m\rangle + \langle n|\rho_S|\dot{m}\rangle$$

Thermal transitions

Non-adiabatic transitions

Non-Markovian Master Equation

$$\dot{\rho}_{nm}(t) \; = \; -i\omega_{nm}\rho_{nm}(t) - \sum_{kl} M_{nmkl}(t)\rho_{kl}(t) - \int_0^t d\tau \sum_{k,l} R_{nmkl}(t-\tau)\rho_{kl}(\tau)$$

Non-adiabatic transitions

Thermal transitions

$$M_{nmkl}(t) \; = \; -\delta_{nk}\langle l|\dot{m}\rangle - \delta_{ml}\langle \dot{n}|k\rangle.$$

$$R_{nmkl}(t) \; = \; \delta_{lm}\sum_r \Gamma^{(+)}_{nrrk}(t) + \delta_{nk}\sum_r \Gamma^{(-)}_{lrrm}(t) - \Gamma^{(+)}_{lmnk}(t) - \Gamma^{(-)}_{lmnk}(t)$$

$$\Gamma^{(+)}_{lmnk}(t) = \frac{e^{-i\omega_{nk}t}}{\hbar^2}\langle \tilde{H}_{I,lm}(t)\tilde{H}_{I,nk}(0)\rangle$$

$$\Gamma^{(-)}_{lmnk}(t) = \frac{e^{-i\omega_{lm}t}}{\hbar^2}\langle \tilde{H}_{I,lm}(0)\tilde{H}_{I,nk}(t)\rangle$$

$$\tilde{H}_{I,nm}(t) = \langle n|e^{iH_B t/\hbar} H_{\text{int}}(t) e^{-iH_B t/\hbar}|m\rangle$$

Non-Markovian Master Equation

$$\dot{\rho}_{nm}(t) \;=\; -i\omega_{nm}\rho_{nm}(t) - \sum_{kl} M_{nmkl}(t)\rho_{kl}(t) - \int_0^t d\tau \sum_{k,l} R_{nmkl}(t-\tau)\rho_{kl}(\tau)$$

Laplace Transformation: $\tilde{R}_{nmkl}(s) = \int_0^\infty dt e^{-st} R_{nmkl}(t)$

$$(s+i\omega_{nm})\tilde{\rho}_{nm}(s) + \sum_{k,l}\left[\tilde{R}_{nmkl}(s) + \tilde{M}_{nmkl(s)}(s)\right]\tilde{\rho}_{kl}(s) = \rho_{nm}(0)$$

Perturbation around Markovian Solution

If the change of $\rho(\tau)$ within the response time of the environment (*τ_B*) is small we can do perturbation expansion in *τ_B/t_f*

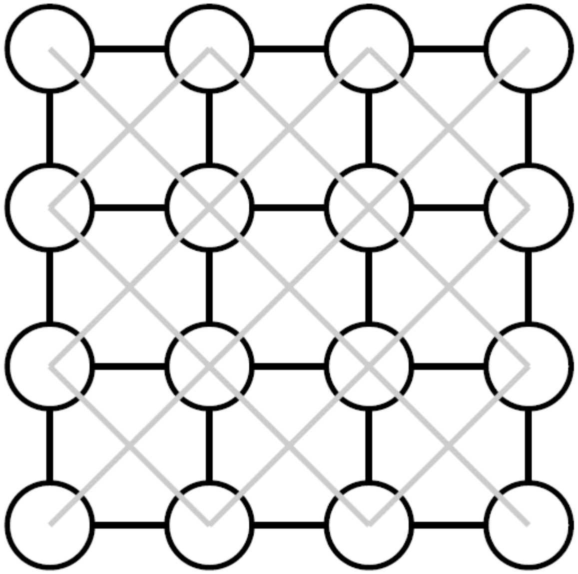
Taylor expansion $\rho(\tau) \approx \rho(t) + (\tau - t)\dot{\rho}(t)$ leads to

$$\boxed{\begin{aligned} \dot{\rho}_{nm}(t) \approx & -i\omega_{nm}\ \rho_{nm}(t) - \sum_{kl} M_{nmkl}(t)\rho_{kl}(t) \\ & - \sum_{k,l}[\tilde{R}_{nmkl}(0)\ \rho_{kl}(t) + \tilde{R}'_{nmkl}(0)\ \dot{\rho}_{kl}(t)] \end{aligned}}$$

Multi-Qubit System

M.H.S Amin, C.J.S. Truncik, D.V. Averin, Phys. Rev. A 80, 022303 (2009)

System (Ising) Hamiltonian:



$$H_S(t) = [1 - s(t)]H_i + s(t)H_f$$

$$\frac{H_i}{E} = -\frac{1}{2}\sum_i \Delta_i \sigma_x^{(i)}$$

$$\frac{H_f}{E} = -\frac{1}{2}\sum_i h_i \sigma_z^{(i)} + \frac{1}{2}\sum_{i>j} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

Random 16 qubit spin glass instances:

- **Randomly choose h_i and J_{ij} from $\{-1,0,1\}$ and $\Delta_i = 1$**
- **Select small gap instances with one solution**

Markovian Approximation

M.H.S Amin, C.J.S. Truncik, D.V. Averin, Phys. Rev. A 80, 022303 (2009)

Interaction Hamiltonian:

$$H_{\rm int} = -\sum_{i=1}^{n} \left(Q_x^{(i)} \sigma_x^{(i)} + Q_z^{(i)} \sigma_z^{(i)} \right)$$

$$\begin{aligned} \Gamma^{(+)}_{lmnk} &= \frac{1}{2} \sum_{i,\alpha} S_\alpha^{(i)}(-\omega_{nk}) \sigma^{(i)}_{\alpha,lm} \sigma^{(i)}_{\alpha,nk} \\ \Gamma^{(-)}_{lmnk} &= \frac{1}{2} \sum_{i,\alpha} S_\alpha^{(i)}(\omega_{lm}) \sigma^{(i)}_{\alpha,lm} \sigma^{(i)}_{\alpha,nk}. \end{aligned}$$

$$\sigma^{(i)}_{\alpha,lm} = \langle l | \sigma_\alpha^{(i)} | m \rangle$$

$$S_\alpha^{(i)}(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \langle Q_\alpha^{(i)}(t) Q_\alpha^{(i)}(0) \rangle$$

Spectral density

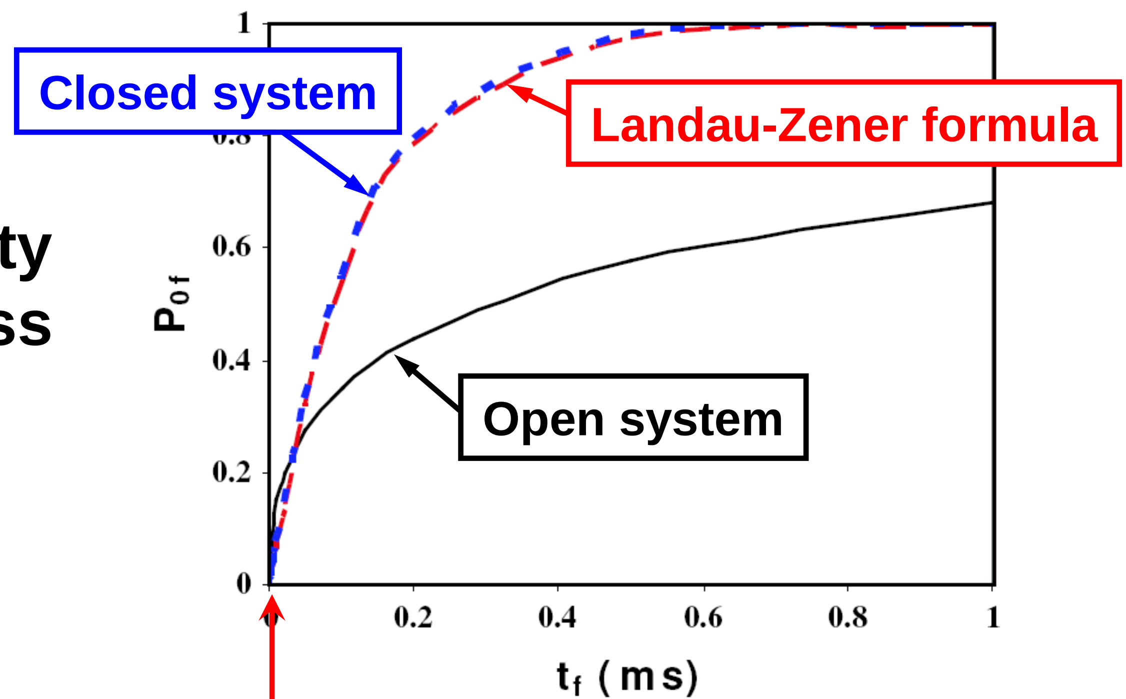
$$= \eta_\alpha^{(i)} \frac{\omega e^{-\omega/\omega_c}}{1 - e^{-\omega/T}}$$

Ohmic baths

Numerical Calculation

M.H.S Amin, C.J.S. Truncik, D.V. Averin, Phys. Rev. A 80, 022303 (2009)

Probability of success



T = 25 mK
η = 0.5
E = 10 GHz
g_{min} = 10 MHz

Single qubit decoherence time $T_2 \sim 1$ ns **time**

Computation time can be much larger than T_2

What Is the Effect of Environment?

Assumptions made:

1. **Final Hamiltonian represents the correct problem**

2. **Coupling to environment is weak (energy levels are distinct except at the anticrossing)**

3. **There are only small number of energy levels near the ground state**

If these assumptions fail to not hold, the conclusion would not hold

The Effect of Environment

- **Low frequency noise changes the final Hamiltonian which leads to solving a wrong problem**

- **With strong coupling to environment, the ground state of system plus environment will not represent the ground state of the system**

- **If there are many states available within energy T above the ground state, they will be occupied lowering the ground state probability**

Open Question

Is there a threshold theorem for AQC?