Quantum Error Correction

Lecture 1

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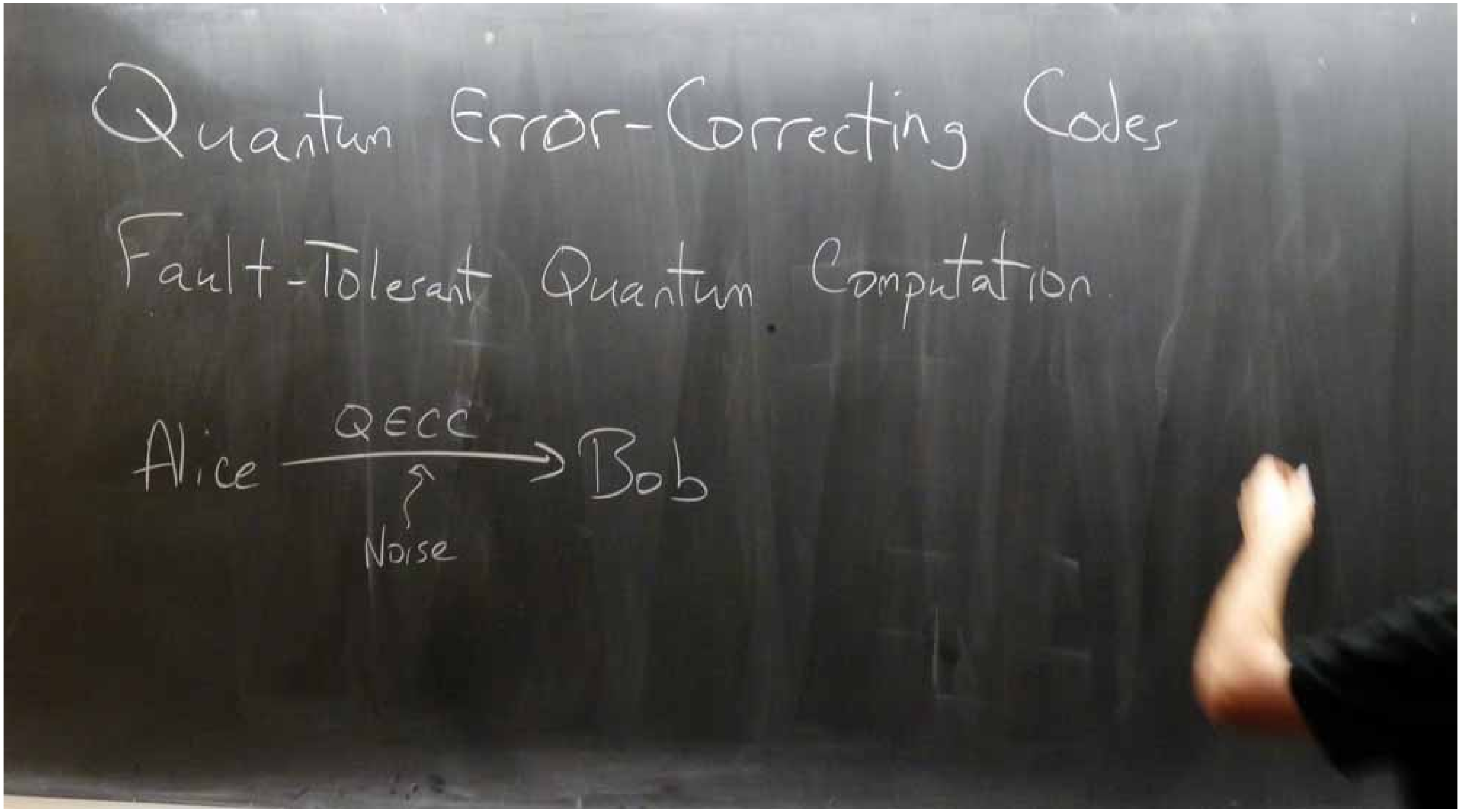

Quantum Error-Correcting Codes
Fault-Tolerant Quantum Computation
QECC
Alice
Bob
Noise

Mixed state:

Prob. p_i , state $|\psi_i\rangle$

Density matrix

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\left(|\psi_i\rangle\langle\psi_i|\right)|\phi\rangle = |\psi_i\rangle\langle\psi_i|\phi\rangle$$

Most general quantum operation

Completely positive, trace preserving map (CPTP)

$$\rho \longmapsto \sum_k A_k \rho A_k^\dagger = \mathcal{O}(\rho) \quad \left\{ \sum_k A_k^\dagger A_k = I \right.$$

① Positive: $\rho \geq 0 \Rightarrow \mathcal{O}(\rho) \geq 0$

② Trace preserving: $\operatorname{tr} \mathcal{O}(\rho) = \operatorname{tr} \rho$

③ Completely Positive: $(\mathcal{O} \otimes I)(\rho) \geq 0$

$\rho \geq 0 \Rightarrow$

p

p

p

Prob. (error on 3 qubits)

$= p^3$

Prob. (1 error) $= 3p(1-p)^2$

Prob. (2 errors) $= 3p^2(1-p)$

Bit flip $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|0\rangle \to |1\rangle$

$|1\rangle \to |0\rangle$

$\rho \to (1-p)\rho + pX\rho X$

Phase flip $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|0\rangle \to |0\rangle$

$|1\rangle \to -|1\rangle$

$\alpha|0\rangle + \beta|1\rangle \to \alpha|0\rangle - \beta|1\rangle$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ$$

$$|0\rangle \to i|1\rangle$$

$$|1\rangle \to -i|0\rangle$$

Amplitude damping

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\epsilon} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\epsilon} \\ 0 & 0 \end{pmatrix}$$

$$\rho \mapsto A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger$$

Repetition code

$0 \longmapsto 000$

$1 \longmapsto 111$

$\rightsquigarrow 010 \longrightarrow 000$

Prob.(0 or 1 errors) $= O(p)$

Prob.(2 or 3 errors) $= O(p^2)$

Prob.(correct) $= 1 - O(p^2)$

Prob.(wrong) $= O(p^2)$

Frequently consider at most t errors

No-Cloning Thm: $\nexists$ quantum operation that $|\psi\rangle \longmapsto |\psi\rangle|\psi\rangle$

$$|\psi\rangle\langle\psi| \longmapsto |\psi\rangle|\psi\rangle \otimes \langle\psi|\langle\psi|$$

Proof: Suppose

$$|\psi\rangle \to |\psi\rangle|\psi\rangle$$

$$|\phi\rangle \to |\phi\rangle|\phi\rangle$$

$$\alpha|\psi\rangle+\beta|\phi\rangle \to (\alpha|\psi\rangle+\beta|\phi\rangle)(\alpha|\psi\rangle+\beta|\phi\rangle)$$

$$\neq \alpha|\psi\rangle|\psi\rangle+\beta|\phi\rangle|\phi\rangle$$

Problems with a QECC

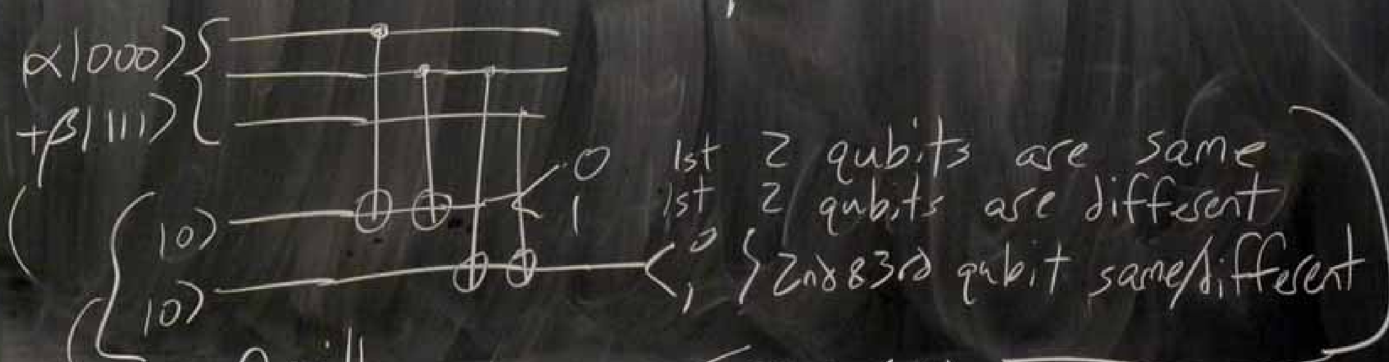
1. No-Cloning Thm.
2. Cannot measure state to locate errors?
3. Deal with phase flips as well as bit flips
4. Infinitely many possible errors

$|0\rangle \rightarrow |000\rangle$

$|1\rangle \rightarrow |111\rangle$

$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$

$X \quad \rightarrow \alpha|010\rangle + \beta|101\rangle$



Ancilla

Error syndrome

$|000\rangle$
$\beta|111\rangle$

$|+\rangle = |0\rangle + |1\rangle$

$|-\rangle = |0\rangle - |1\rangle$

$|0\rangle \rightarrow |+\rangle|+\rangle|+\rangle$

$|1\rangle \rightarrow |-\rangle|-\rangle|-\rangle$

$Z|+\rangle = |-\rangle, \quad Z|-\rangle = |+\rangle$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H|0\rangle = |+\rangle, \quad H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle, \quad H|-\rangle = |1\rangle$

Bit flip $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|0\rangle \rightarrow |1\rangle$

$|1\rangle \rightarrow |0\rangle$

$\rho \rightarrow (1-p)\rho + pX\rho X$

Phase flip $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|0\rangle \rightarrow |0\rangle$

$|1\rangle \rightarrow -|1\rangle$

$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$