Topological Quantum Computing

Nick Bonesteel, Florida State University

Main original sources:

Fault Tolerant Quantum Computation by Anyons,
A. Yu. Kitaev, Annals Phys. 303, 2 (2003). (quant-ph/9707021)

A Modular Functor Which is Universal for Quantum Computation,
M.H. Freedman, M. Larsen and Z. Wang, Comm. Math. Phys. 227, 605 (2002).

Some excellent reviews:

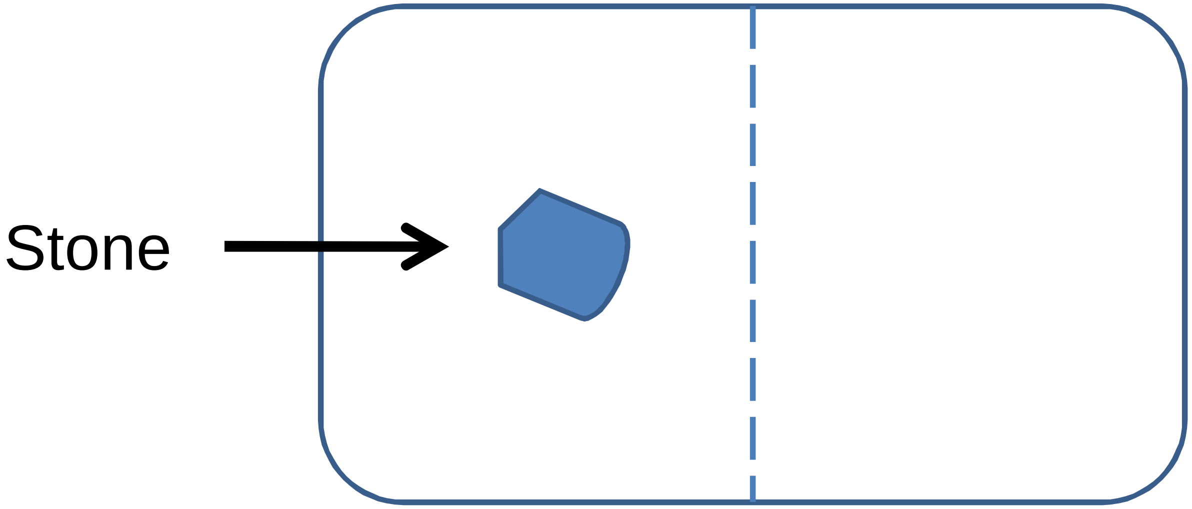
Non-Abelian Anyons and Topological Quantum Computation,
C. Nayak et al., Rev. Mod. Phys. 80, 1083 (2008). (arXiv:0707.1889v2)

Lectures on Topological Quantum Computation,
J. Preskill, Available online at: www.theory.caltech.edu/~preskill/ph219/topological.pdf

Also:

NEB, L. Hormozi, G. Zikos, S.H. Simon, Phys. Rev. Lett. 95 140503 (2005).
S.H. Simon, NEB, M.Freedman, N, Petrovic, L. Hormozi, Phys. Rev. Lett. 96, 070503 (2006).
L. Hormozi, G. Zikos, NEB, and S.H. Simon, Phys. Rev. B 75, 165310 (2007).
L. Hormozi, NEB, and S.H. Simon, Phys. Rev. Lett. 103, 160501 (2009).

Early Digital Memory



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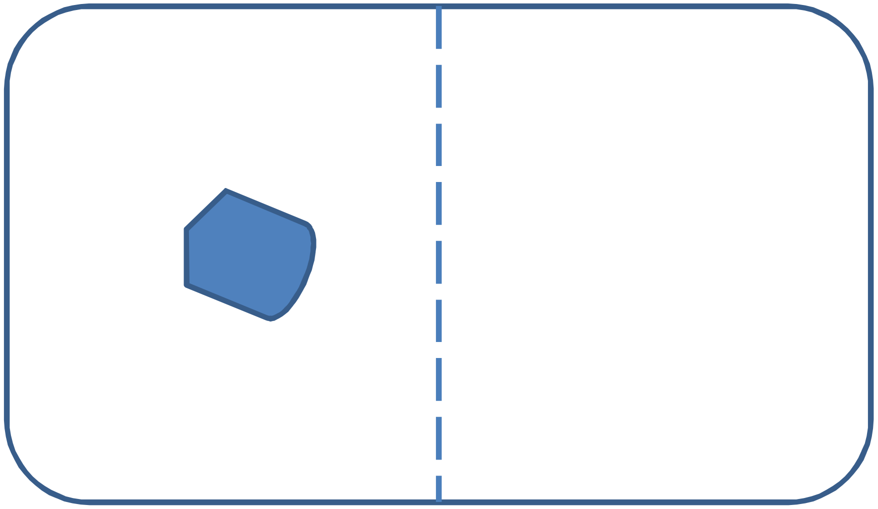

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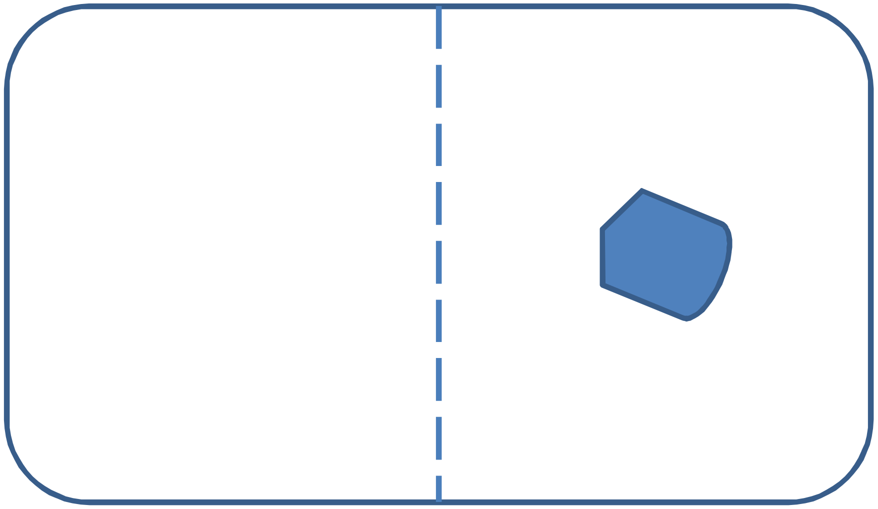

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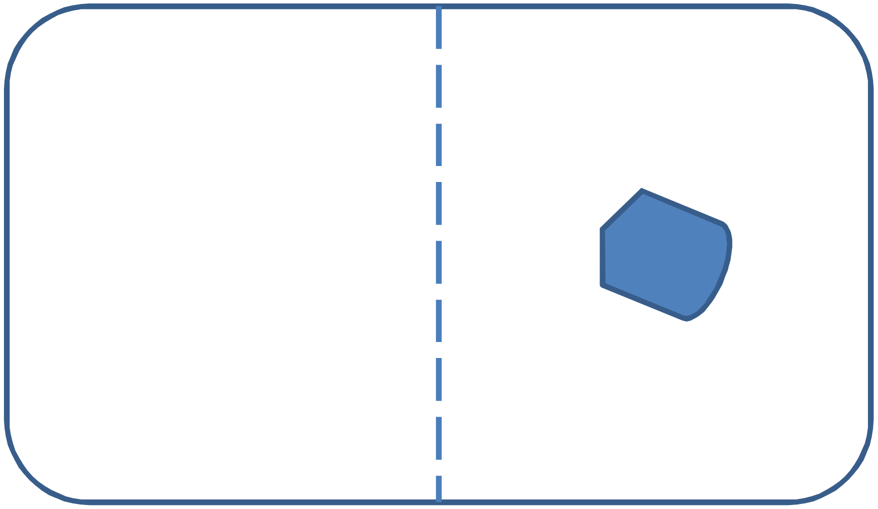

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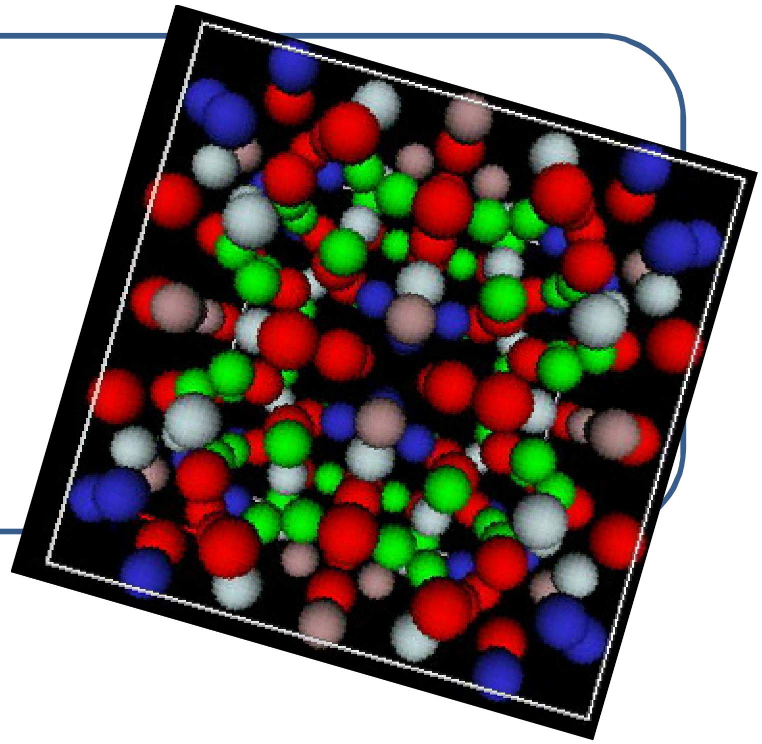
The iStone



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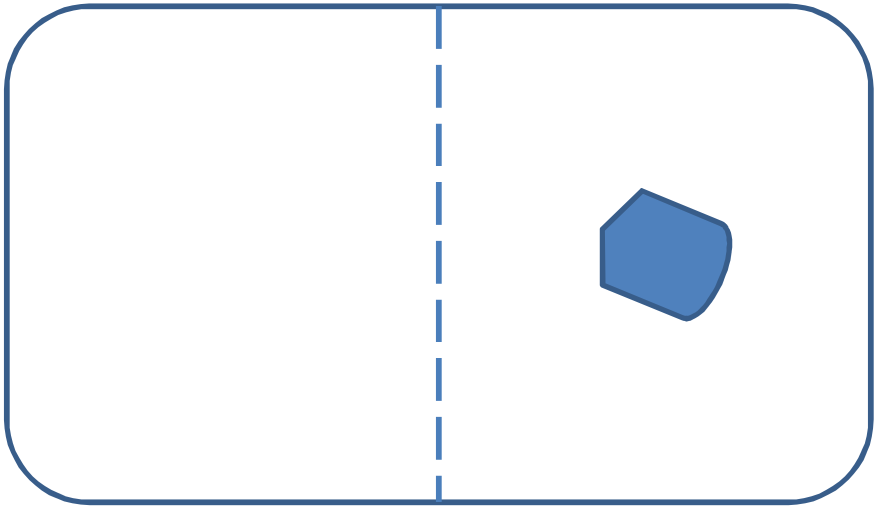
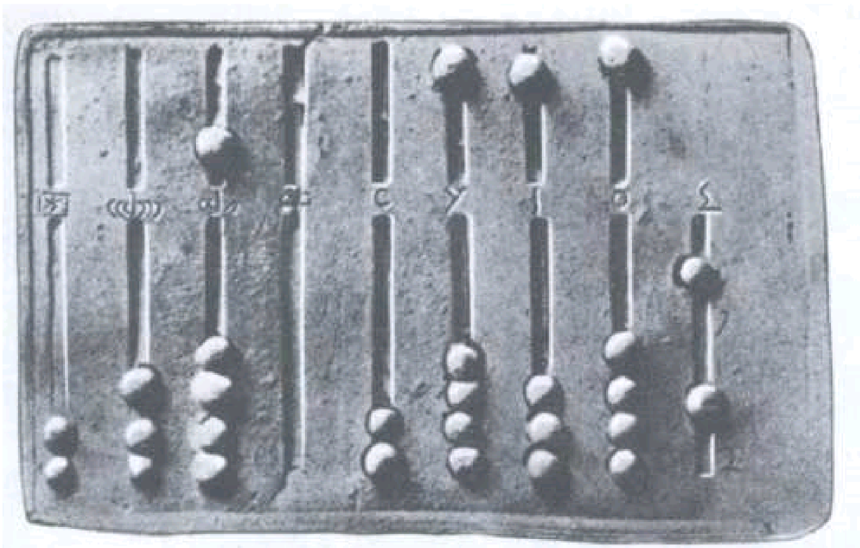


The iStone: 1 bit

Early Digital Memory



The iStone 4: ~ 20 bits

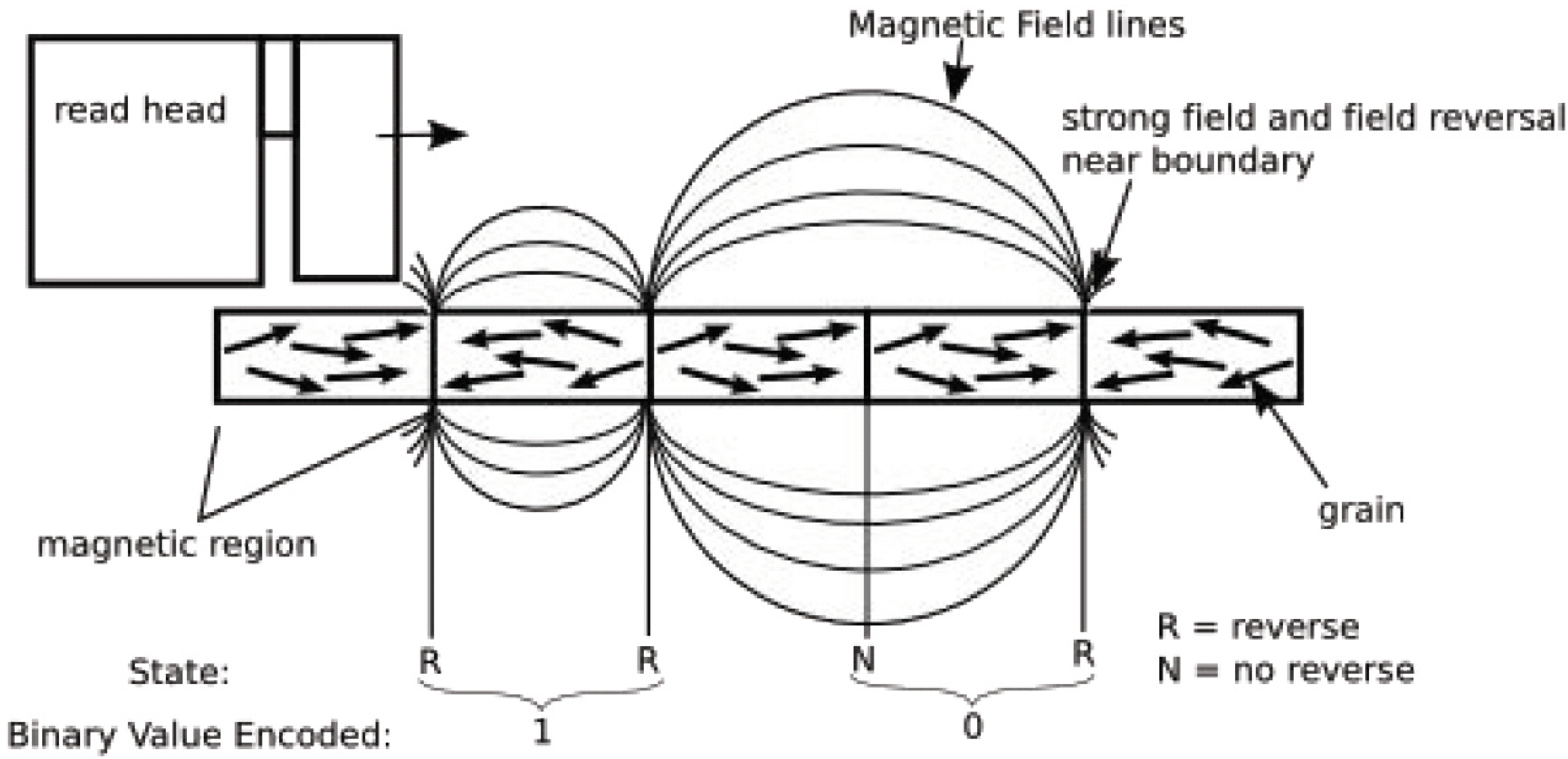
Modern Digital Memory

The iPhone 4: ~ 2.6×10^{11} bits

Modern Digital Memory

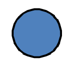
The iPod: ~ 1.4 x 10^{12} bits

Modern Digital Memory



http://en.wikipedia.org/wiki/Hard_disk_drive

Magnetic Order

A spin-1/2 particle: 

“spin up”

“spin down”

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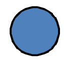
A spin-1/2 particle: 

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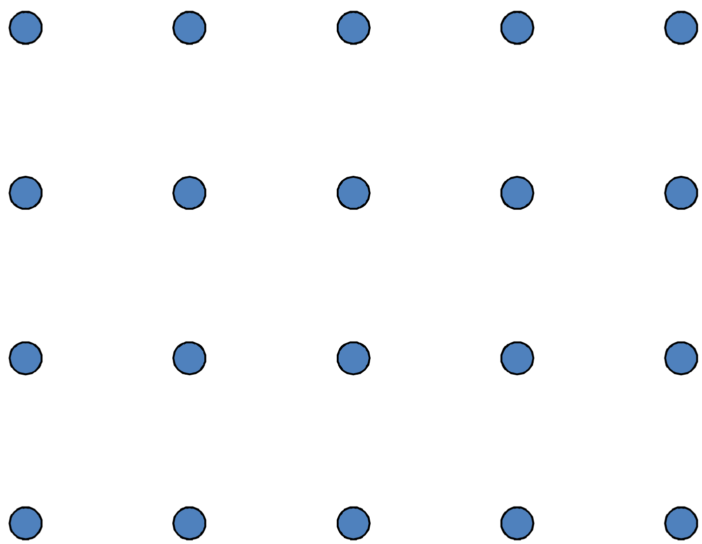
Many spin-1/2 particles:



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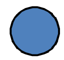
A spin-1/2 particle: 

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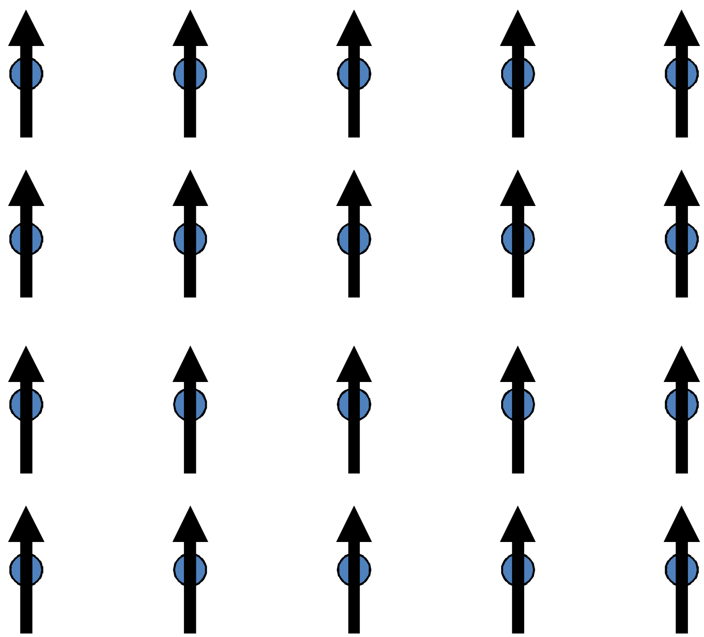

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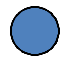
A spin-1/2 particle: 

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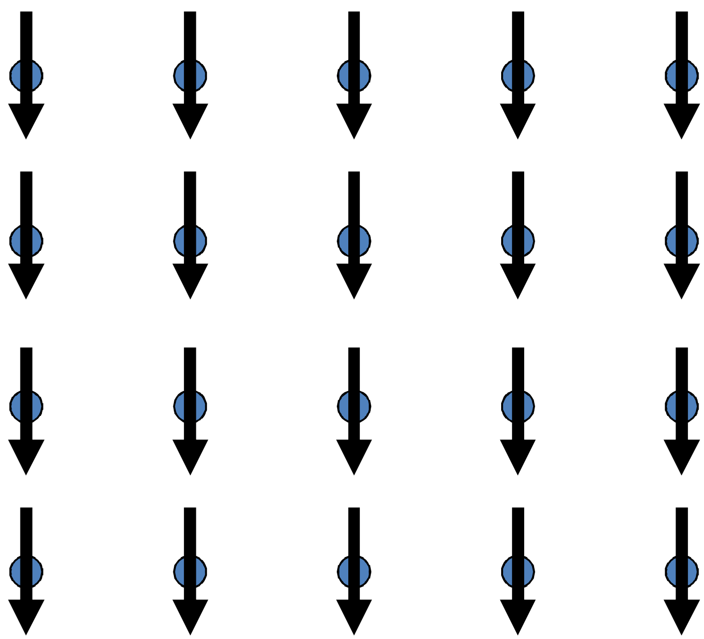

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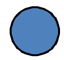
A spin-1/2 particle: 

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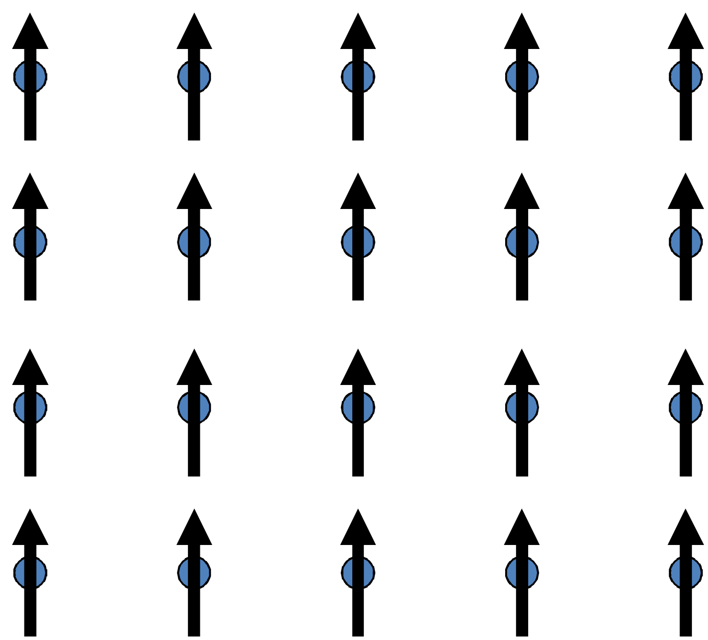

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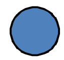
Magnetic Order

A spin-1/2 particle: 

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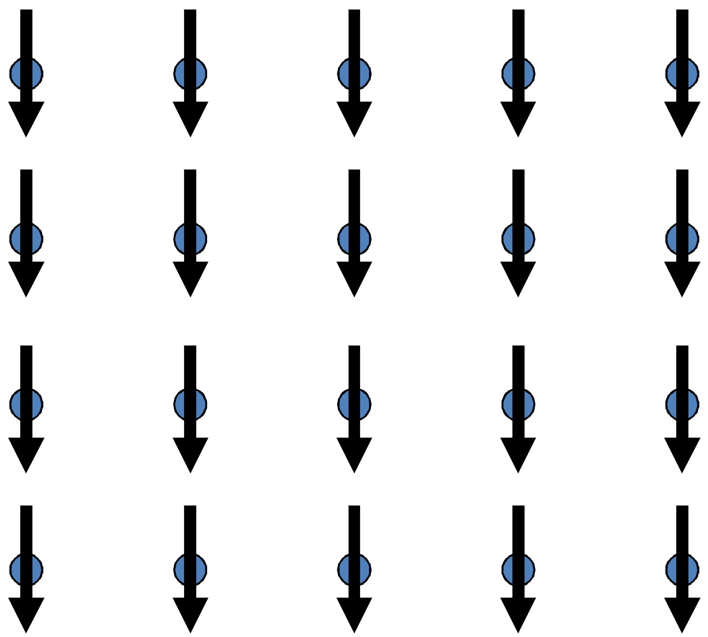

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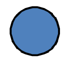
Magnetic Order

A spin-1/2 particle: 

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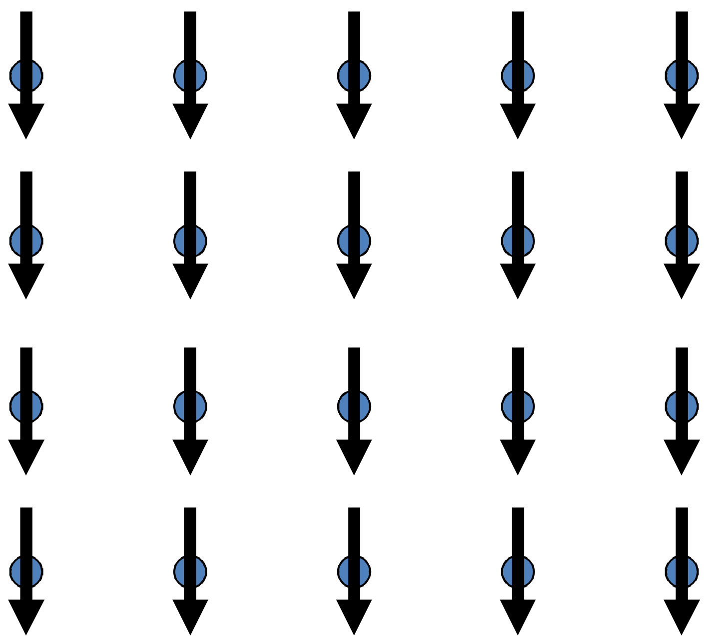

= 1

Terrific for storing classical information, but useless for quantum Information.

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$

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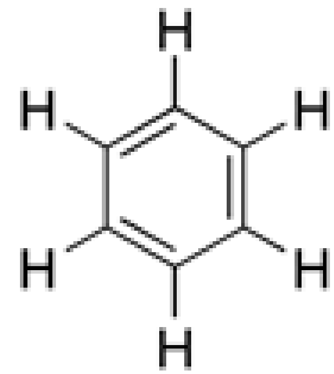

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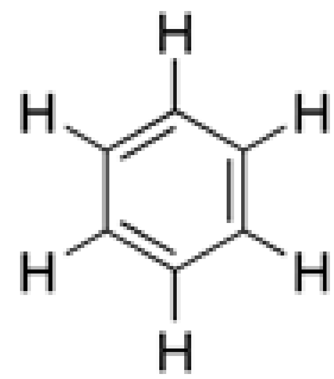

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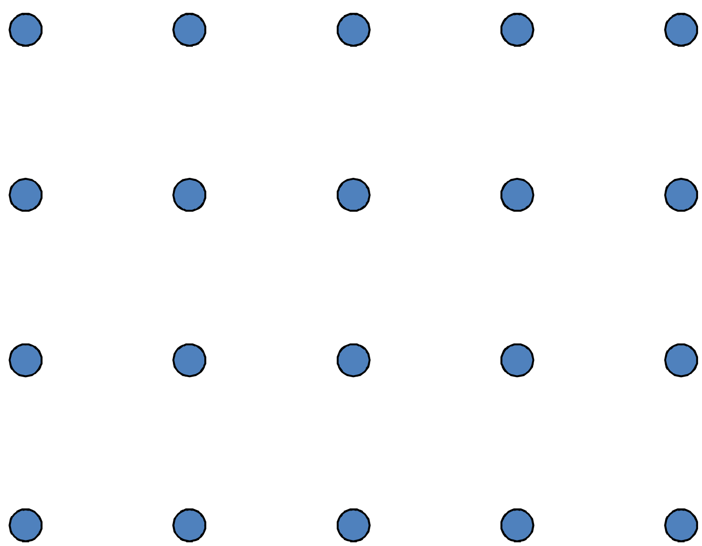

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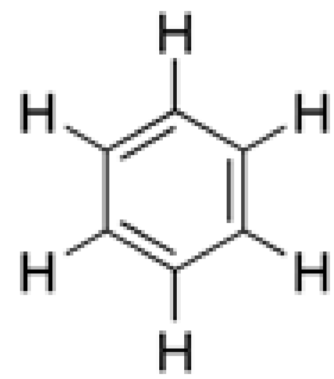

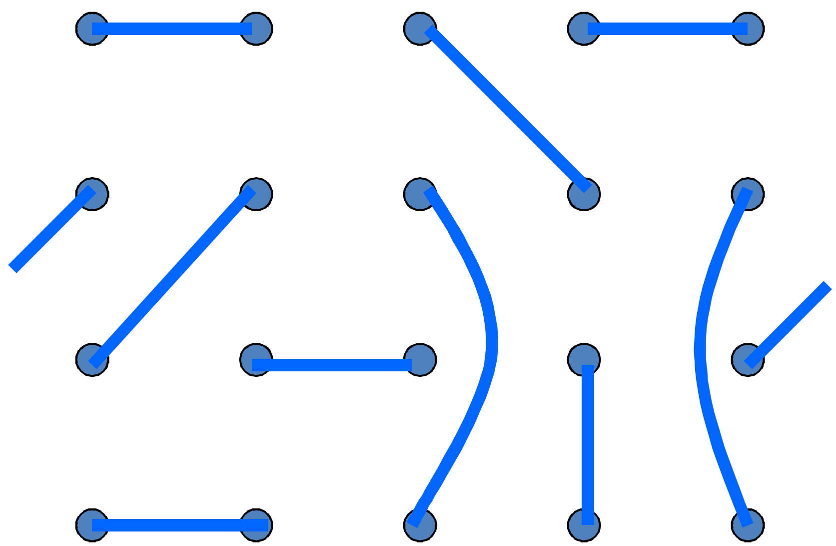

Use periodic boundary conditions

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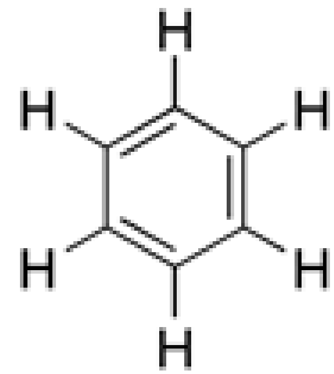

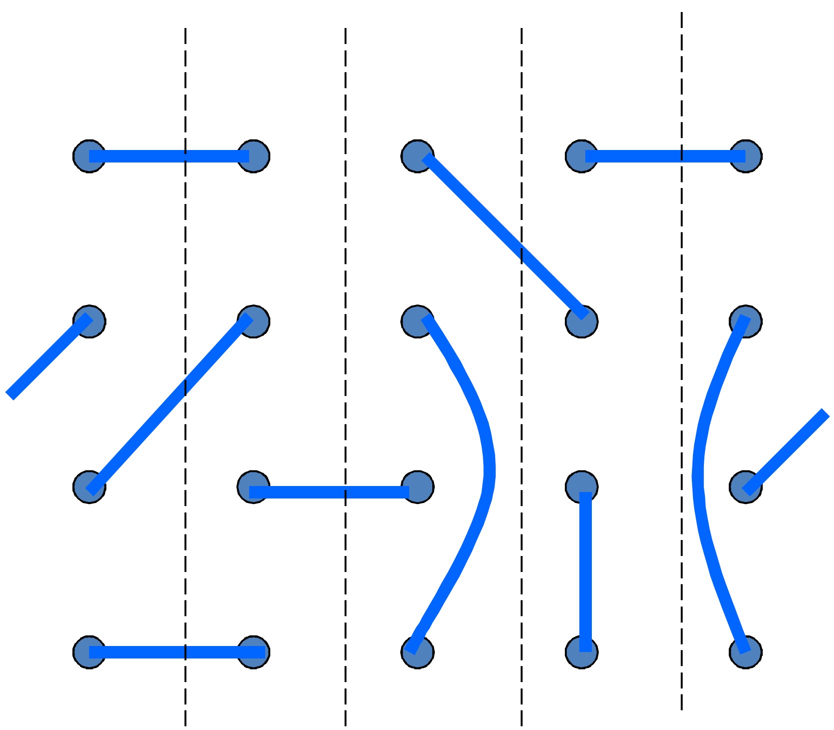

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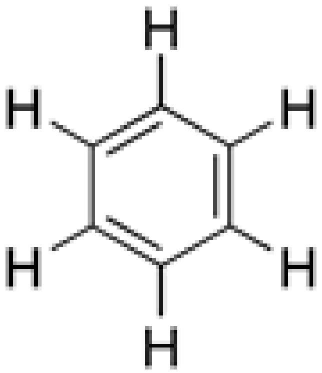

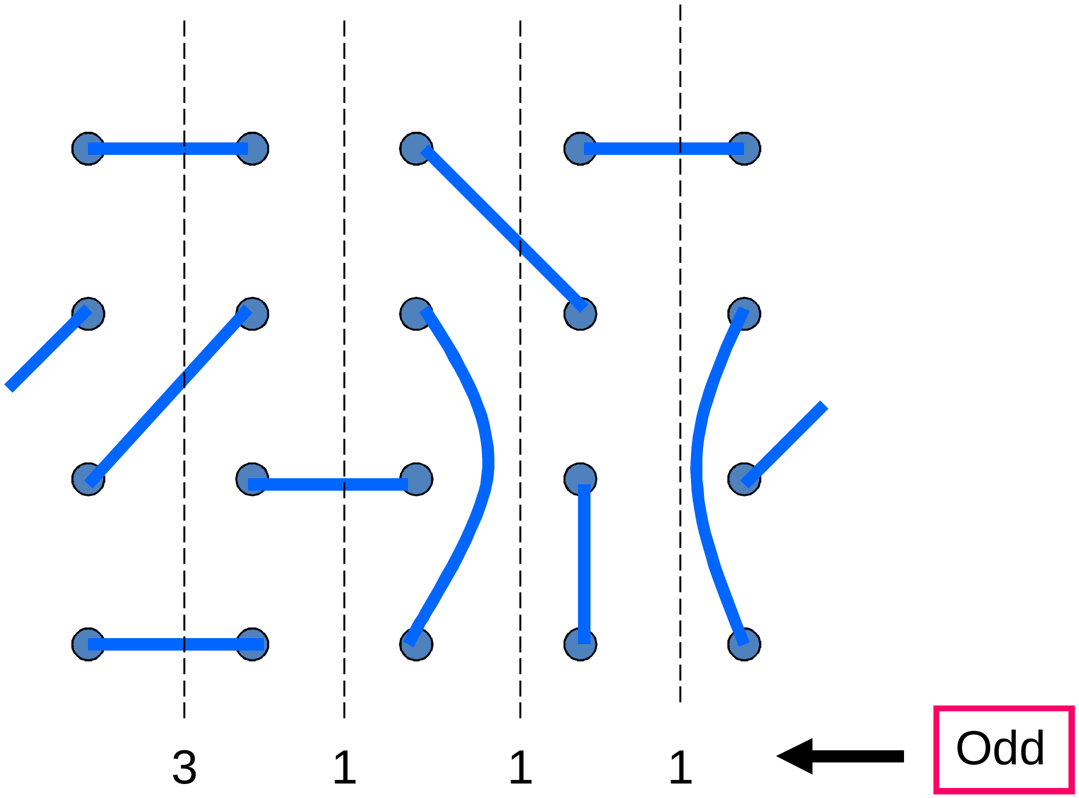

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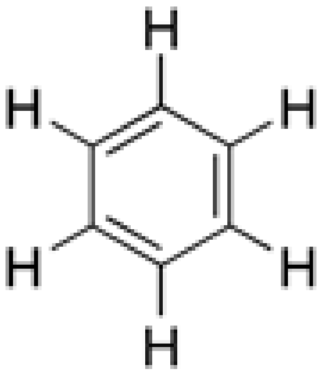
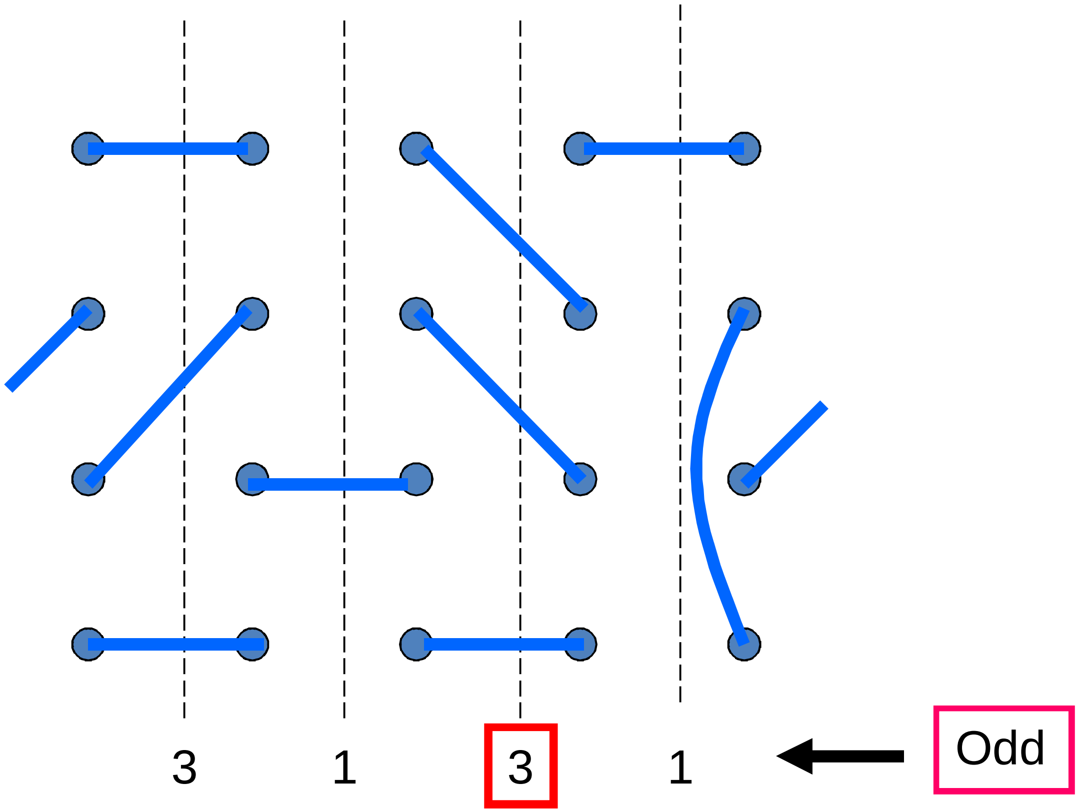
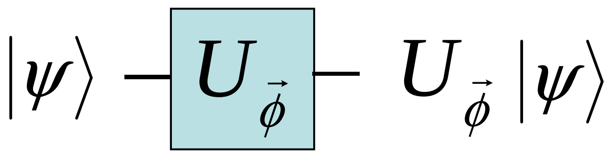


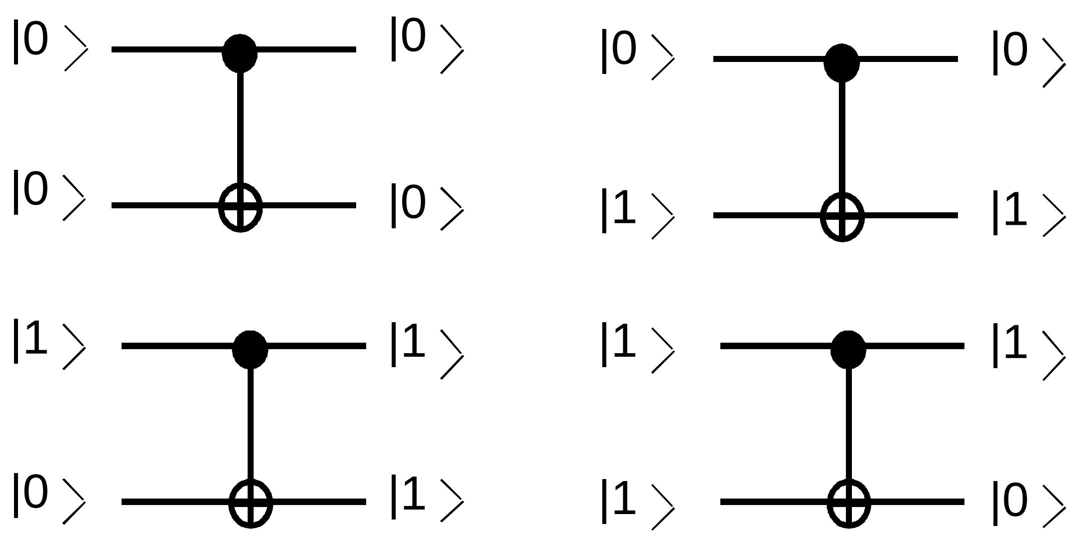


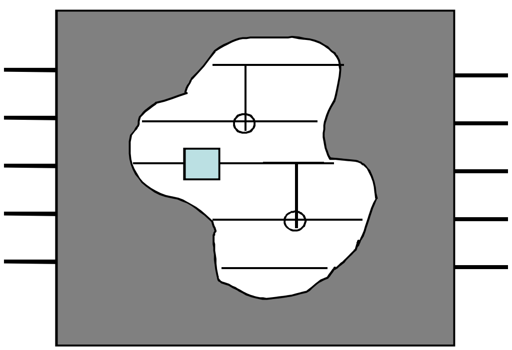
Universal Quantum Gates

Single Qubit Rotation



Controlled-Not





Any N qubit operation can be carried out using these two gates.

$$\left| \Psi_f \right\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} \left| \Psi_i \right\rangle$$

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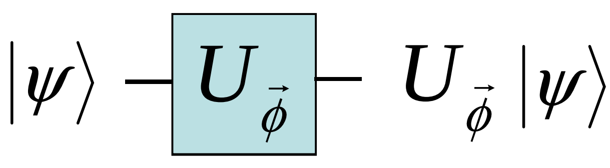

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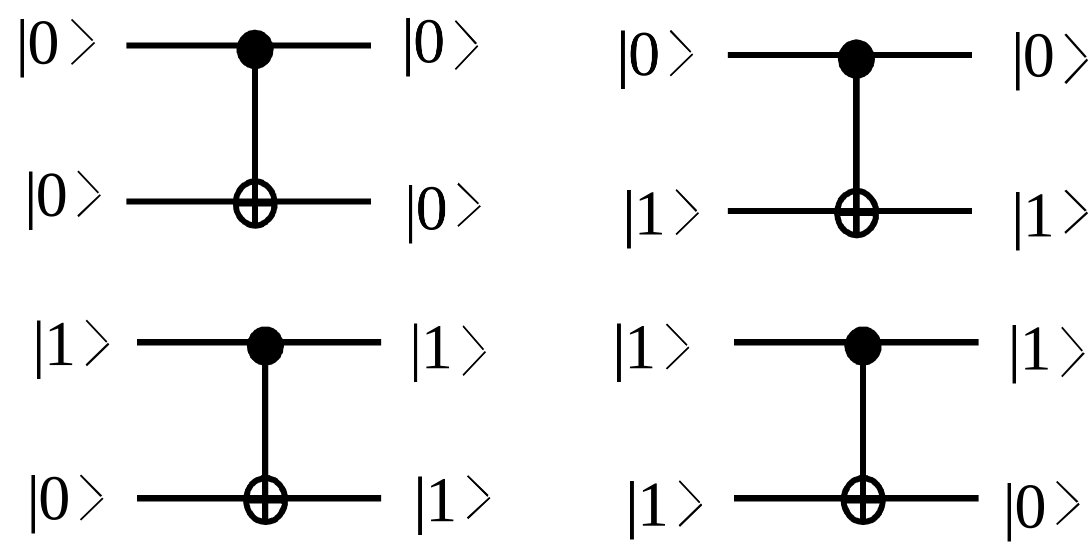

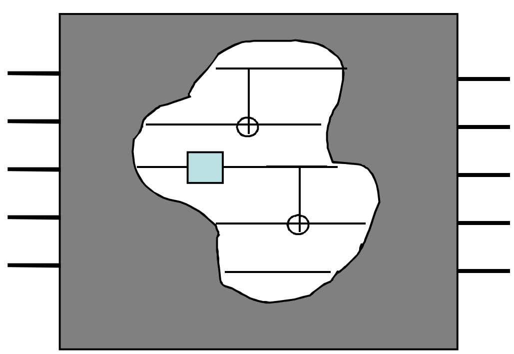

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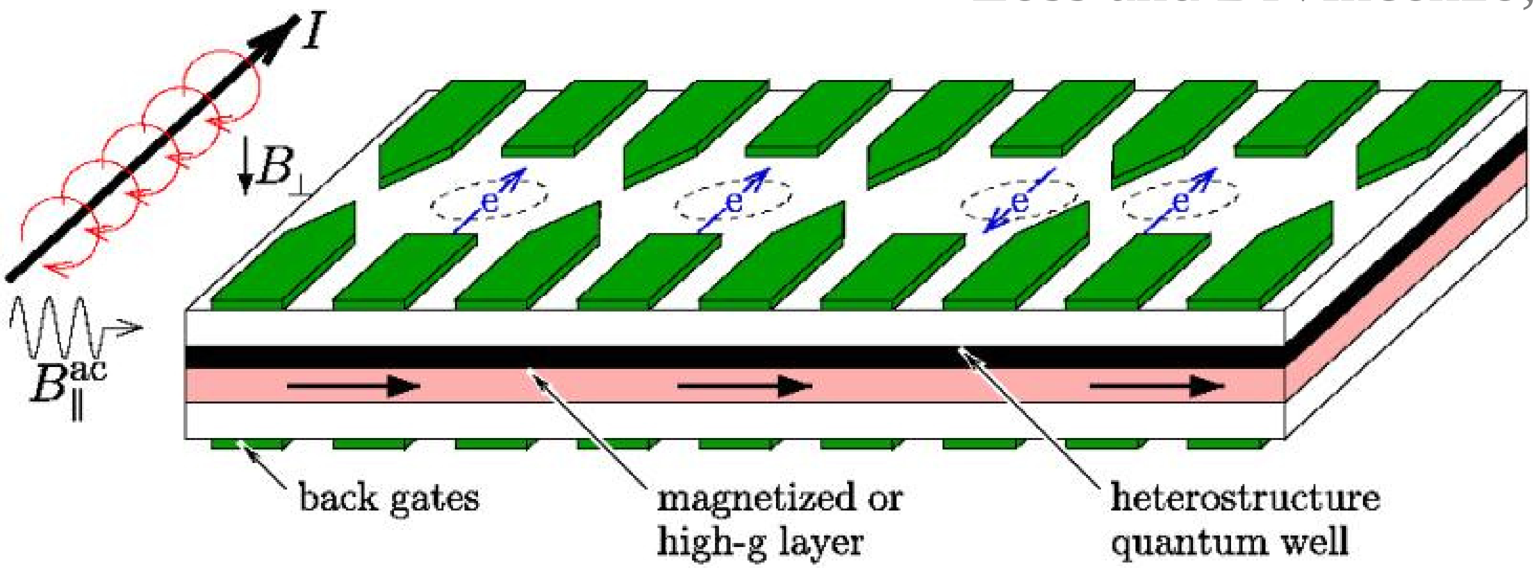
One way to go… $|0\rangle = \uparrow$ $|1\rangle = \downarrow$

Loss and DiVincenzo, '98

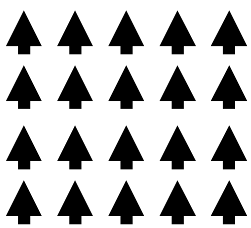


Manipulate electron spins with electric and magnetic fields to carry out quantum gates.

Problem: Errors and Decoherence! May be solvable, but it won't be easy!

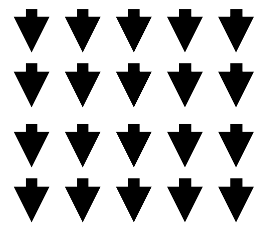
Topological Order (Wen & Niu, PRB 41, 9377 (1990))

Conventionally Ordered States: Multiple "broken symmetry" ground states characterized by a locally observable order parameter.



magnetization

$$m = \langle S_z \rangle = +\frac{1}{2}$$

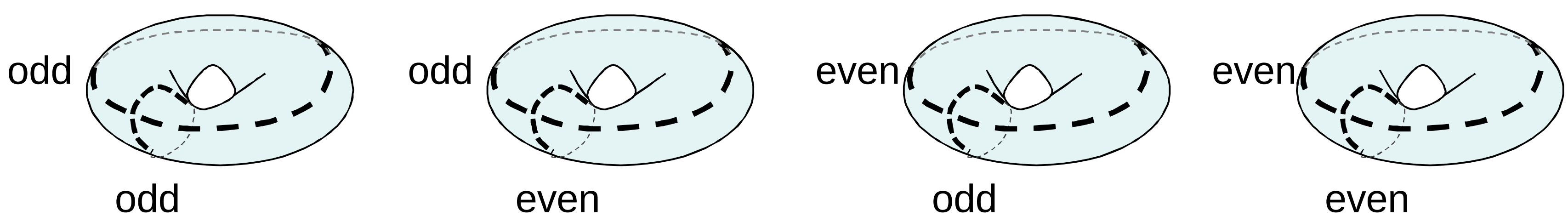


magnetization

$$m = \langle S_z \rangle = -\frac{1}{2}$$

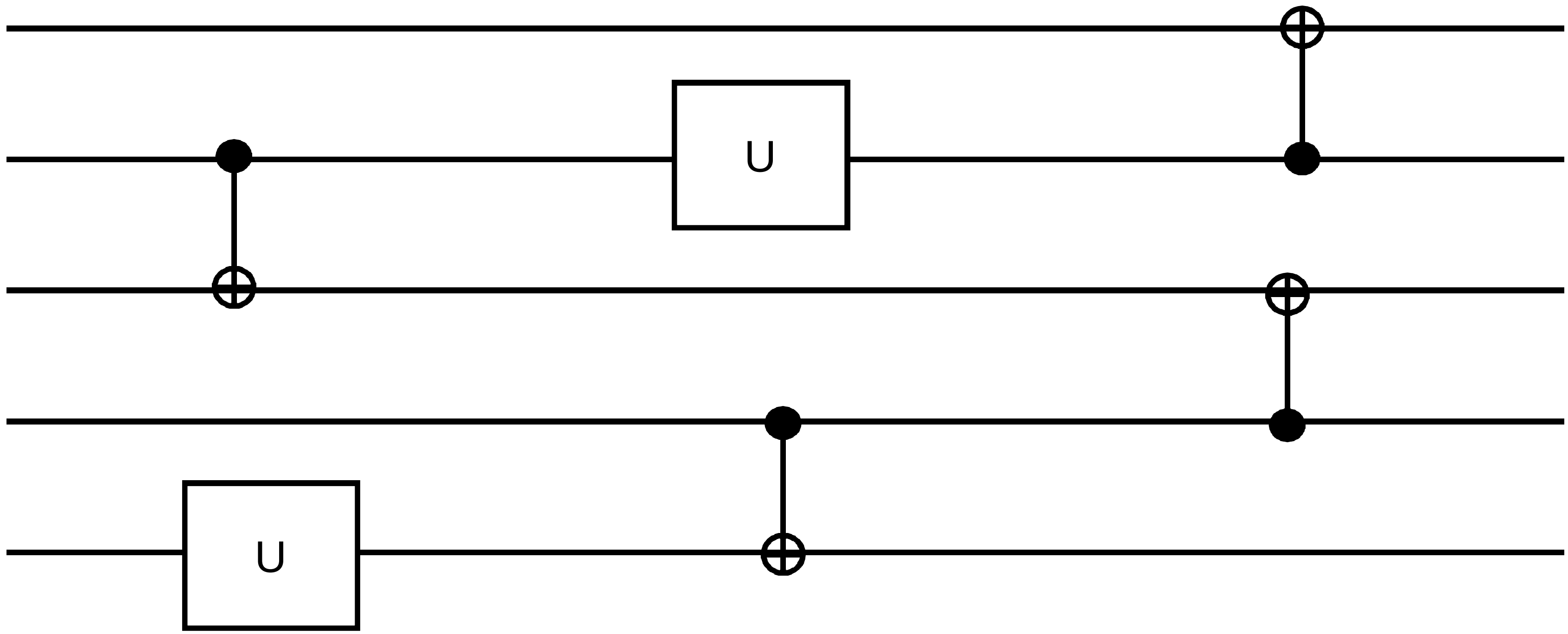
Nature's classical error correcting codes !

Topologically Ordered States: Multiple ground states on topologically nontrivial surfaces with no locally observable order parameter.



Nature's quantum error correcting codes ?

Quantum Circuit

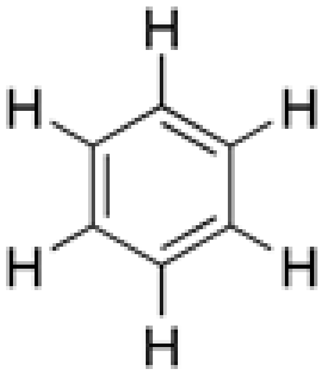


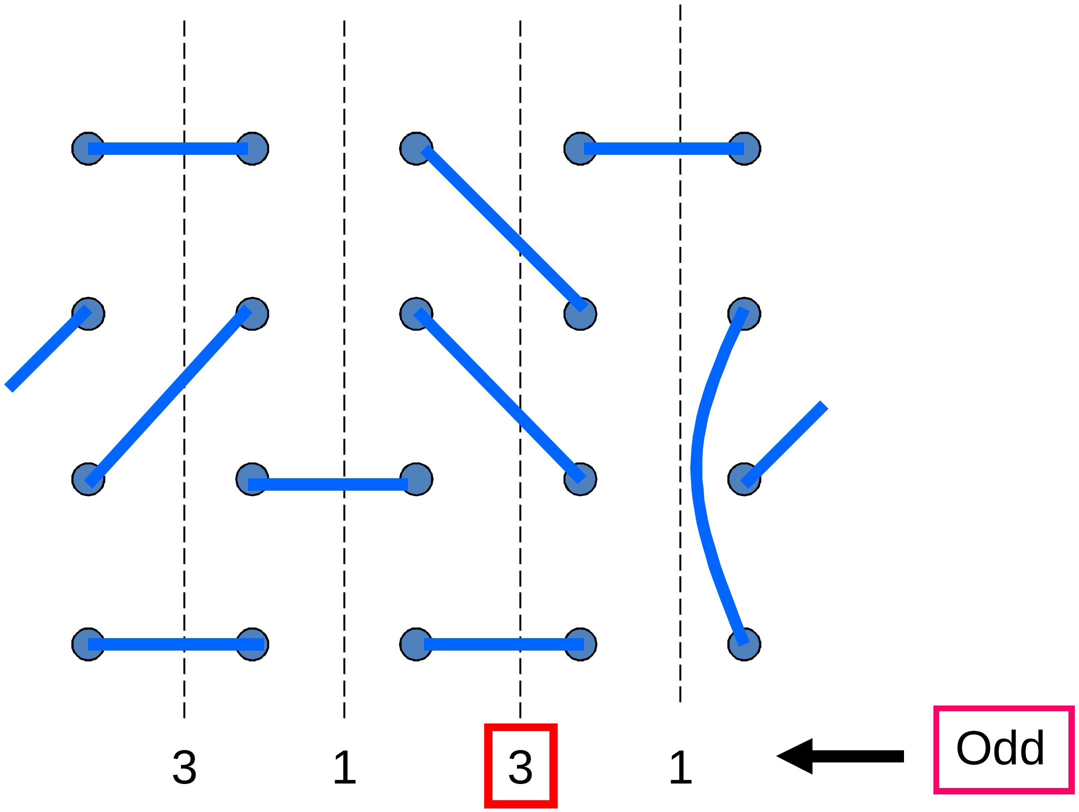
What braid corresponds to this circuit?

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$





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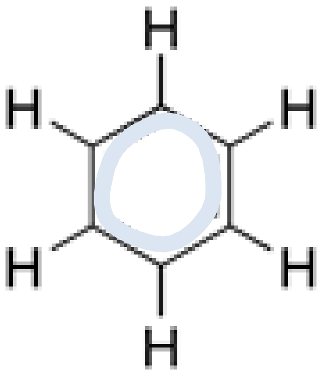

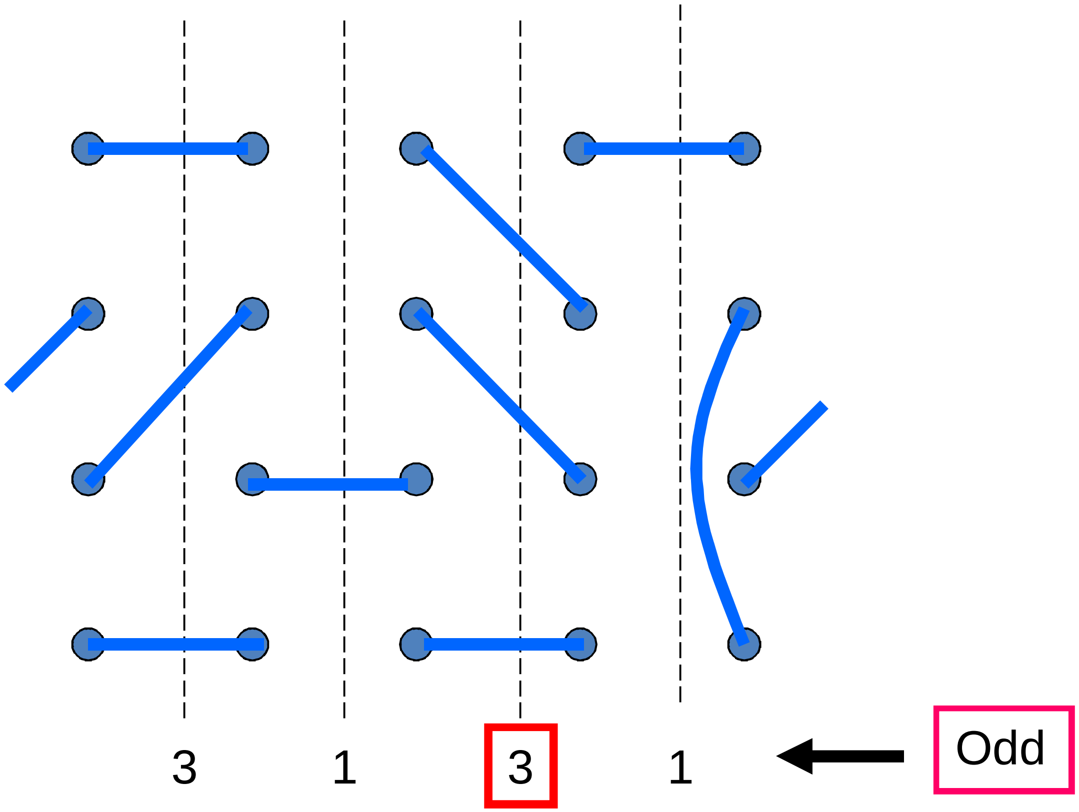

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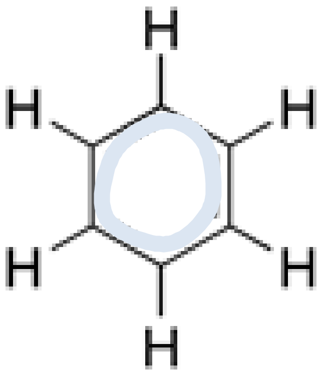

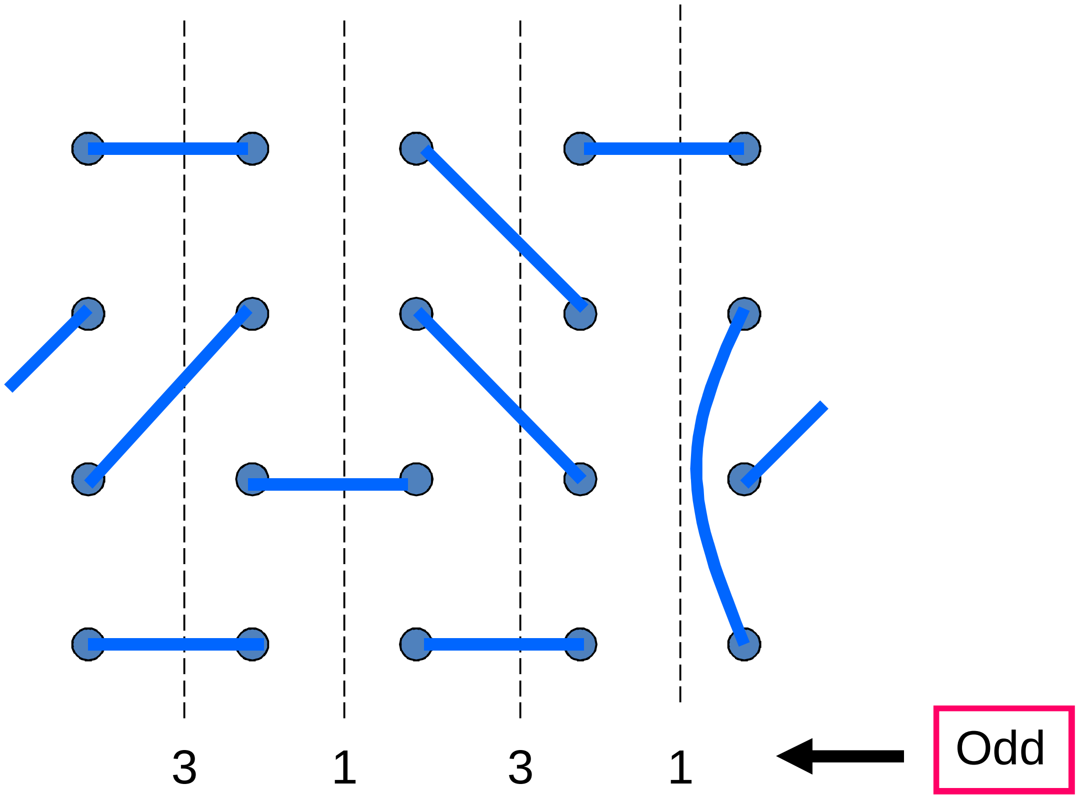

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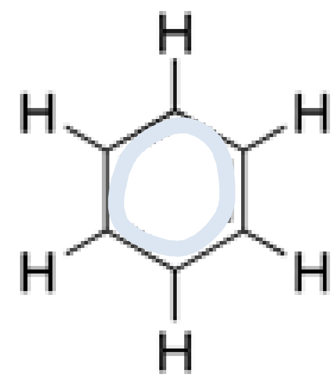

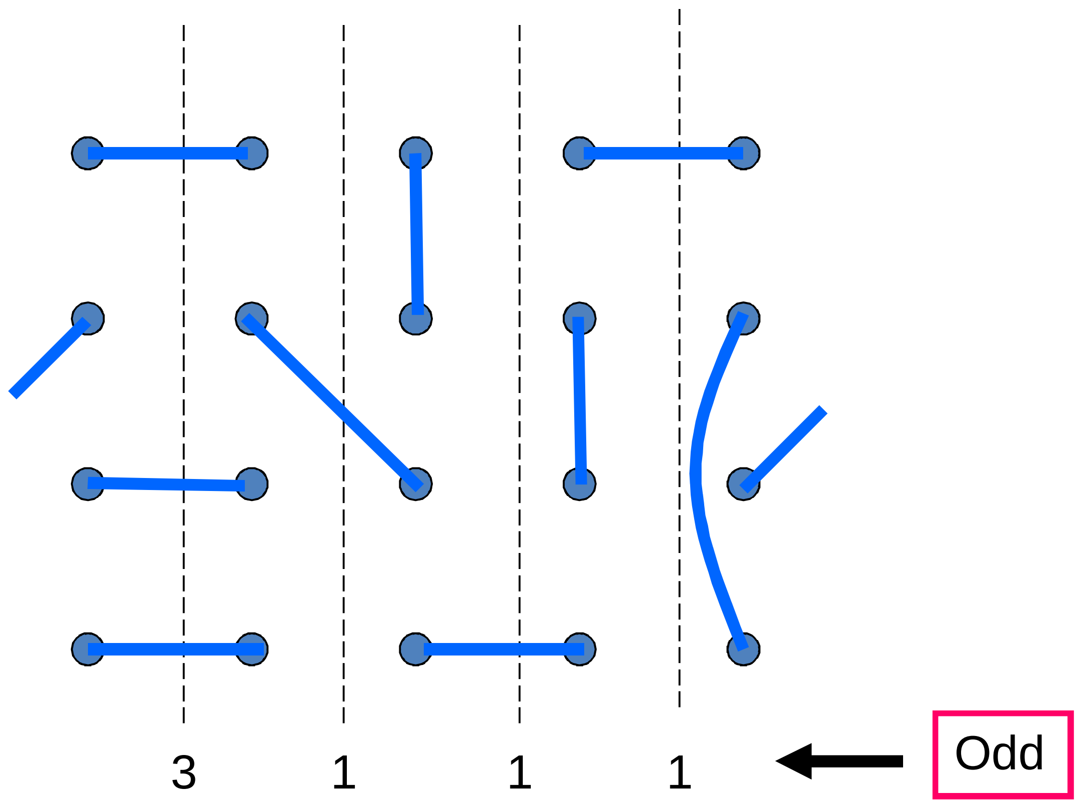

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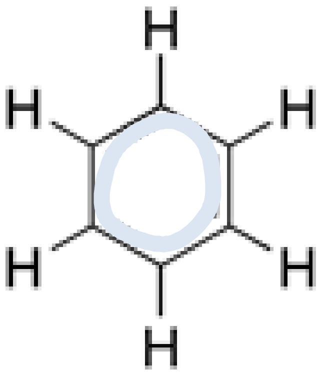

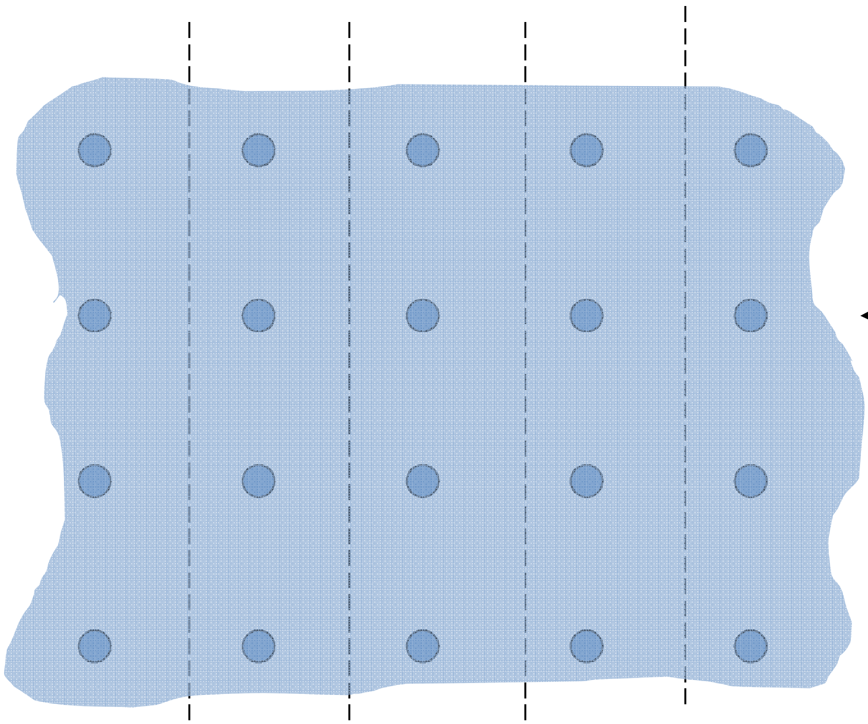

Quantum superposition of many valence-bond states: A "**spin liquid**."

Odd

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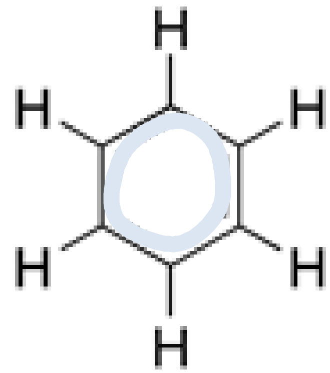

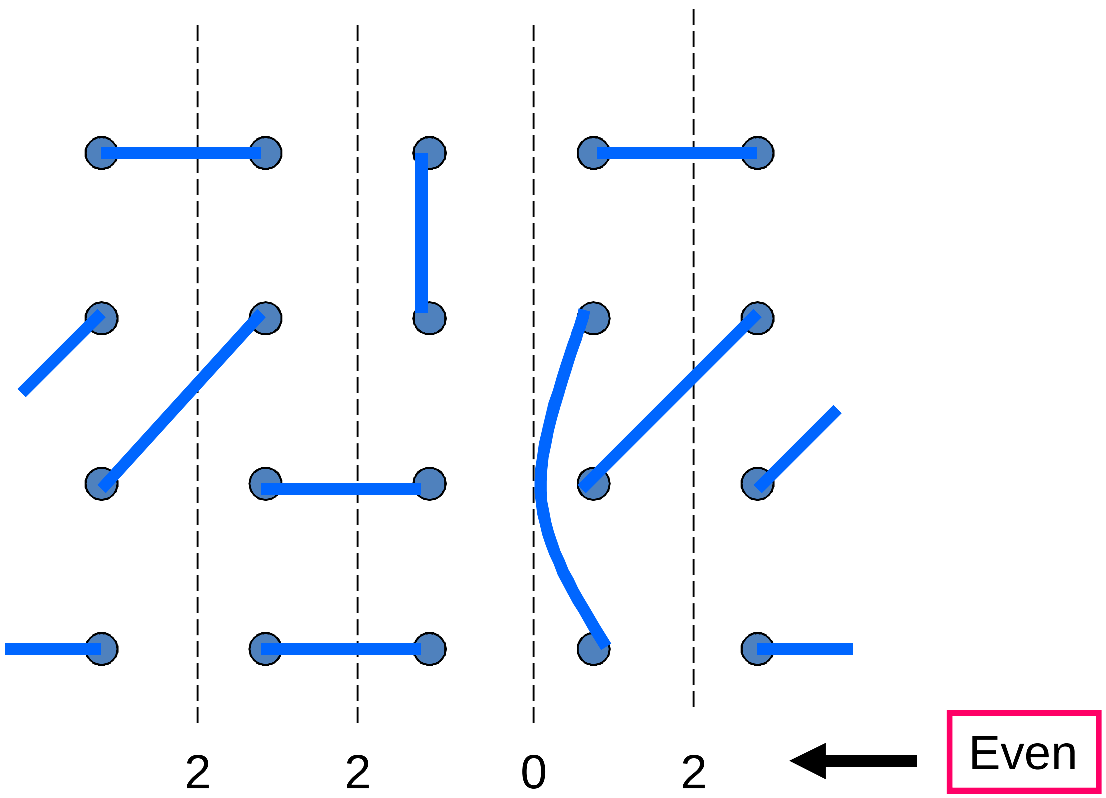

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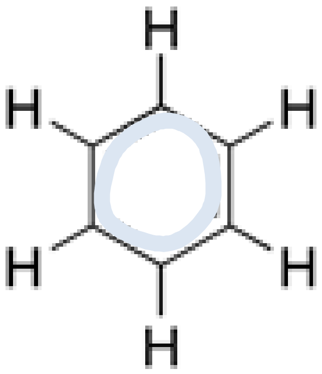

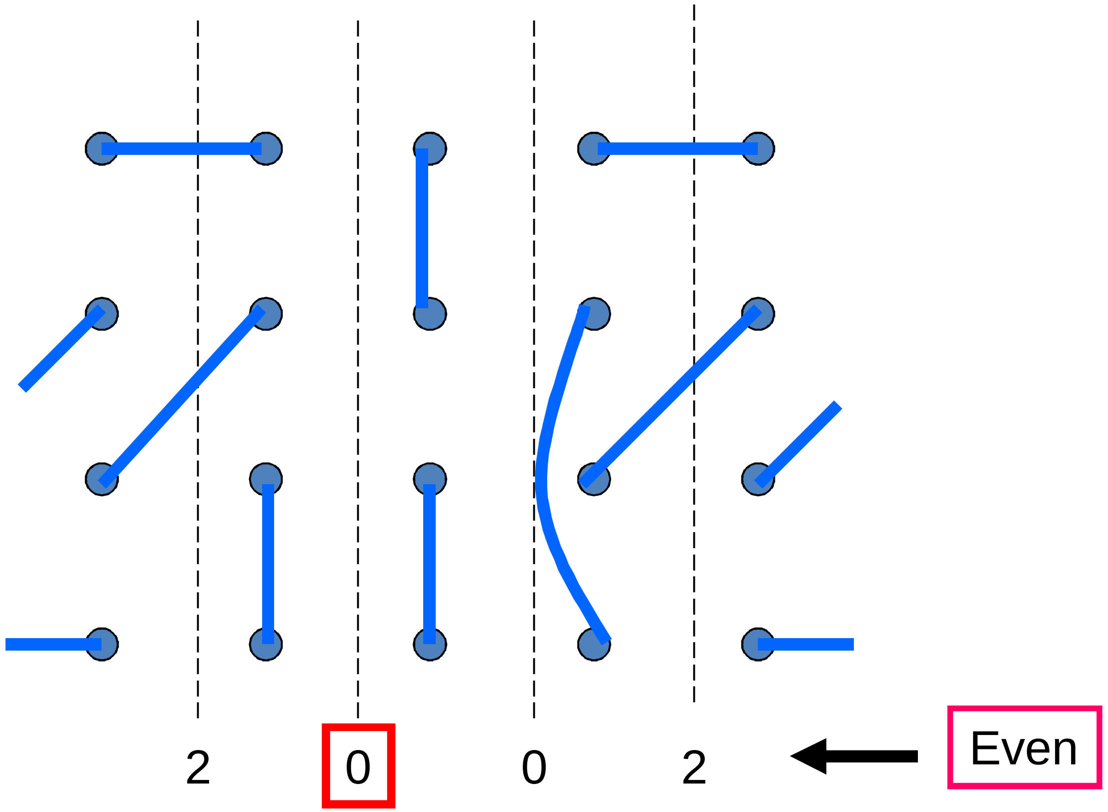

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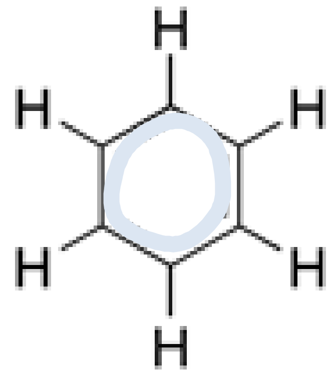

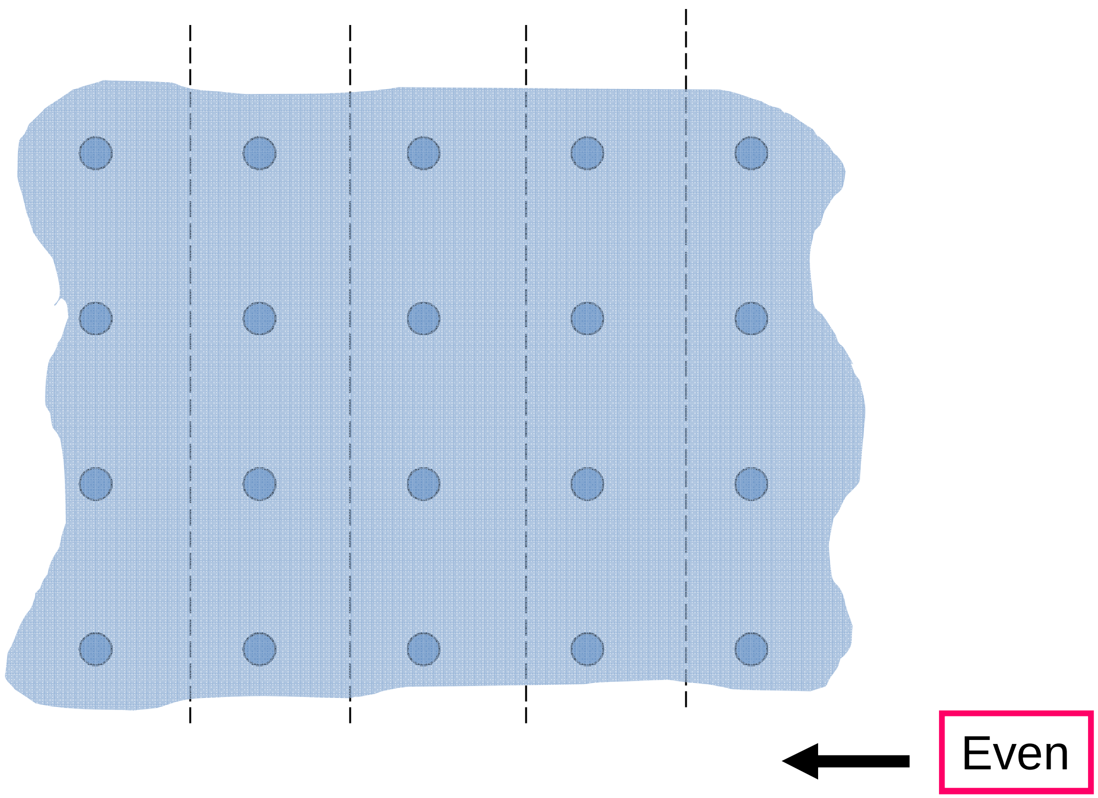

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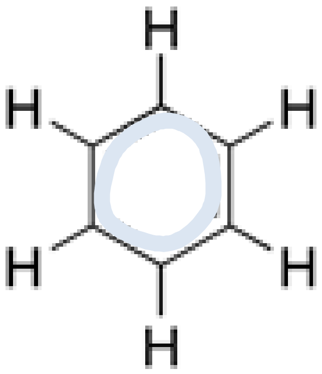
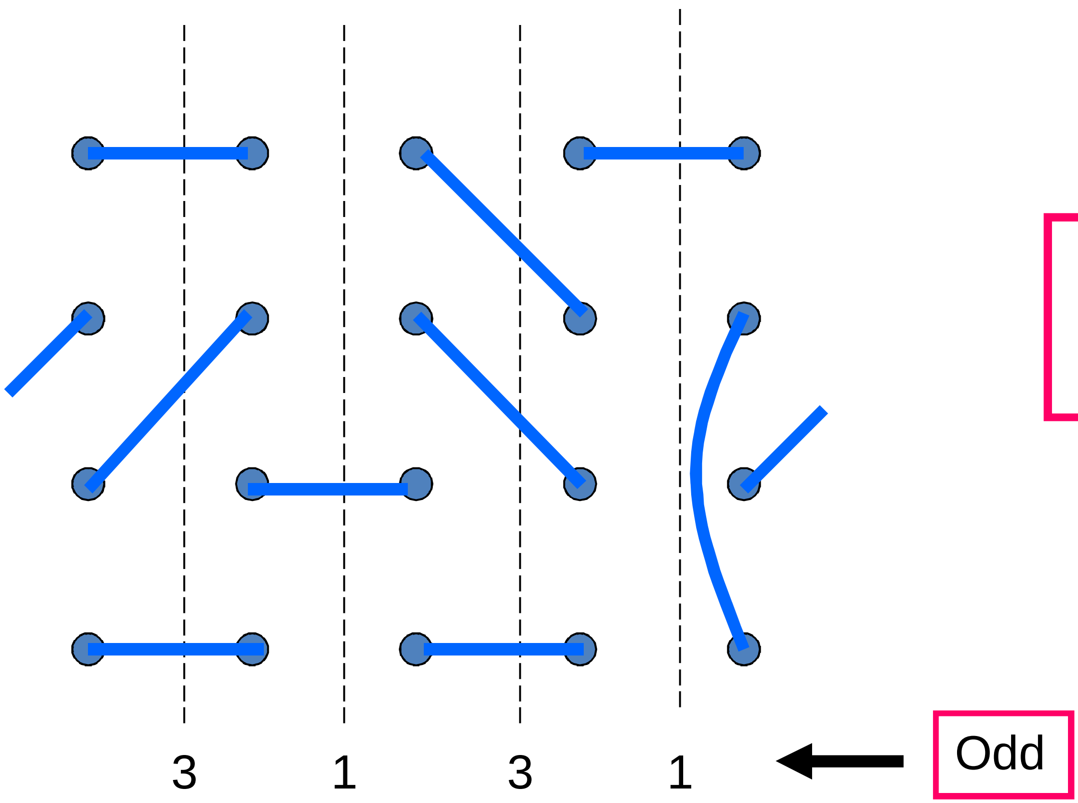


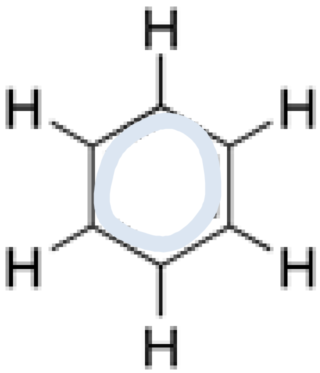


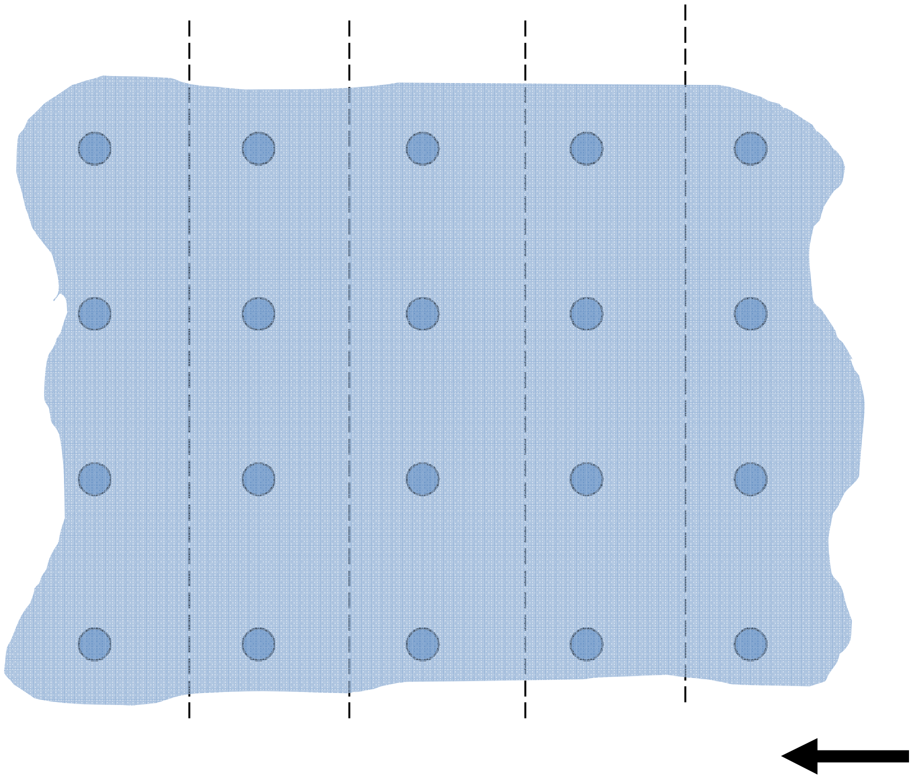
$|0\rangle$

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$





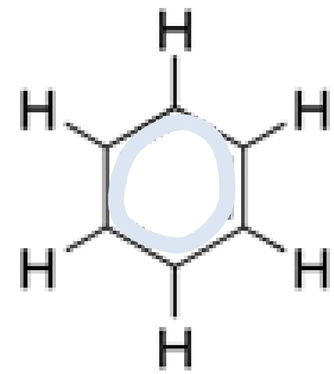
$|0\rangle$

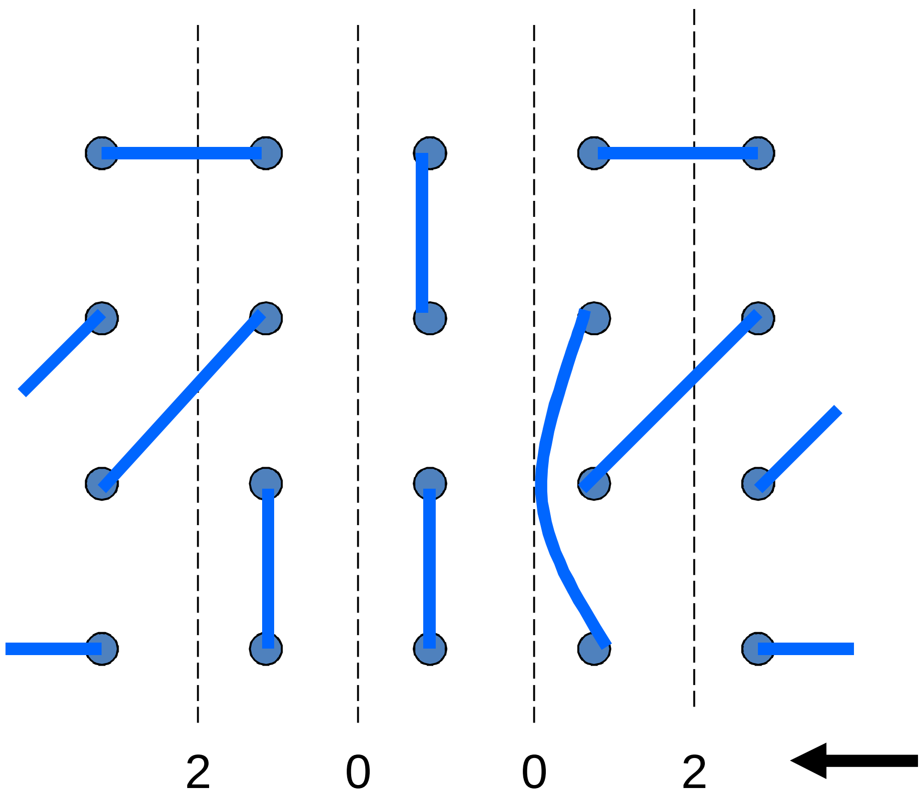
Odd

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$





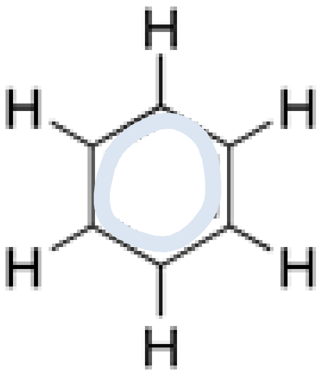
Even

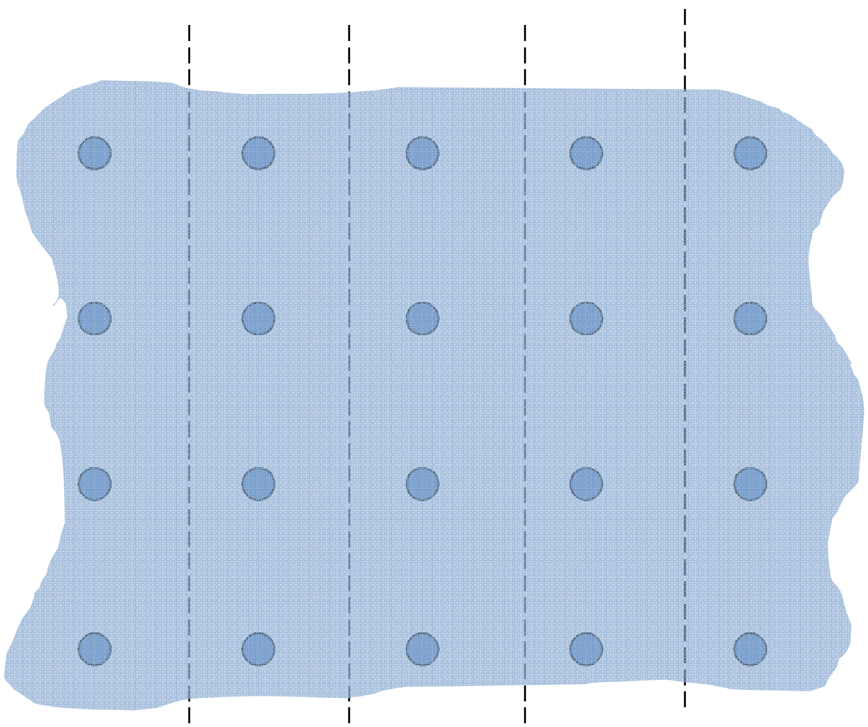
$|1\rangle$

Another Kind of Order

A valence bond:

$$= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$$



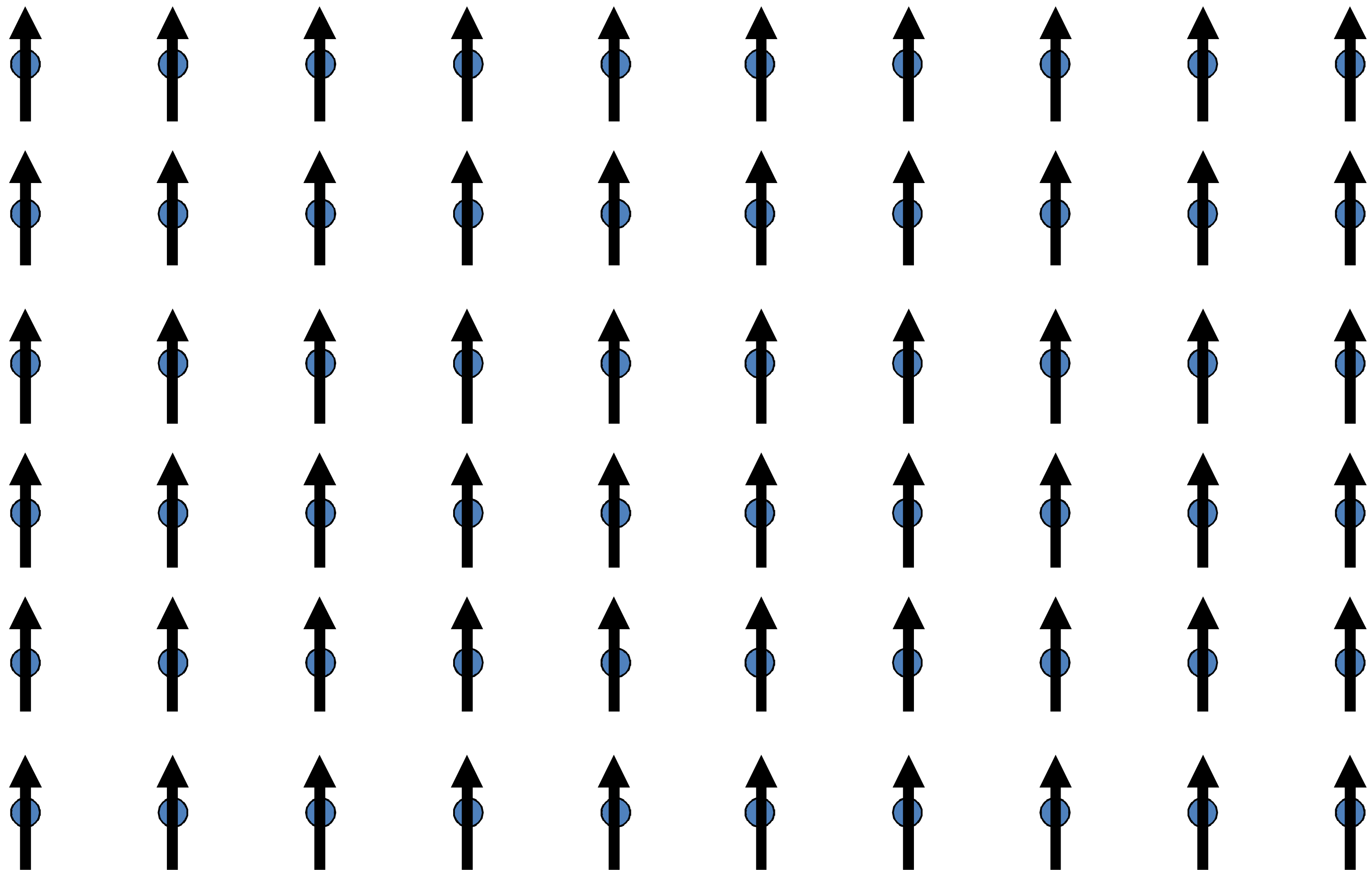


$|1\rangle$

Even

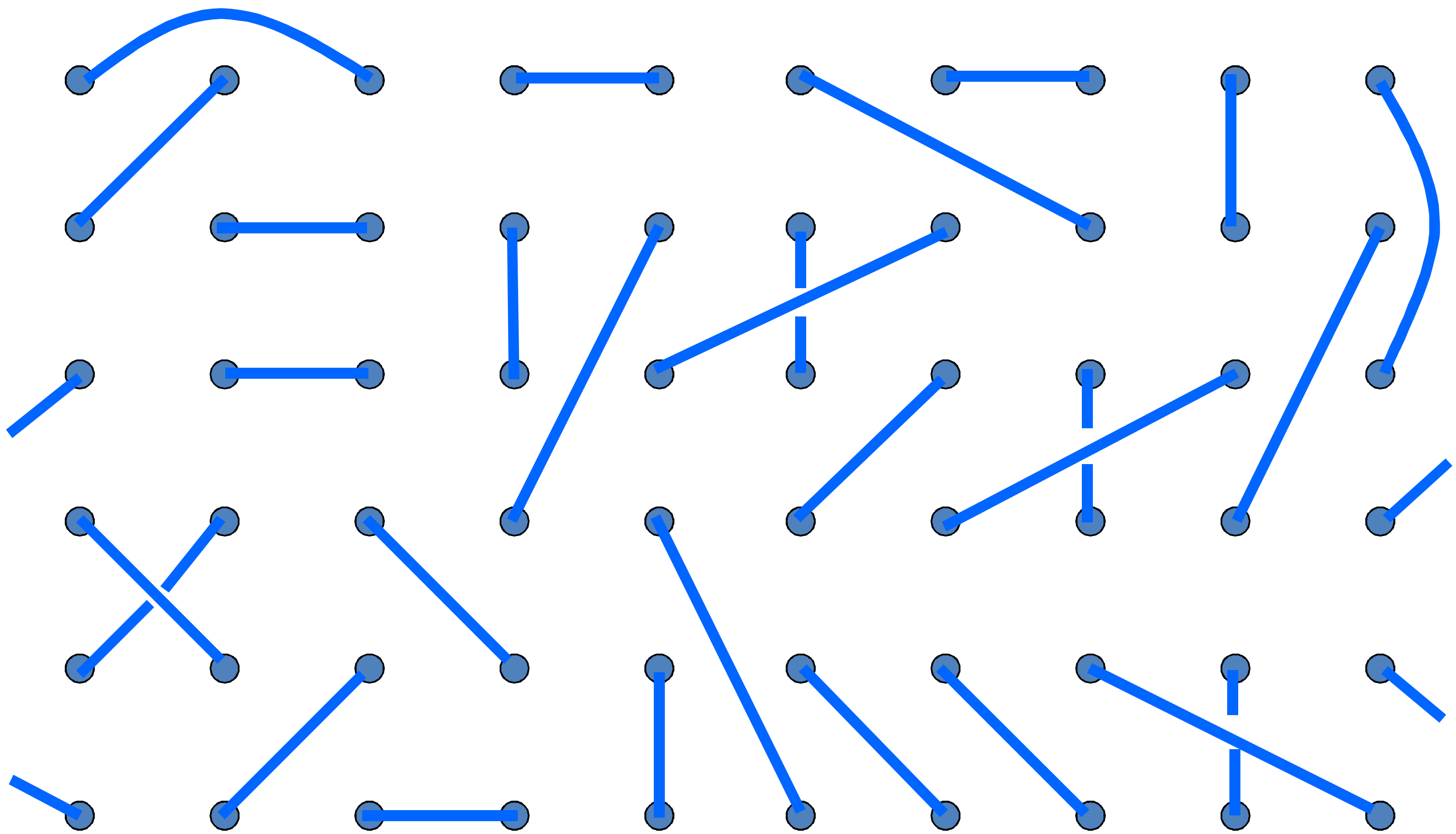
Is it a 0 or a 1?

Is it a 0 or a 1?

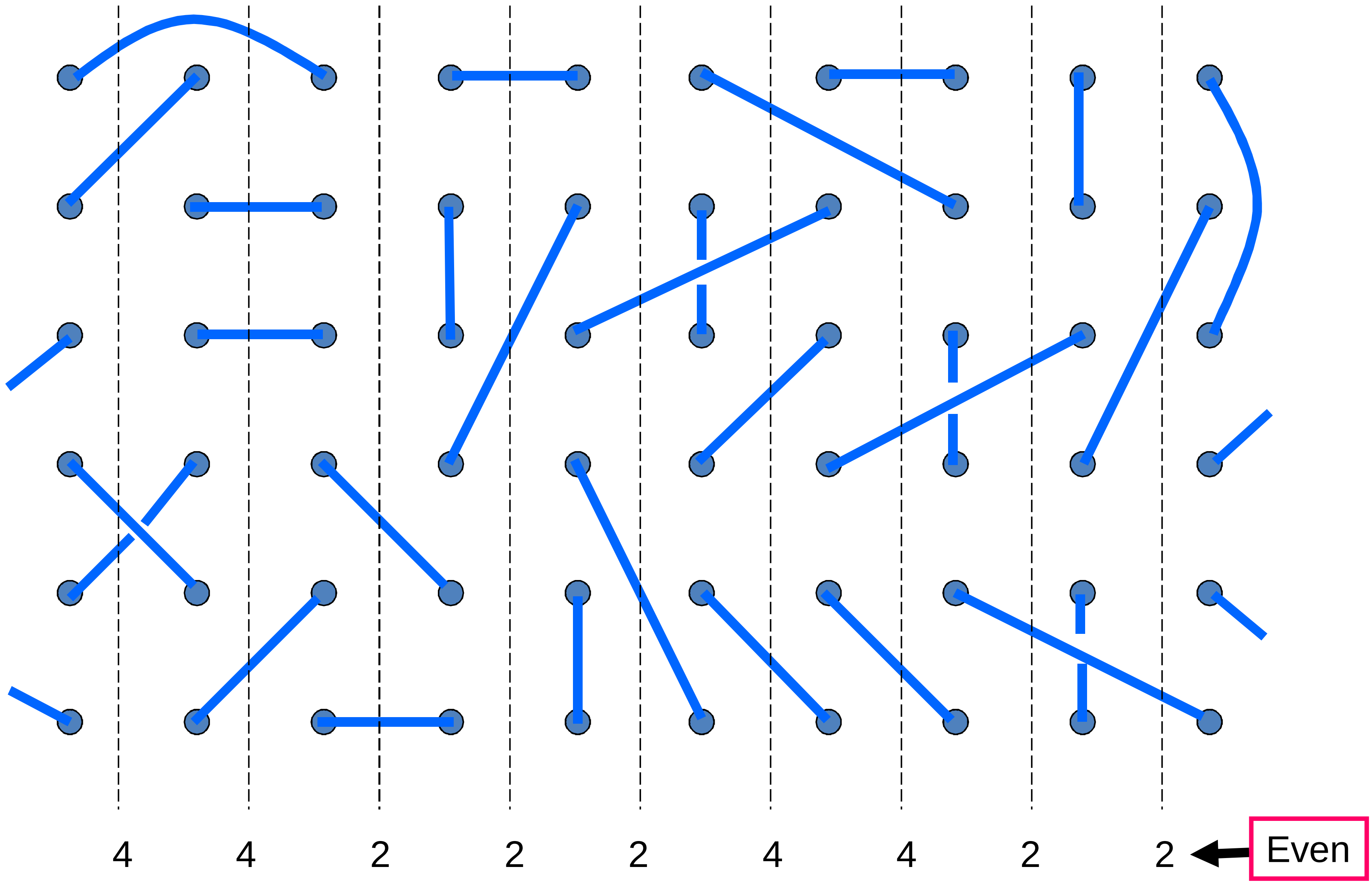


Is it a $|0\rangle$ or a $|1\rangle$?

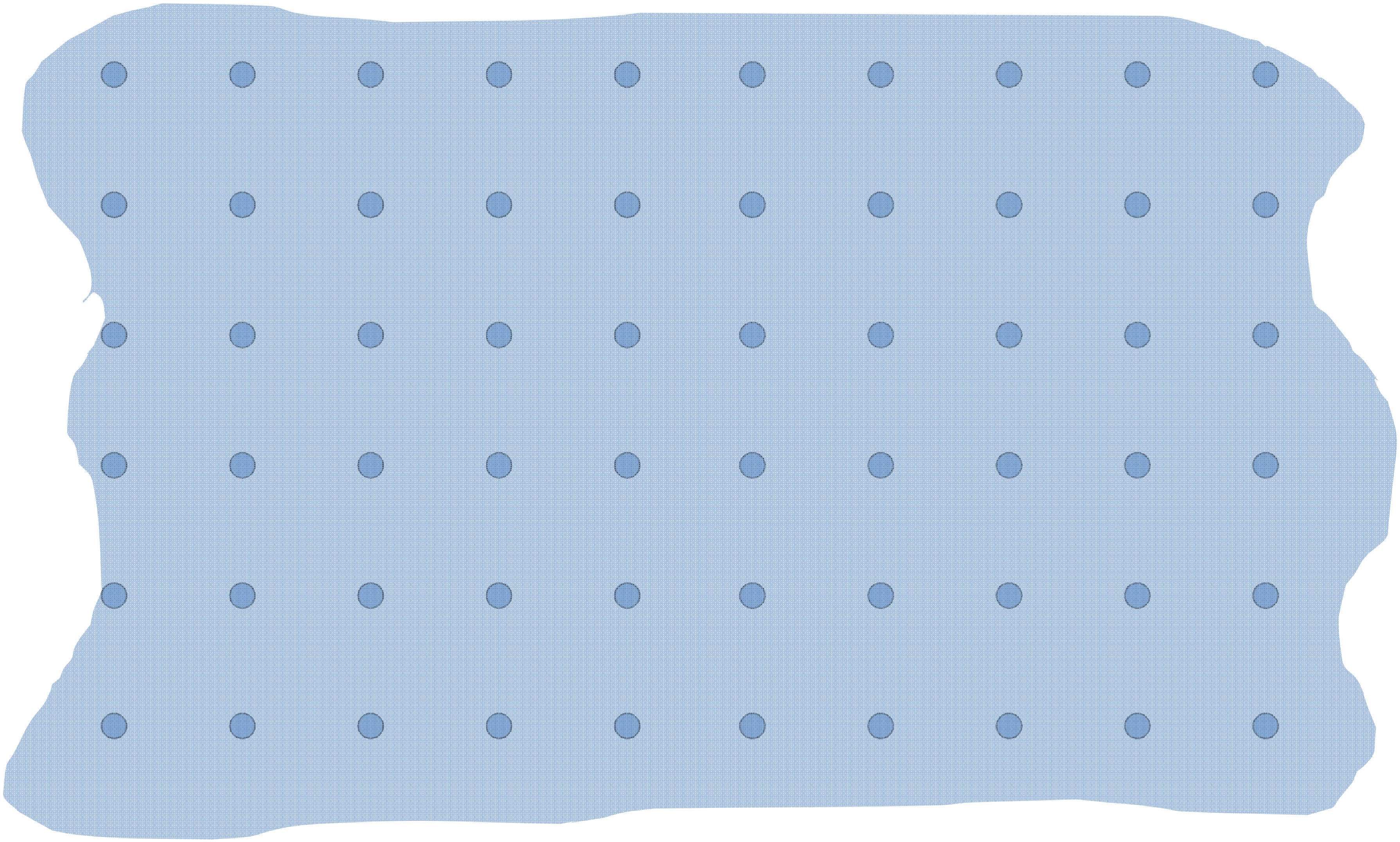
Is it a $|0\rangle$ or a $|1\rangle$?



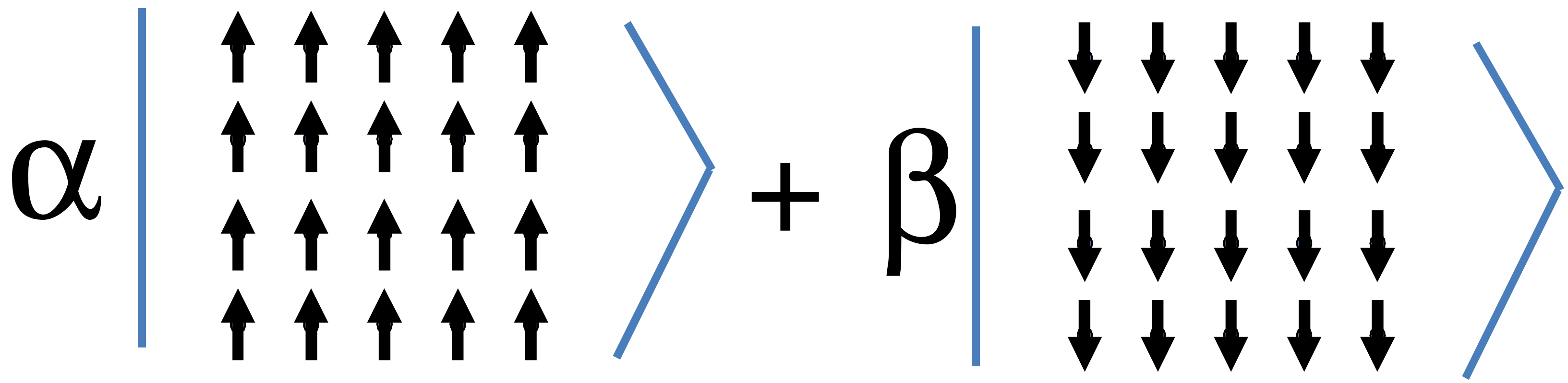
Is it a |0⟩ or a |1⟩?



Is it a $|0\rangle$ or a $|1\rangle$?



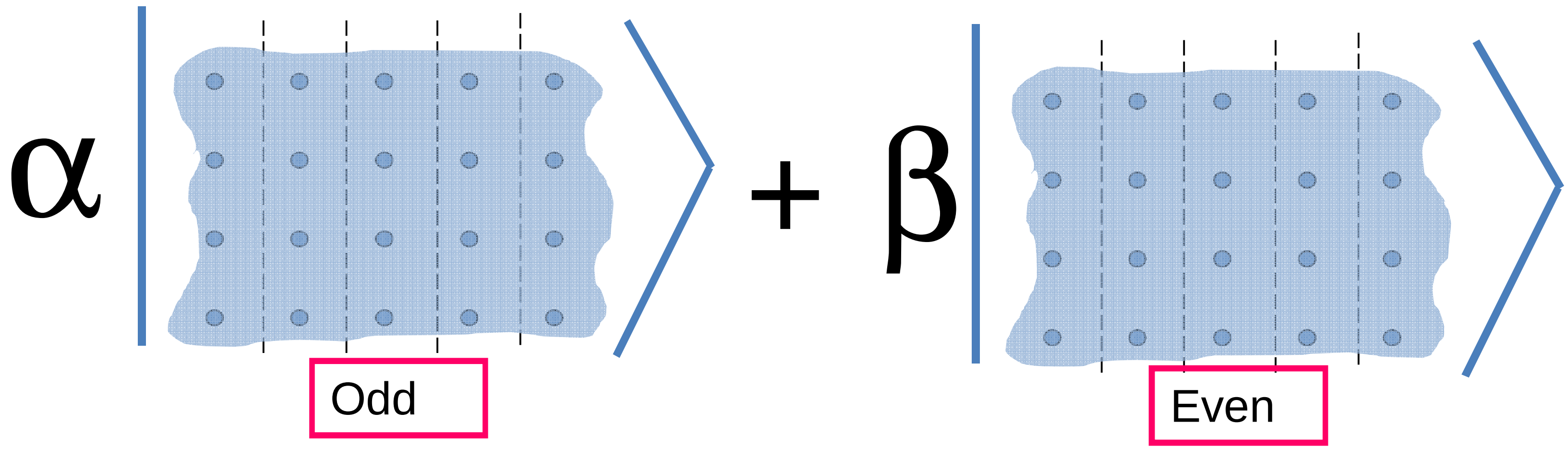
Storing a Qubit



Environment can measure the state of the qubit by a local measurement – any quantum superposition will decohere almost instantly.

Bad Qubit!

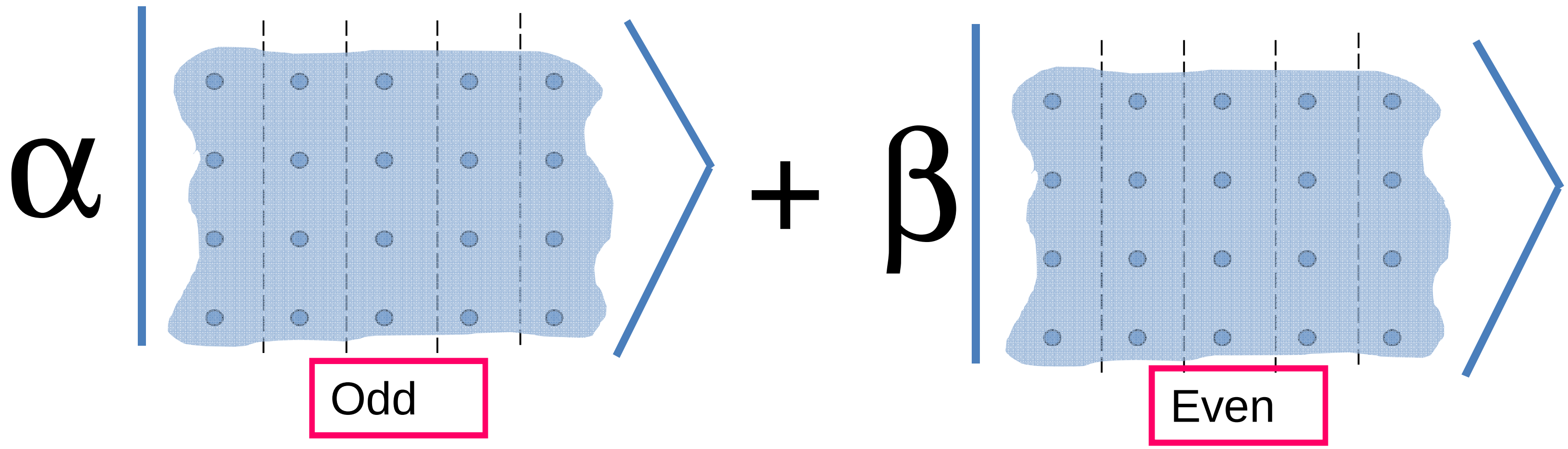
Storing a Qubit



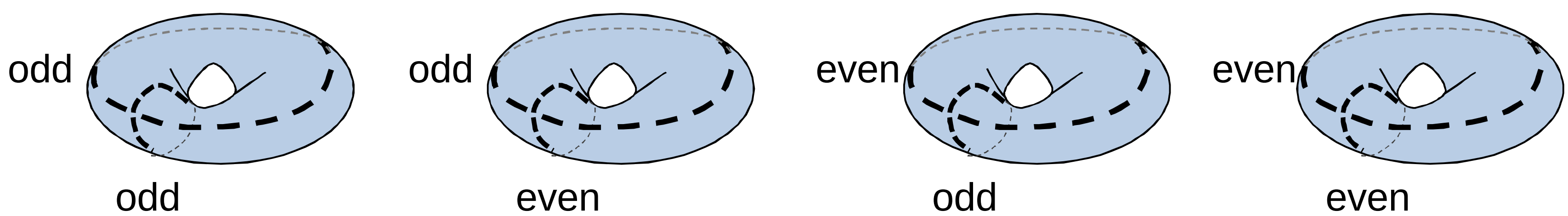
Environment can only measure the state of the qubit by a global measurement – quantum superposition should have long coherence time.

Good Qubit!

Storing a Qubit

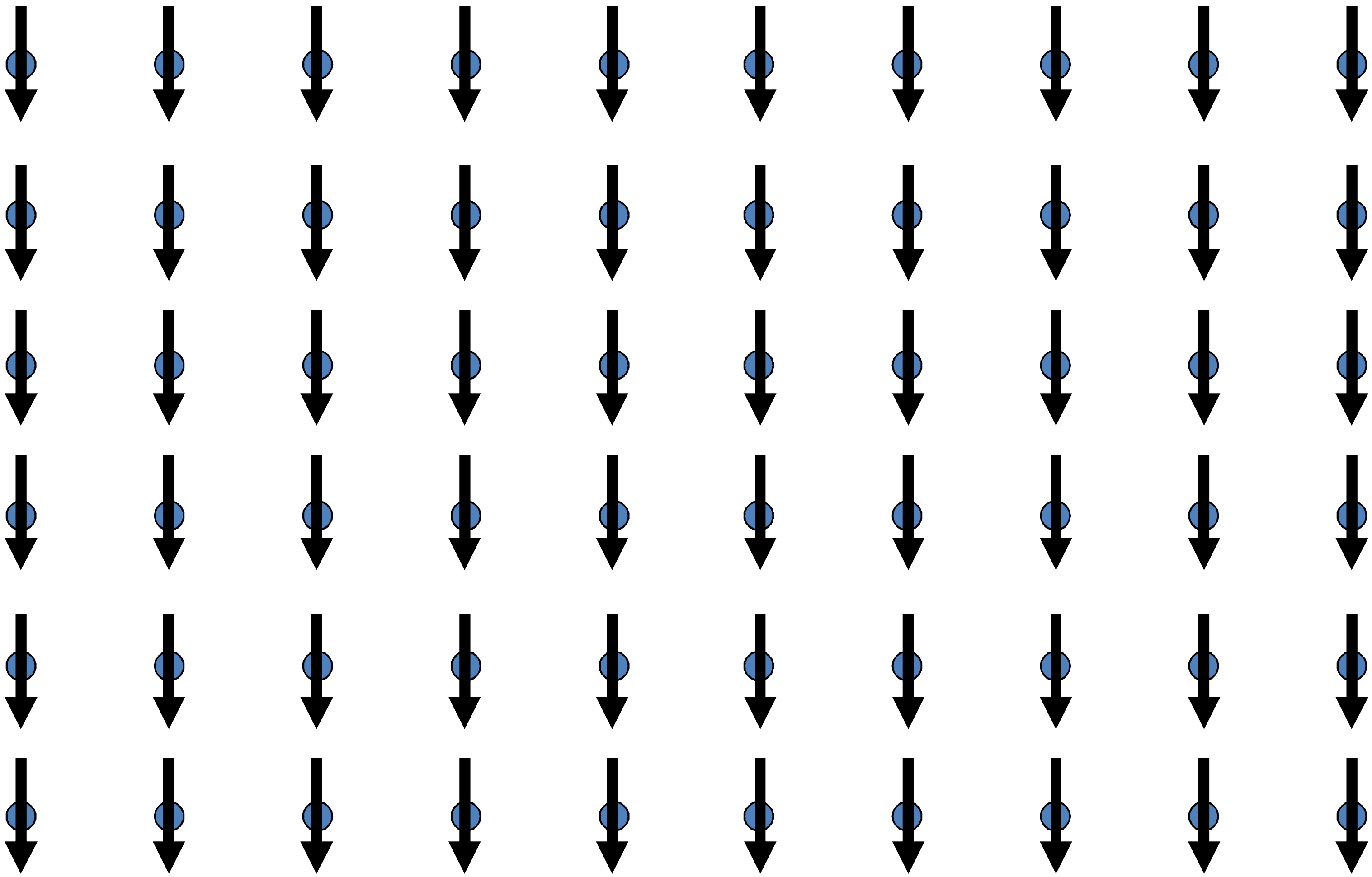


Topologically Ordered States (Wen & Niu, '90) : Multiple ground states on topologically nontrivial surfaces with no locally observable order parameter.

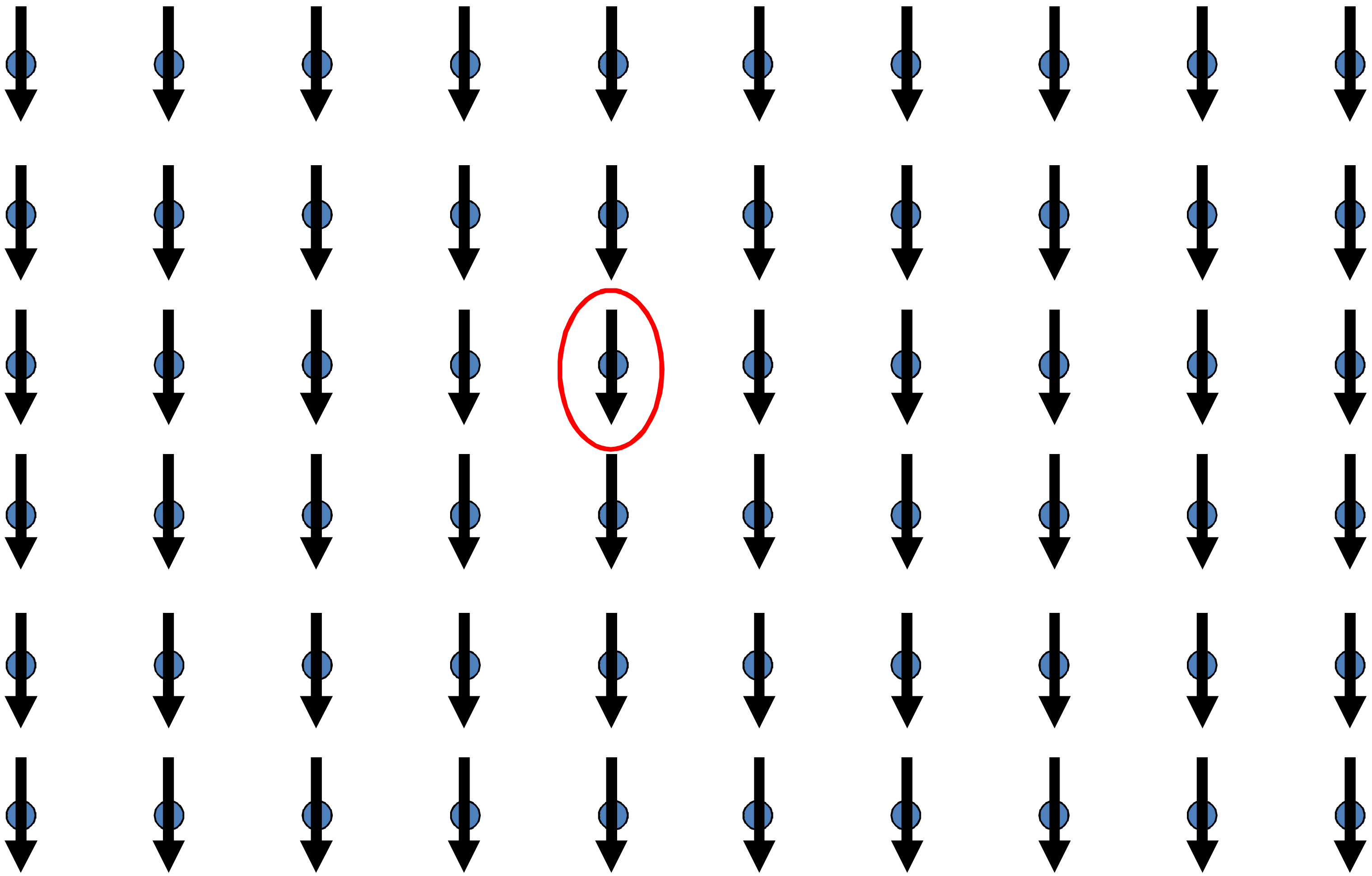


Nature's quantum error correcting codes ?

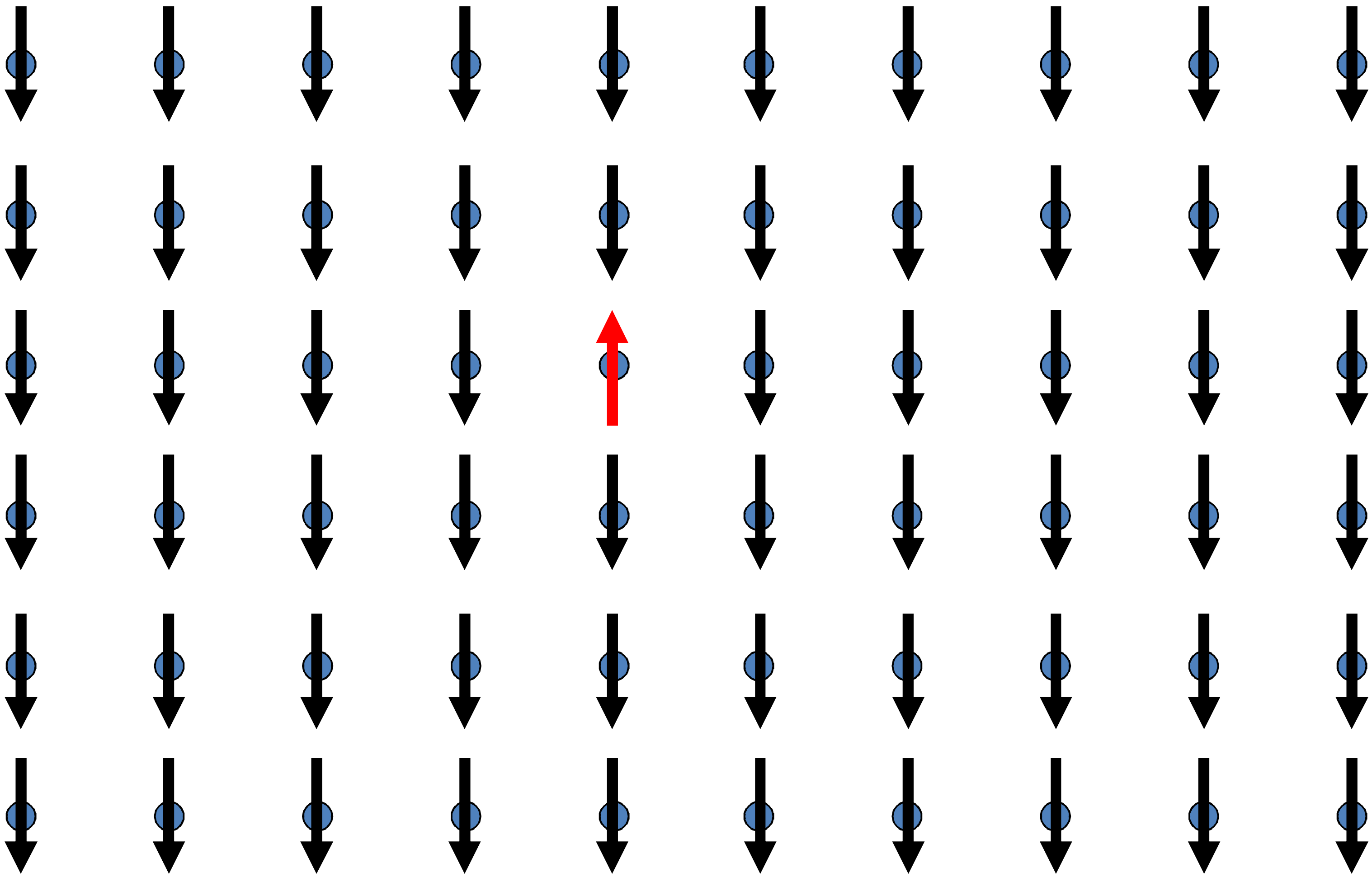
Conventional Order: Excitations



Conventional Order: Excitations

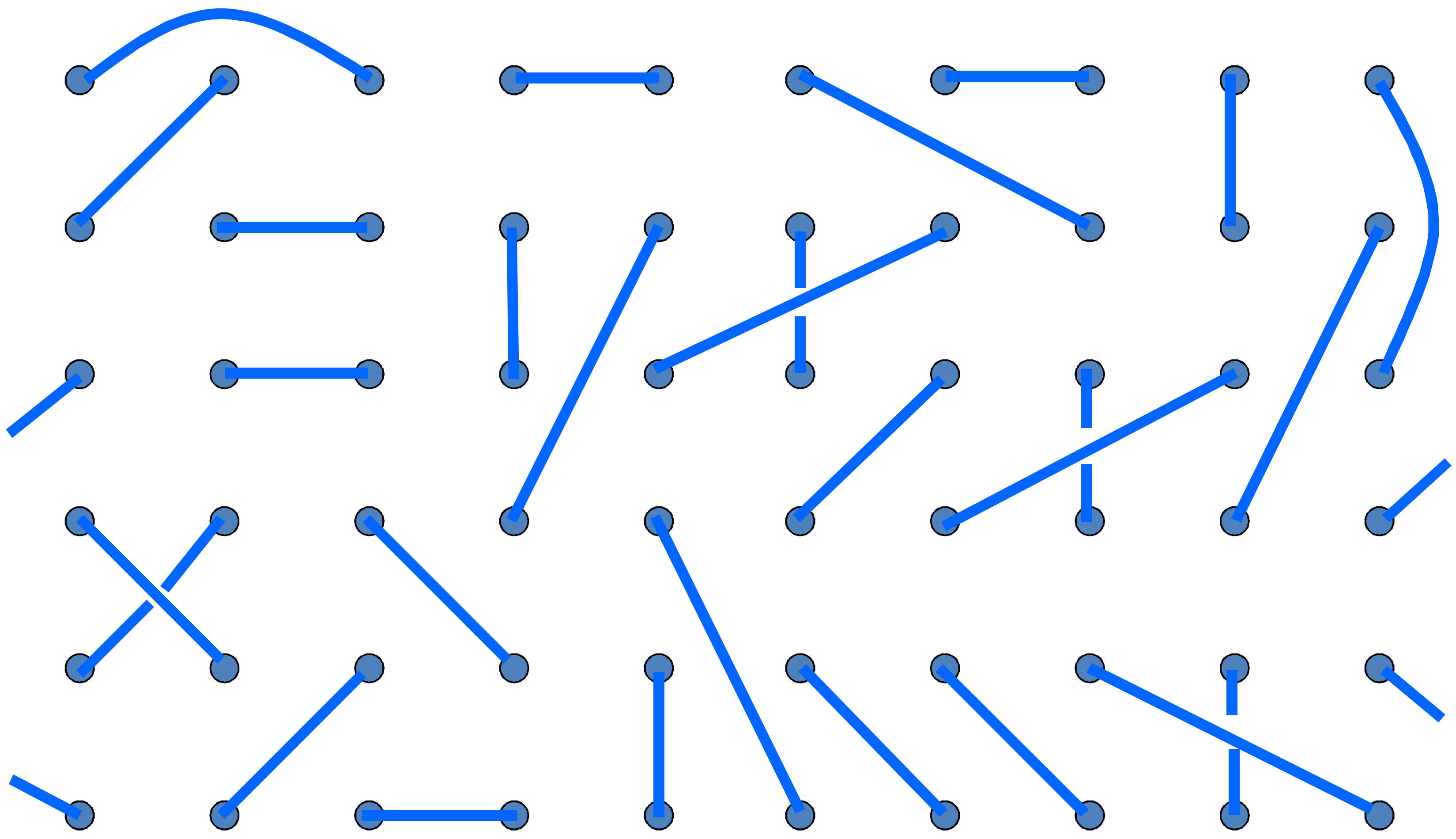


Conventional Order: Excitations

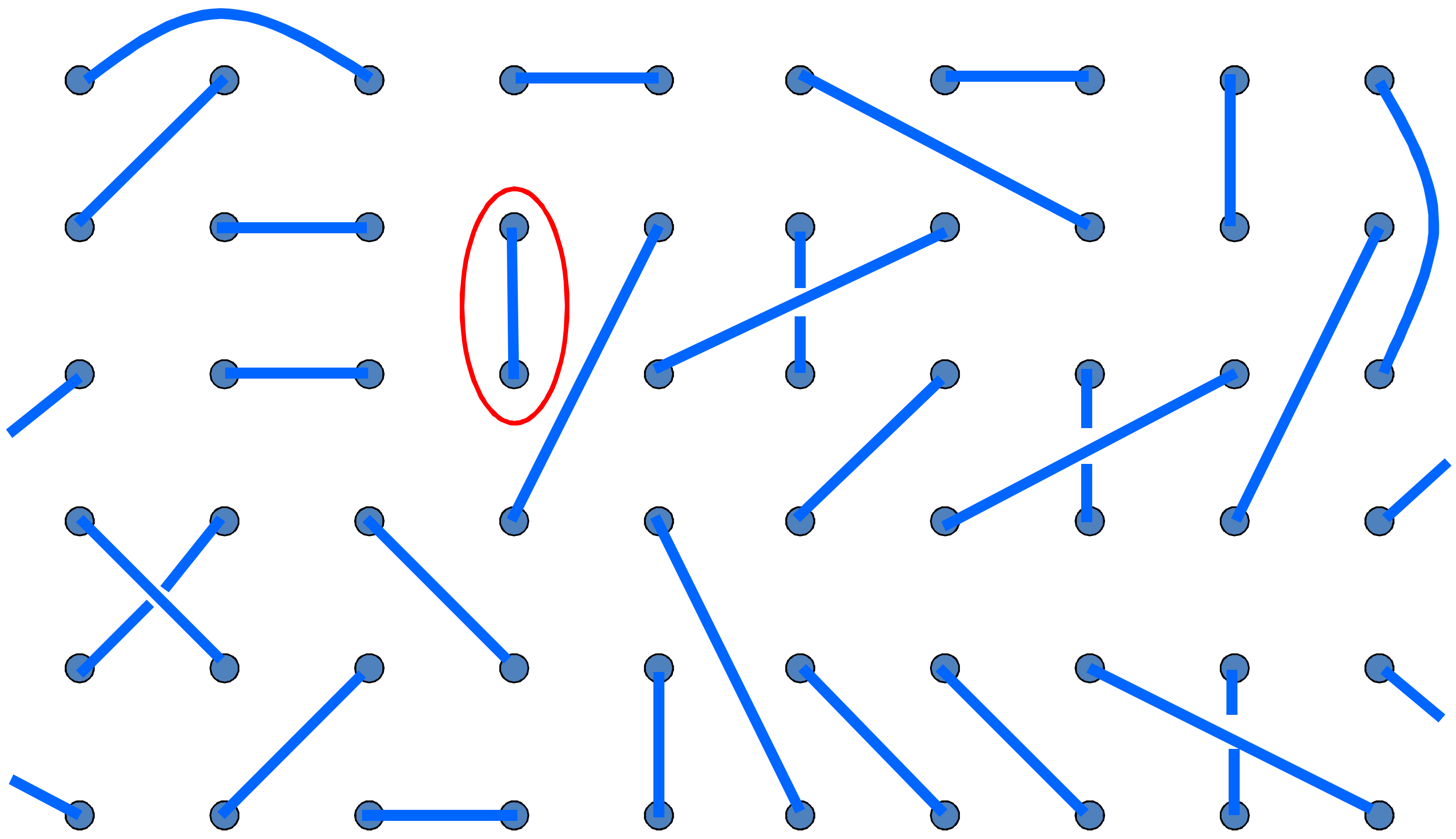


Spin flip: "**quasiparticle**" with total S_z =+1

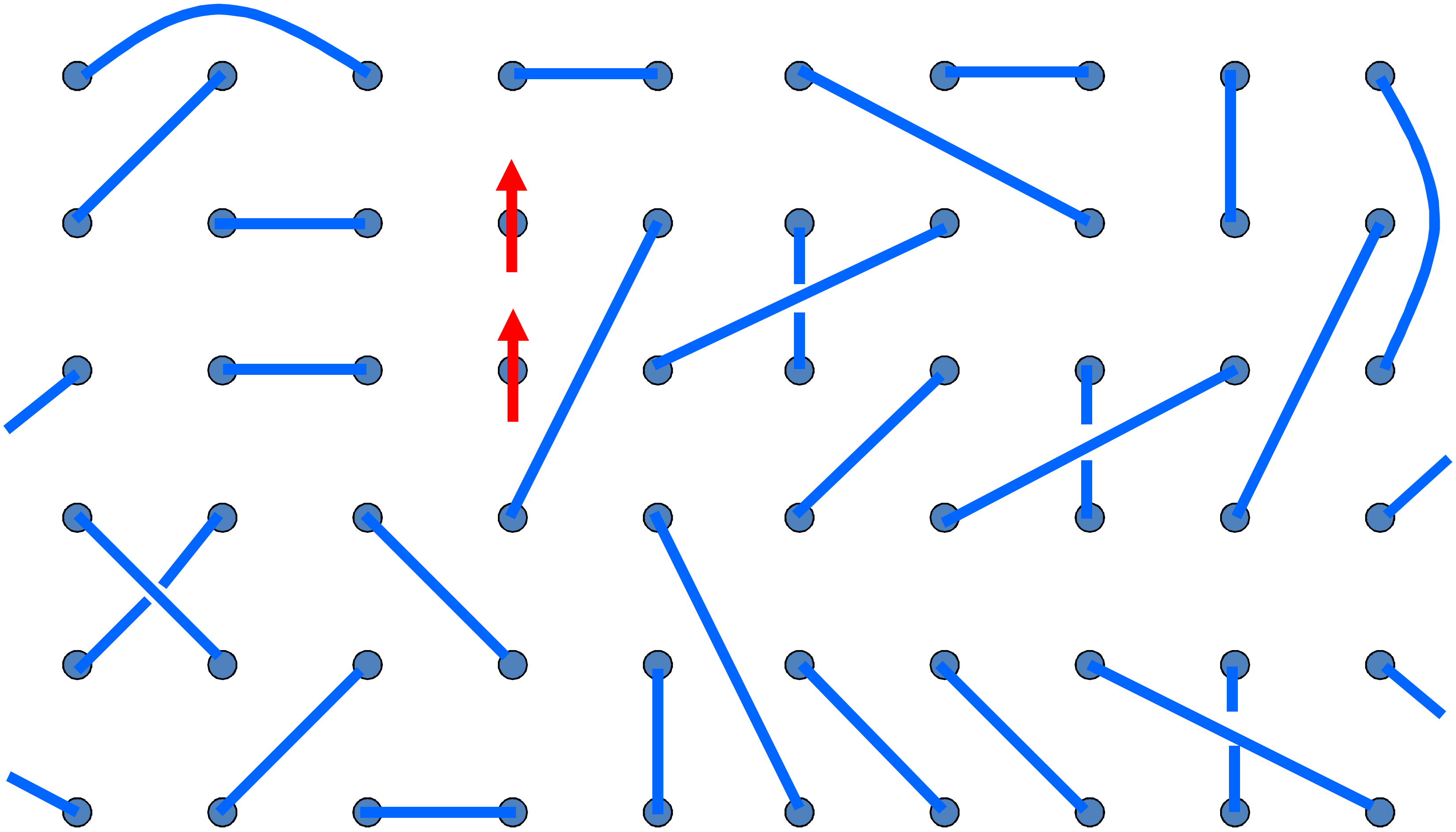
Topological Order: Excitations



Topological Order: Excitations

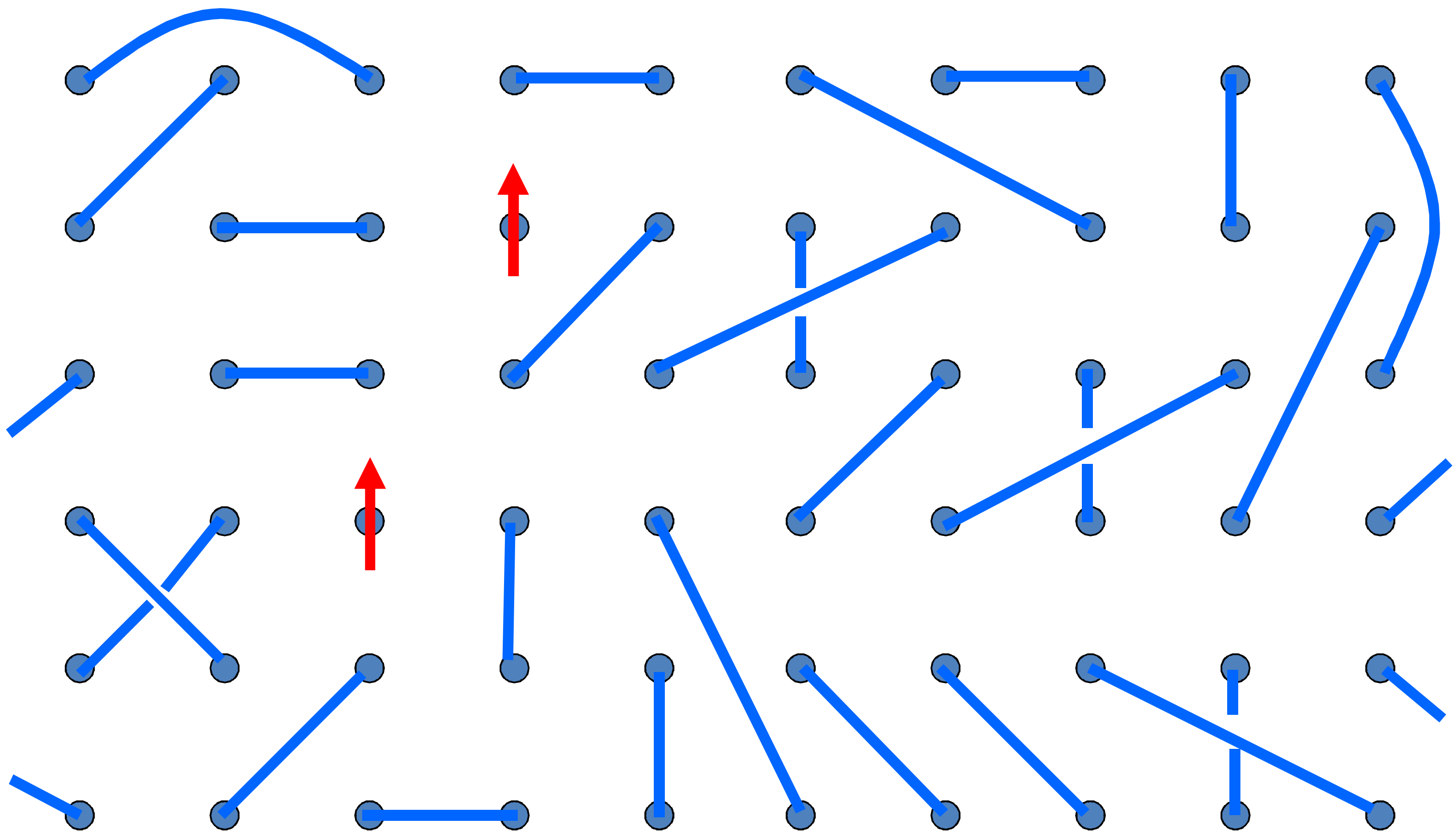


Topological Order: Excitations



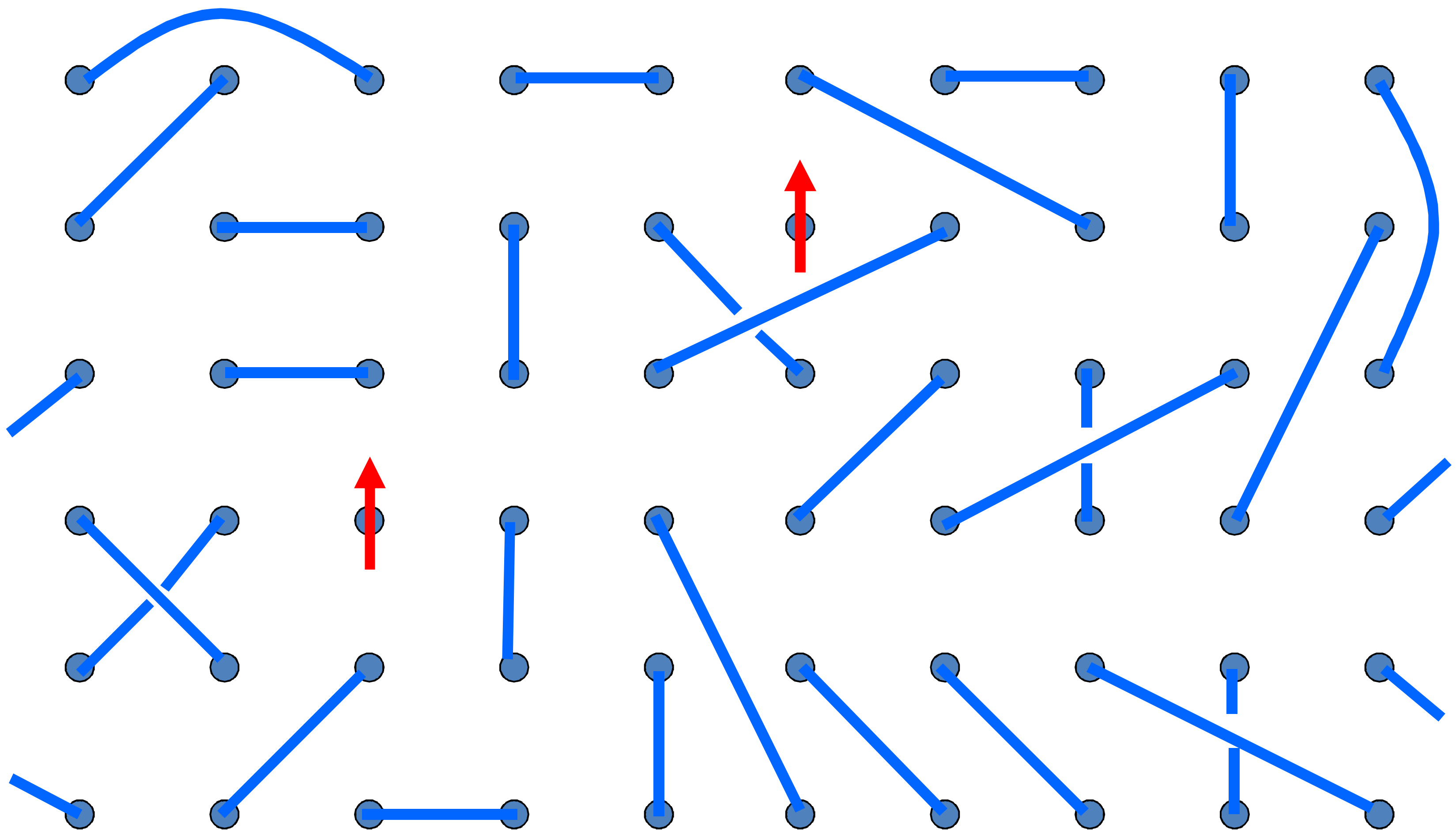
Breaking a bond creates an excitation with $S_z = 1$

Topological Order: Excitations



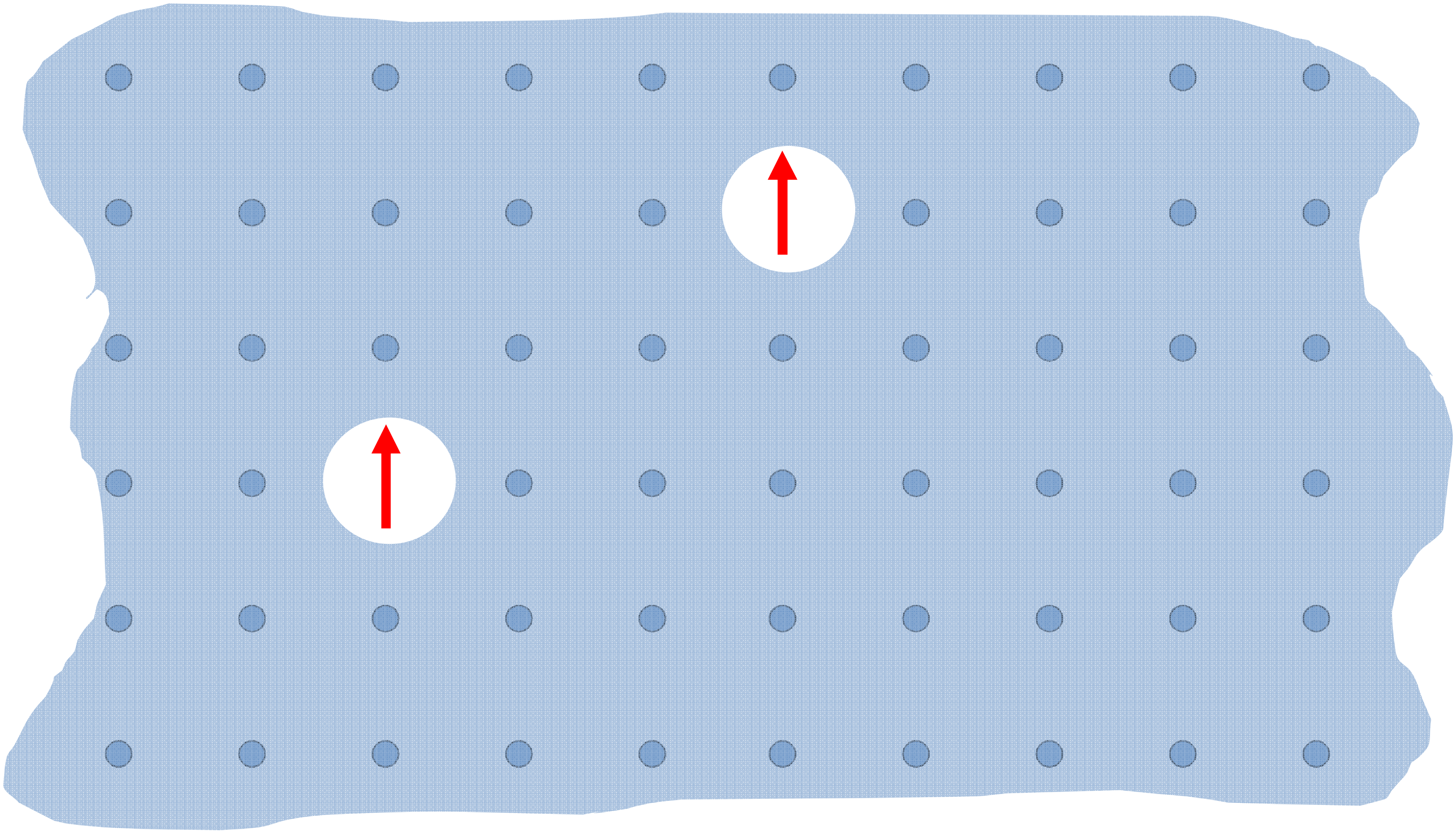
Breaking a bond creates an excitation with $S_z = 1$

Topological Order: Excitations



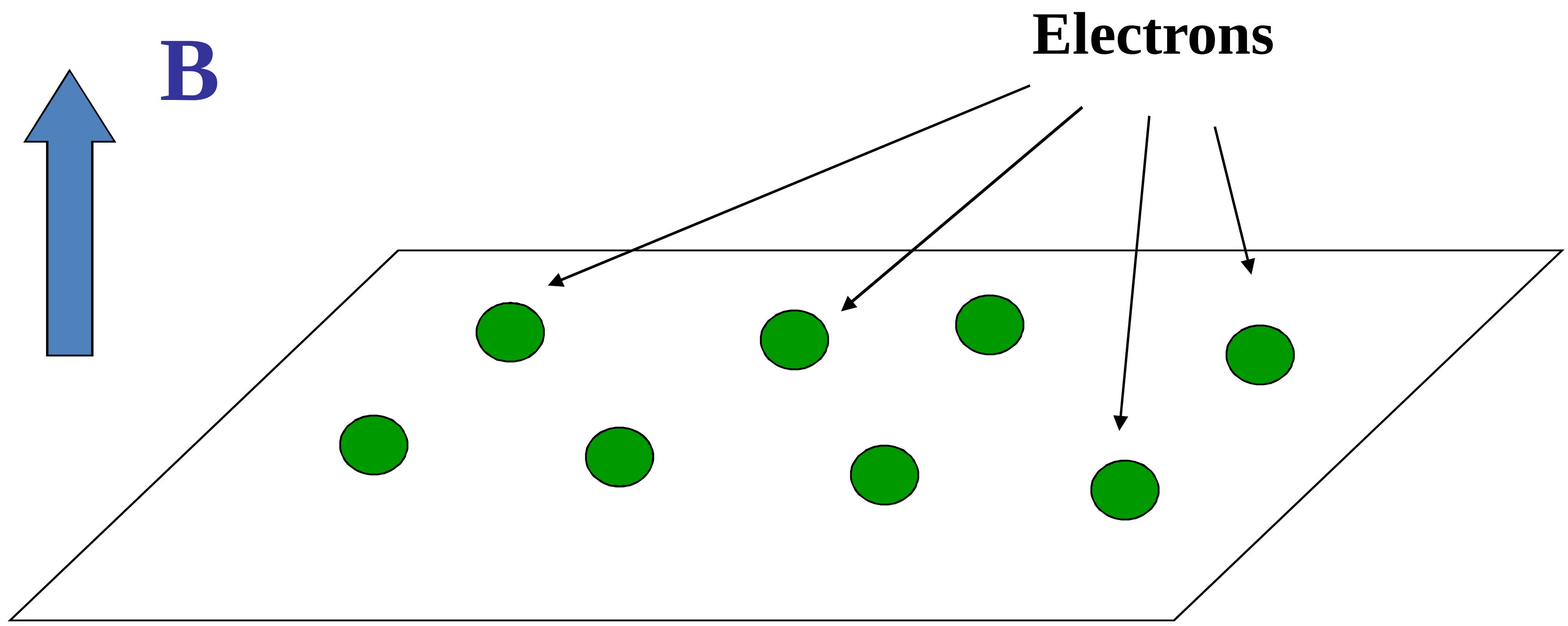
Breaking a bond creates an excitation with $S_z = 1$

Fractionalization



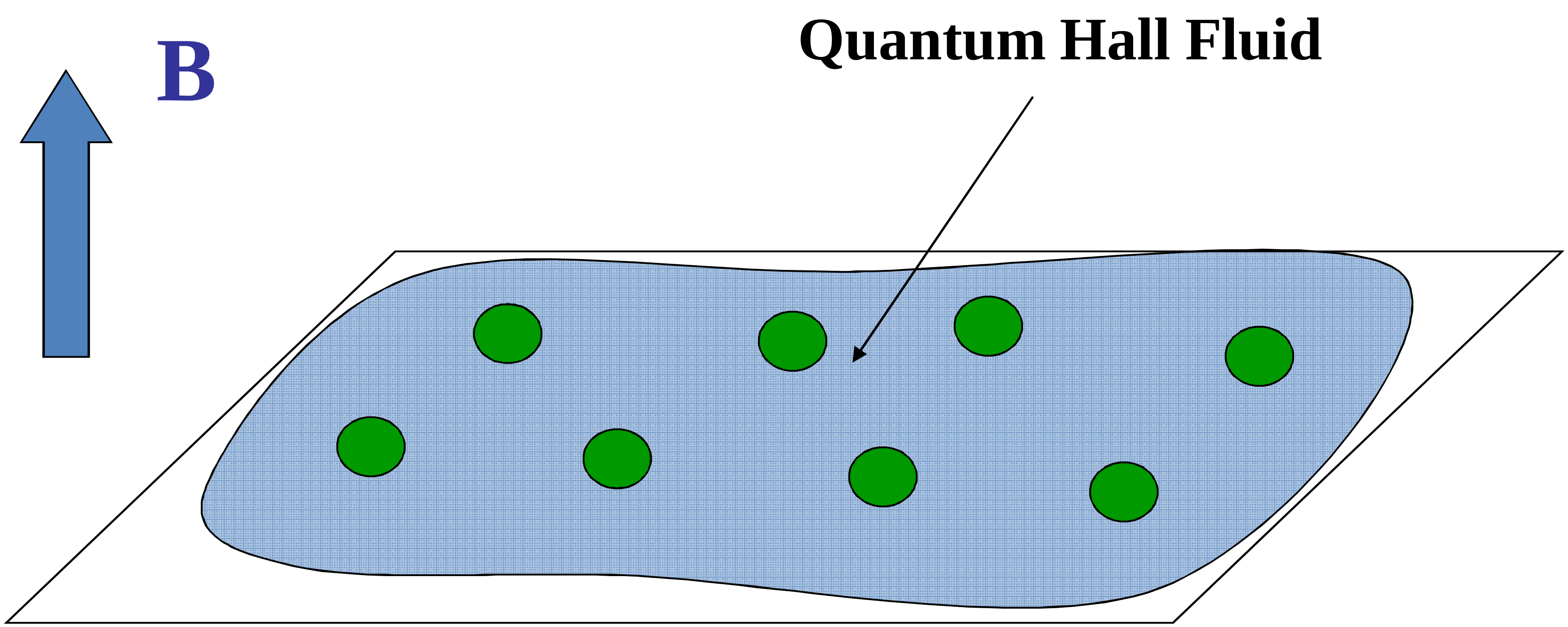
S_z = 1 excitation ***fractionalizes*** into two S_z = ½ quasiparticles.

Fractional Quantum Hall States



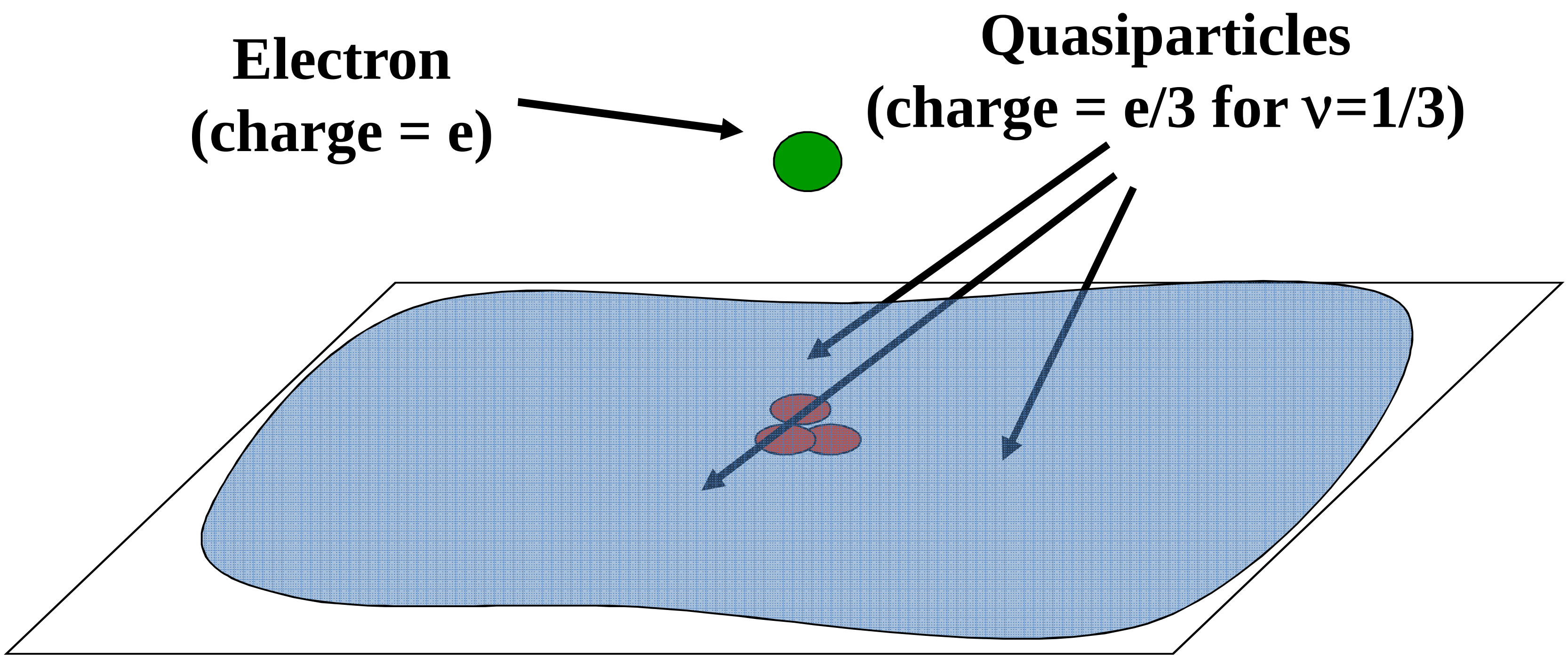
A two dimensional gas of electrons in a strong magnetic field **B**.

Fractional Quantum Hall States



An **incompressible quantum liquid** can form when the Landau level filling fraction $\nu = n_{elec}(hc/eB)$ is a rational fraction.

Charge Fractionalization

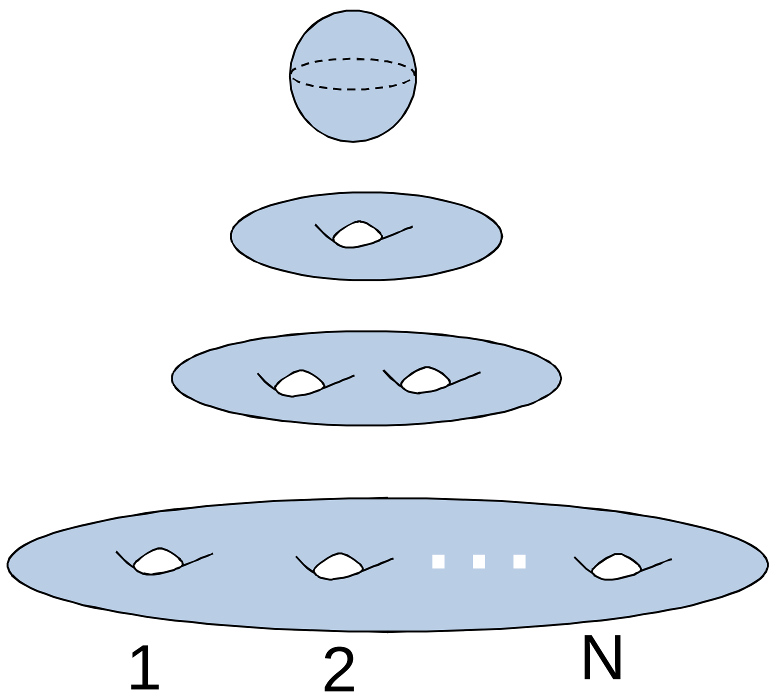


When an electron is added to a FQH state it can be **fractionalized** --- i.e., it can break apart into **fractionally charged quasiparticles.**

Topological Degeneracy (Wen & Niu, PRB 41, 9377 (1990))

As in our spin-liquid example, FQH states on **topologically nontrivial surfaces** have degenerate ground states which **can only be distinguished by global measurements**.

For the ν **= 1/3** state:



Degeneracy

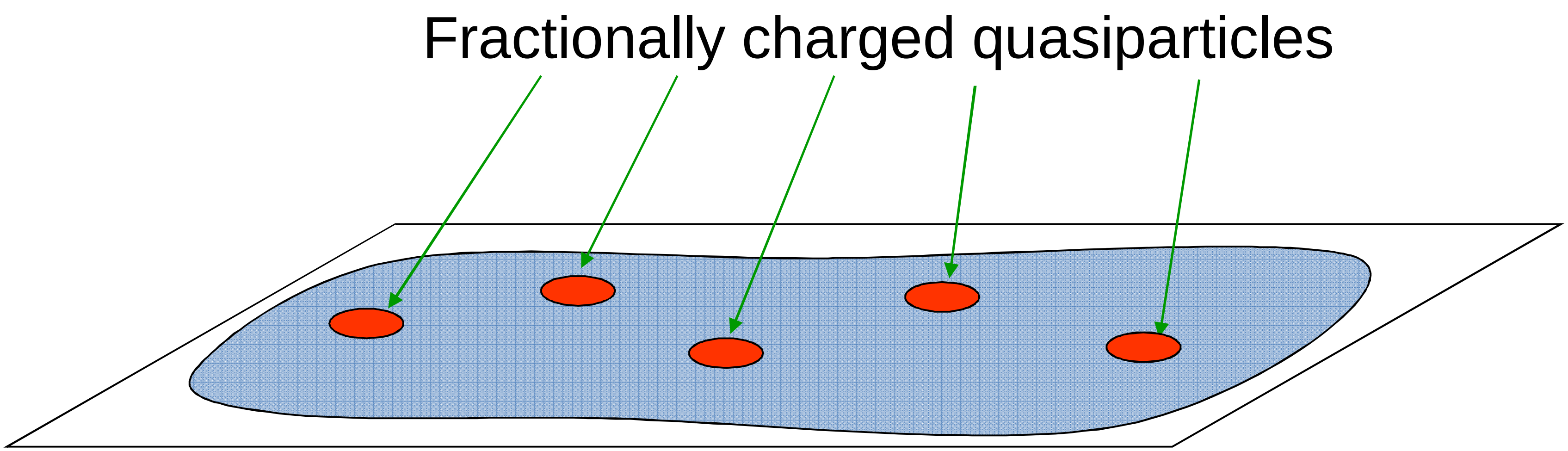
1

3

9

3^N

"Non-Abelian" FQH States (Moore & Read '91)



Essential features:

A degenerate Hilbert space whose dimensionality is **exponentially large in the number of quasiparticles**.

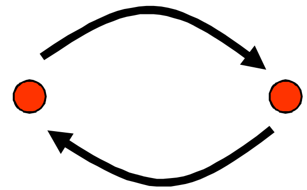
States in this space **can only be distinguished by global measurements** provided quasiparticles are far apart.

→ ***A perfect place to hide quantum information!***

Identical Quantum Particles

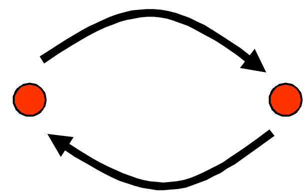
$$\left|\psi(r_1, r_2)\right\rangle$$

One exchange



$$\left|\psi(r_2, r_1)\right\rangle = \lambda \left|\psi(r_1, r_2)\right\rangle$$

A second exchange



$$\left|\psi(r_1, r_2)\right\rangle = \lambda^2 \left|\psi(r_2, r_1)\right\rangle$$

Two exchanges = Identity → $\lambda^2 = 1$

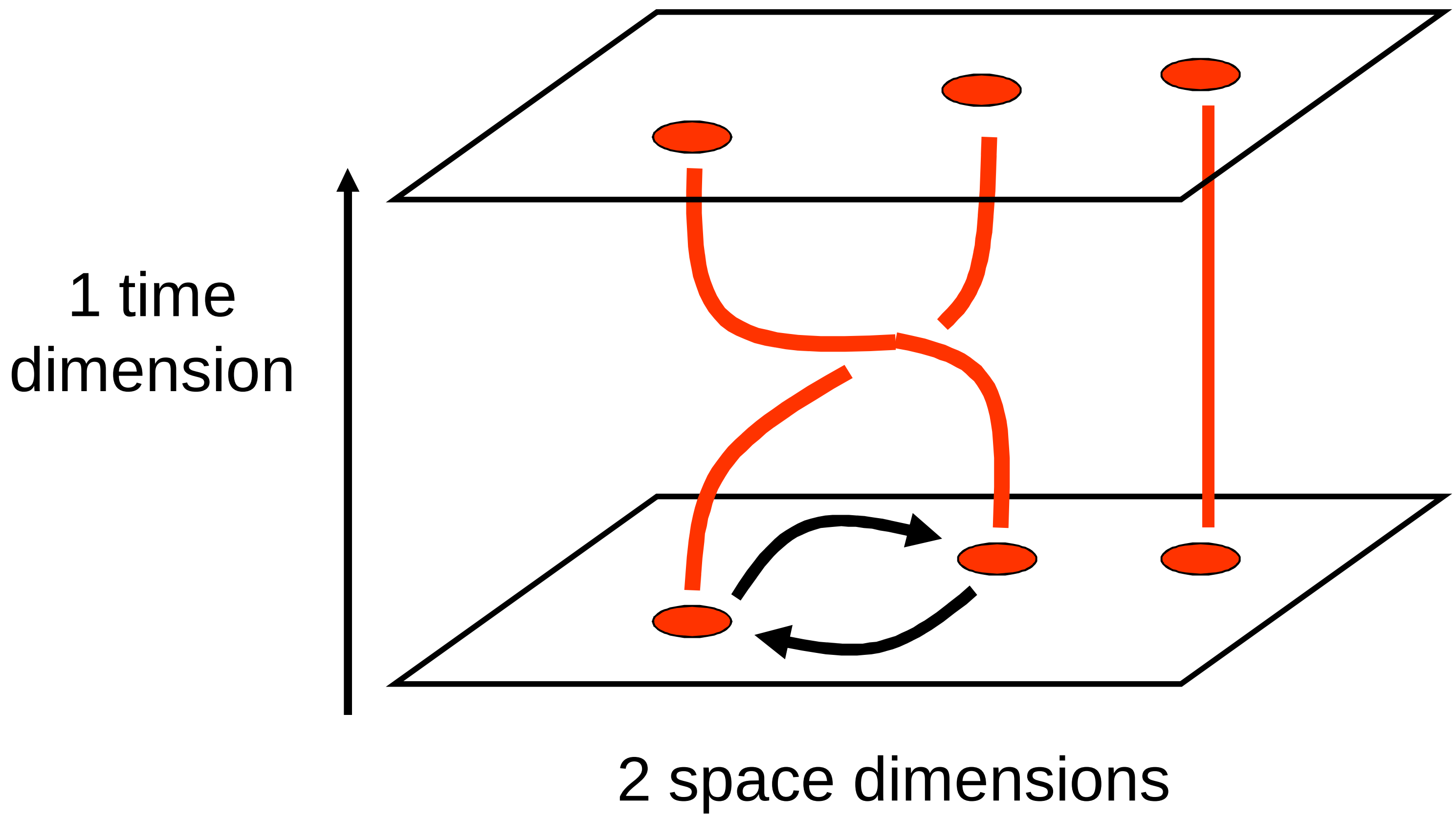
$\lambda = +1$ Bosons

Photons, He^4 atoms, Gluons…

$\lambda = -1$ Fermions

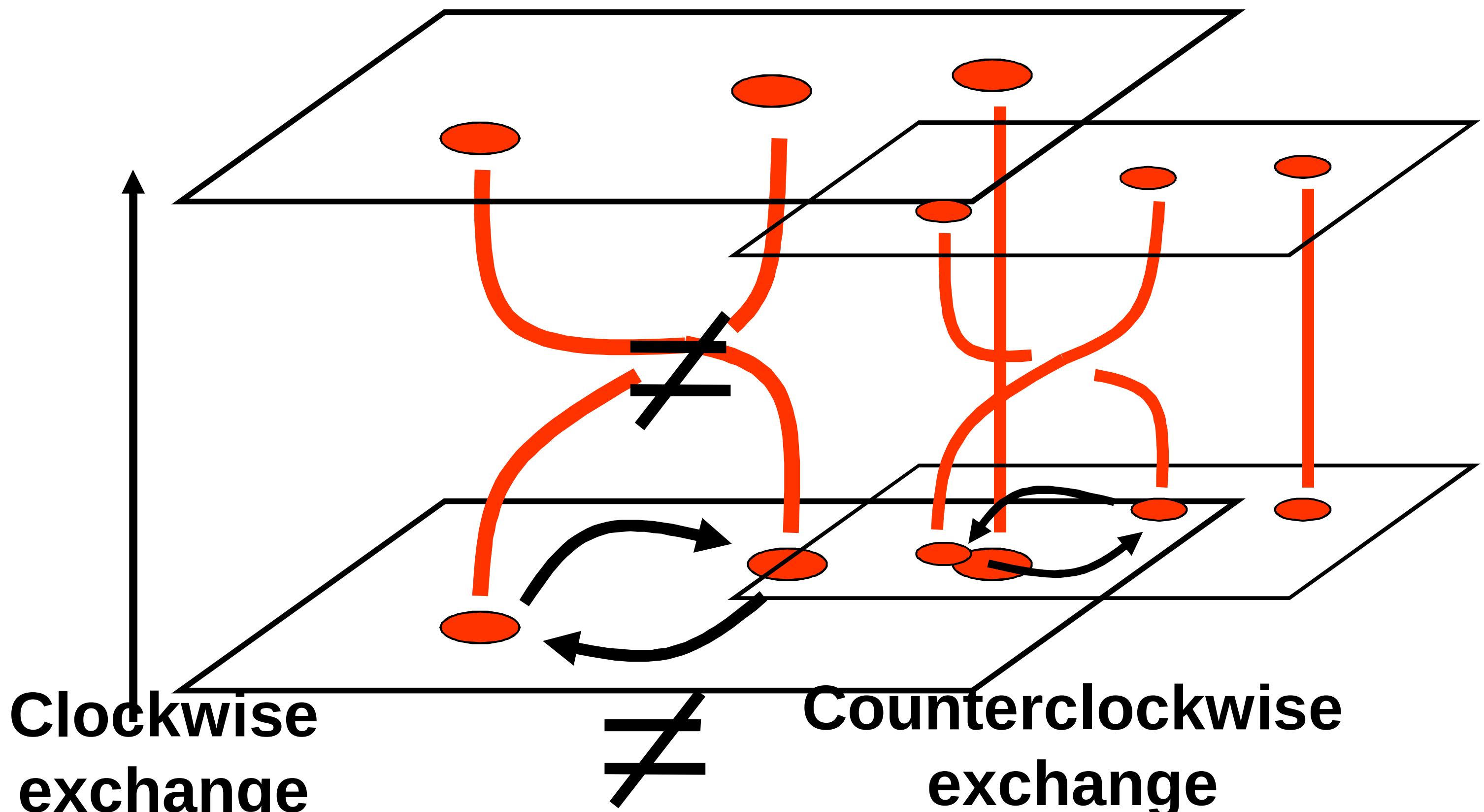
Electrons, Protons, Neutrons…

Particle Exchange in 2+1 Dimensions



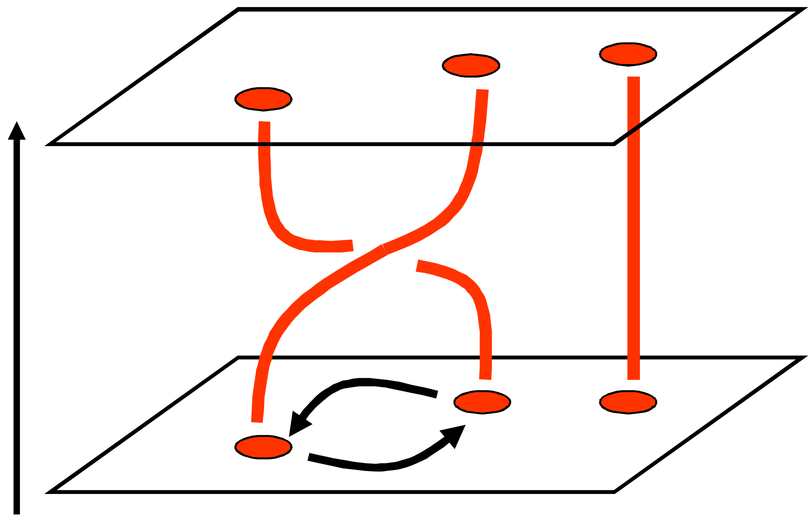
Particle “world-lines” form **braids** in 2+1 (=3) dimensions

Particle Exchange in 2+1 Dimensions



Particle “world-lines” form **braids** in 2+1 (=3) dimensions

Fractional (Abelian) Statistics



$$\left|\psi_f\right\rangle = e^{i\vartheta}\left|\psi_i\right\rangle$$

Phase

$$\left|\psi_i\right\rangle$$

$\theta = 0$ Bosons

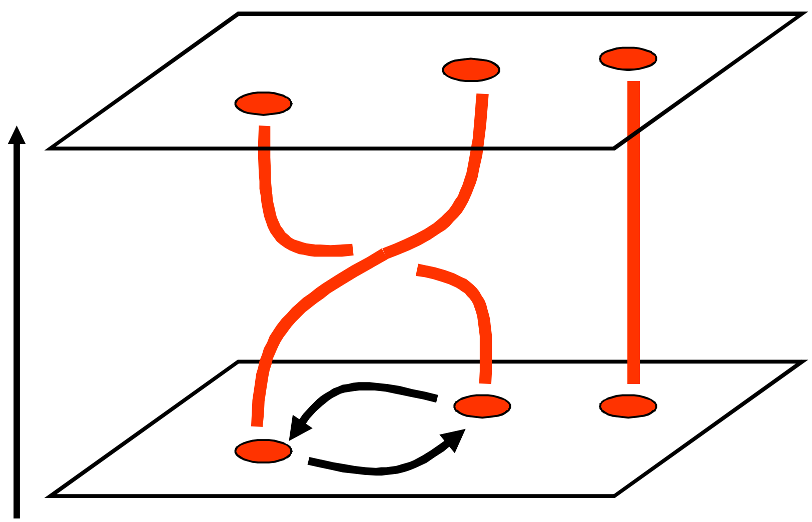
$\theta = \pi$ Fermions

$\theta = \pi/3$ ν=1/3 quasiparticles

Anyons

Only possible for particles in 2 space dimensions.

Non-Abelian Statistics (Moore & Read '91)



$$|\psi_f\rangle = \tilde{\alpha}|\psi_0\rangle + \tilde{\beta}|\psi_1\rangle$$

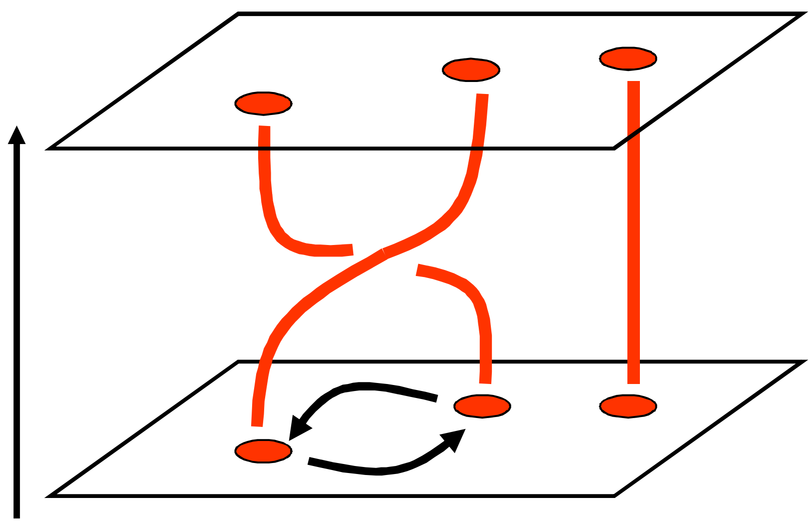
$$\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$|\psi_i\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

degenerate states

Non-Abelian Statistics (Moore & Read '91)



$$|\psi_f\rangle = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

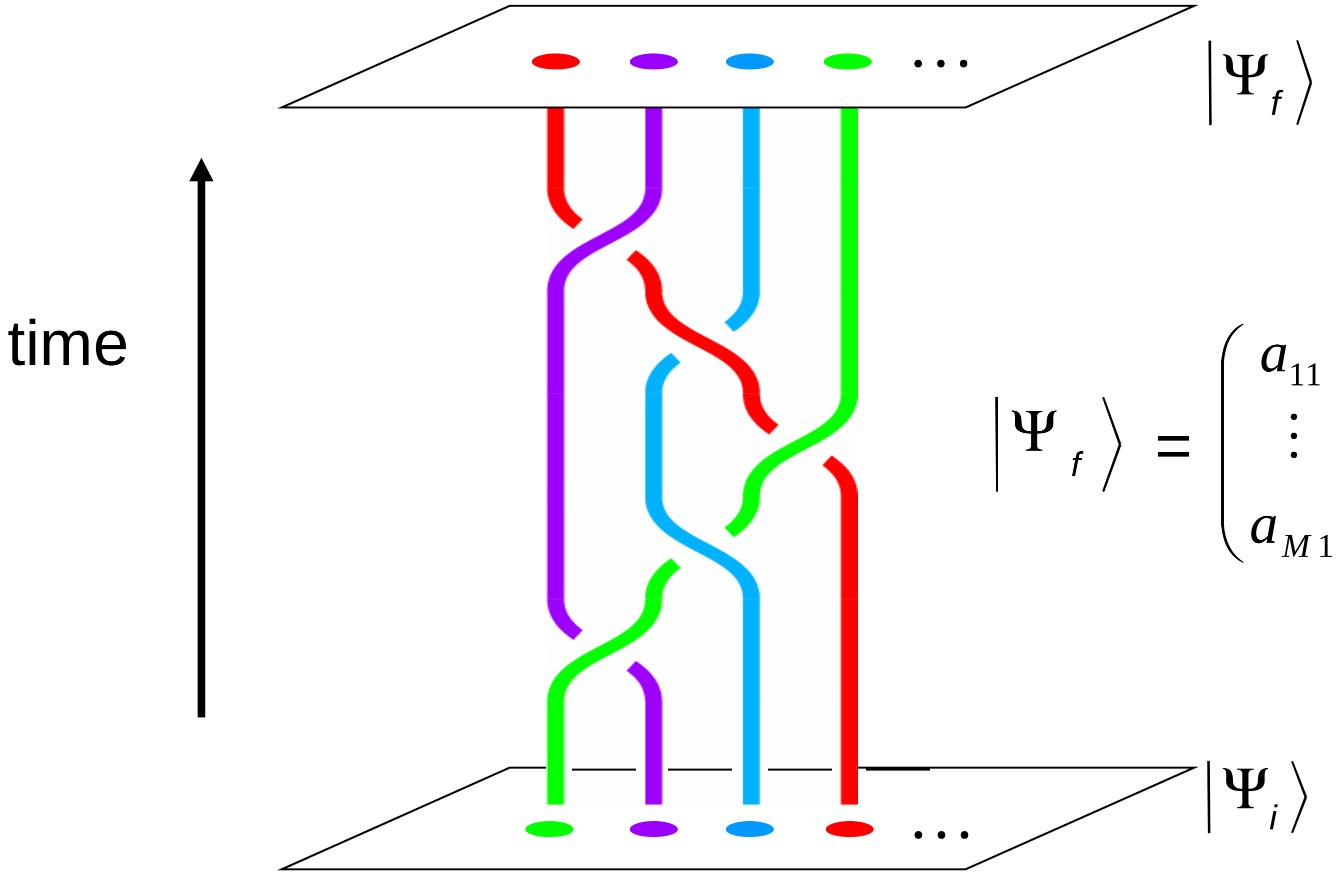
$$|\psi_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Matrix!

Matrices form a **non-Abelian** representation of the **braid group**.

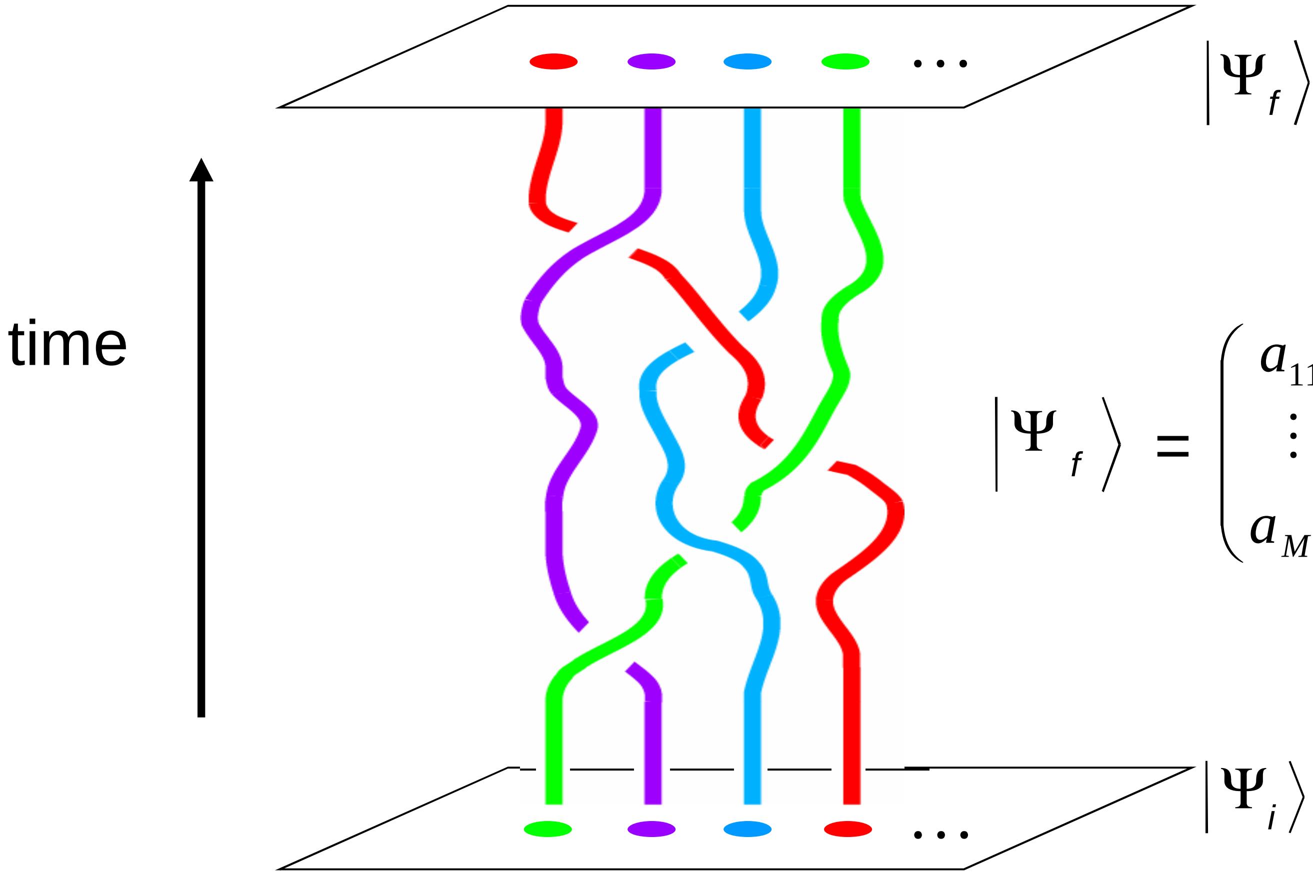
(Related to the Jones Polynomial, TQFT (Witten), Conformal Field Theory (Moore, Seiberg), etc.)

Many Non-Abelian Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Many Non-Abelian Anyons



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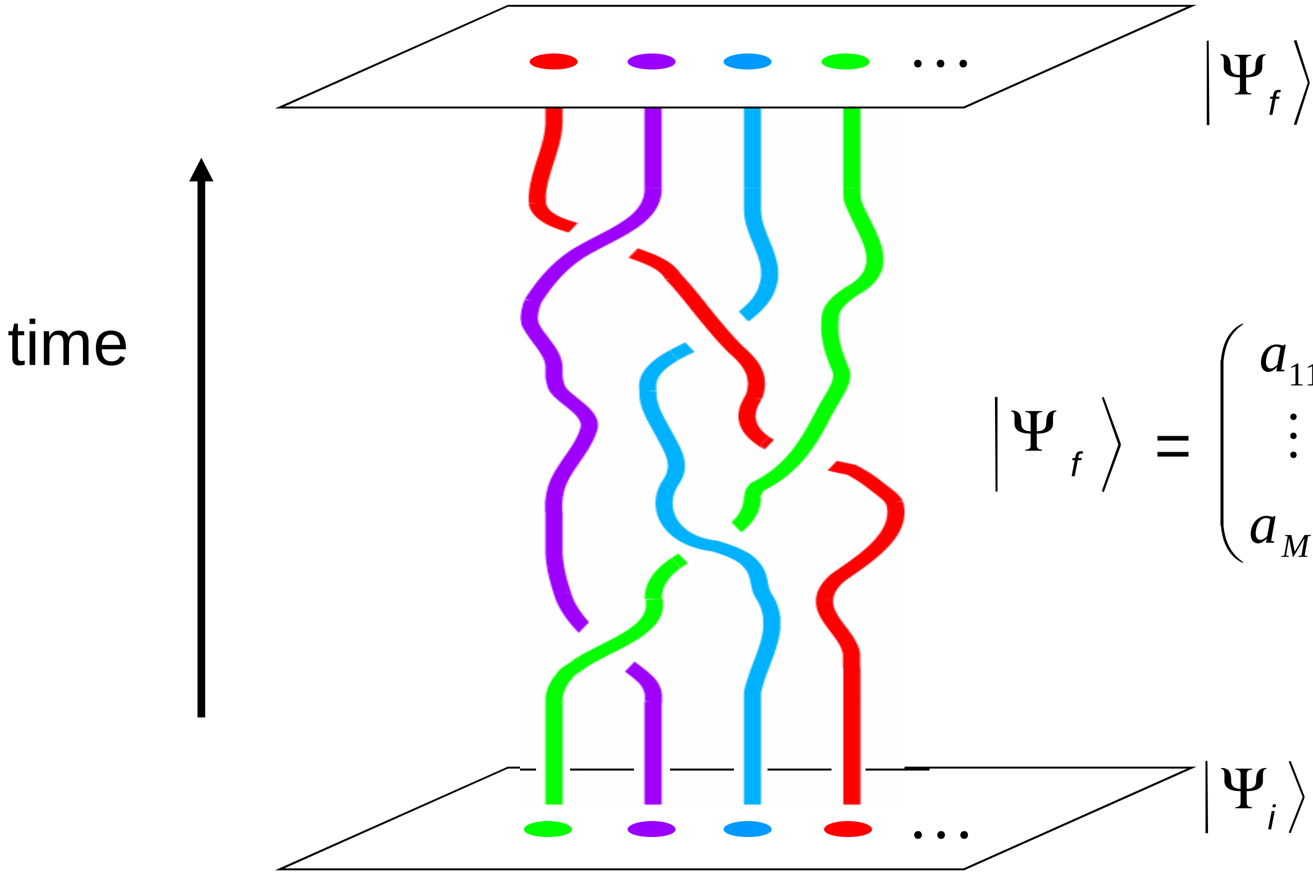


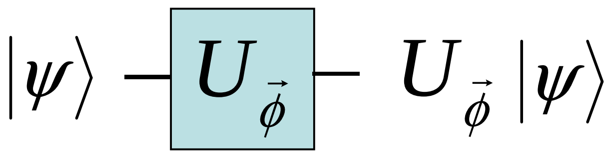
$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Matrix depends only on the topology of the braid swept out by anyon world lines!

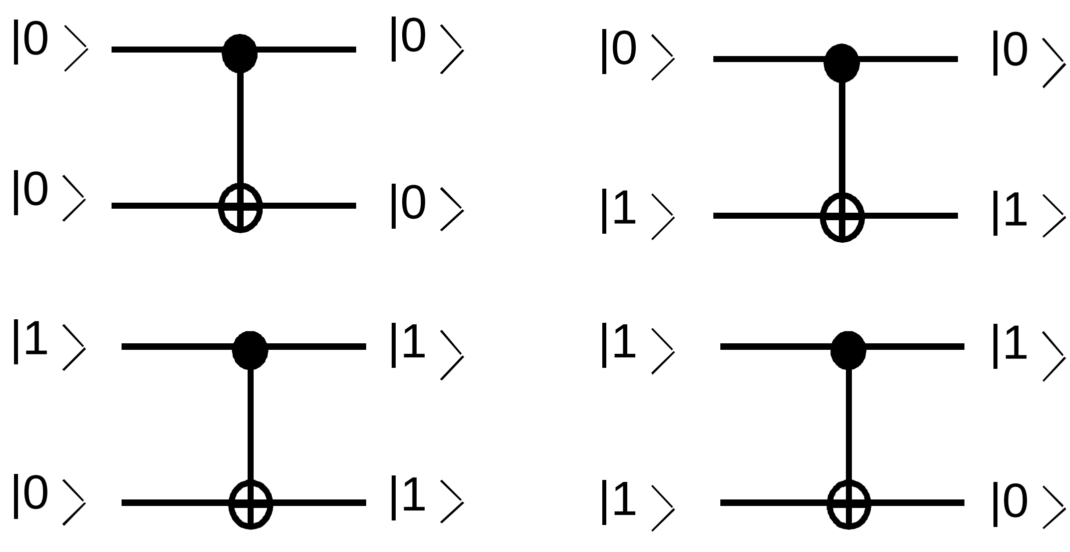
Robust quantum computation?

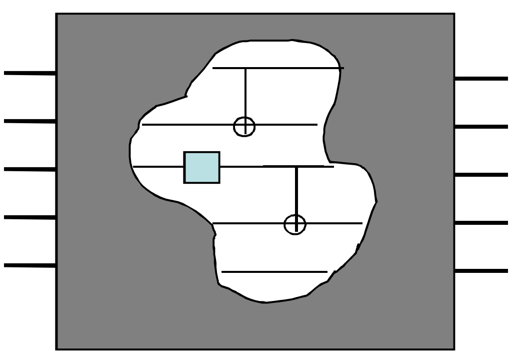
Universal Quantum Gates

Single Qubit Rotation



Controlled-Not

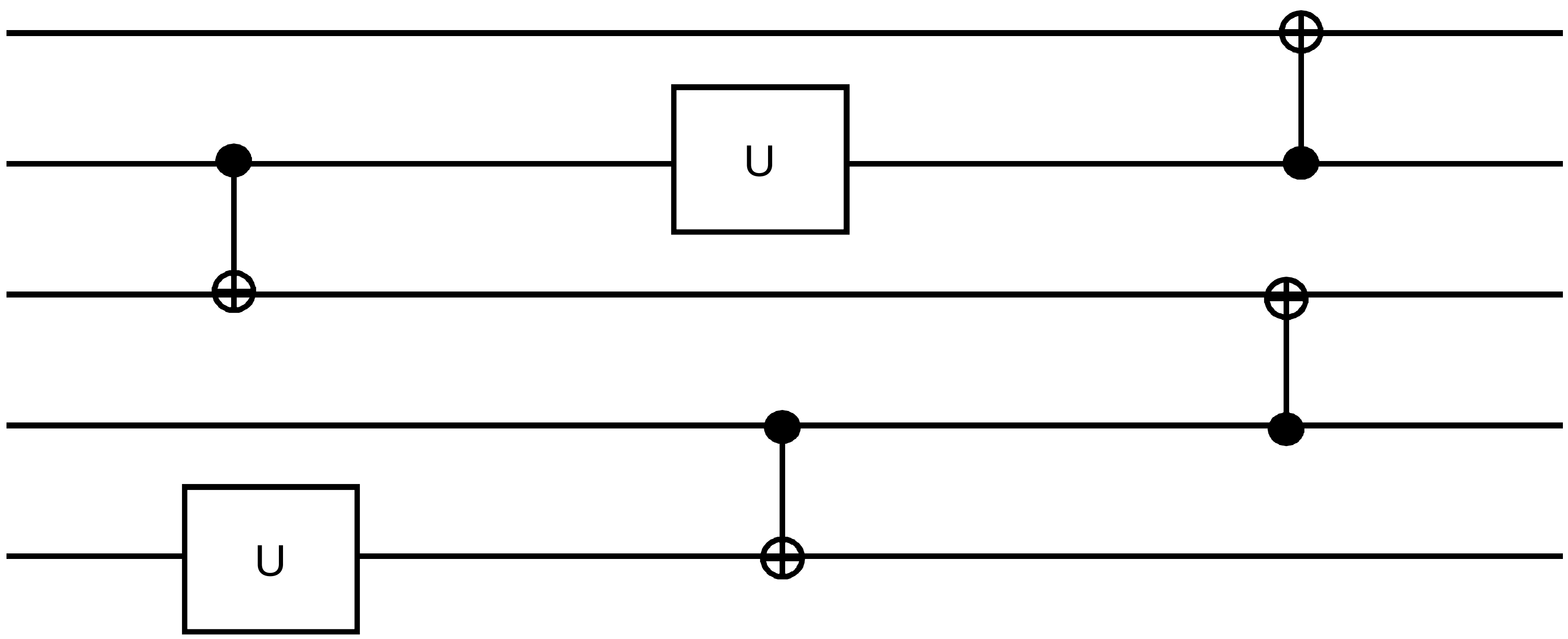




Any N qubit operation can be carried out using these two gates.

$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

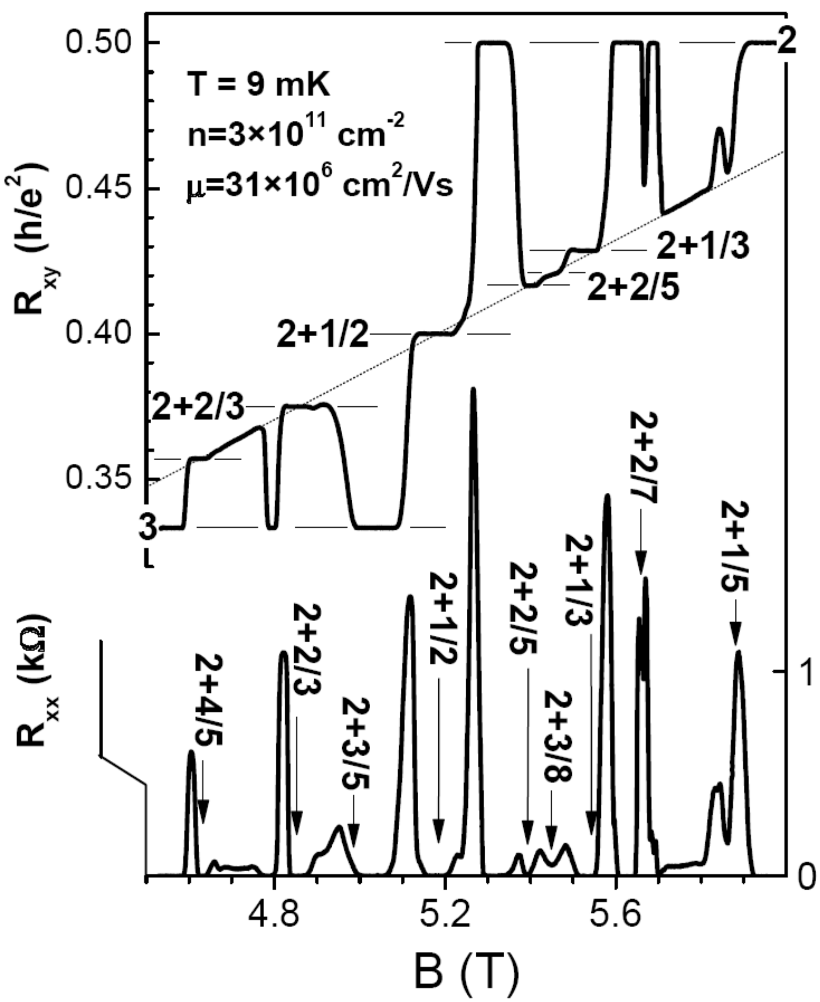
Quantum Circuit



What braid corresponds to this circuit?

Possible Non-Abelian FQH States

J.S. Xia et al., PRL (2004).



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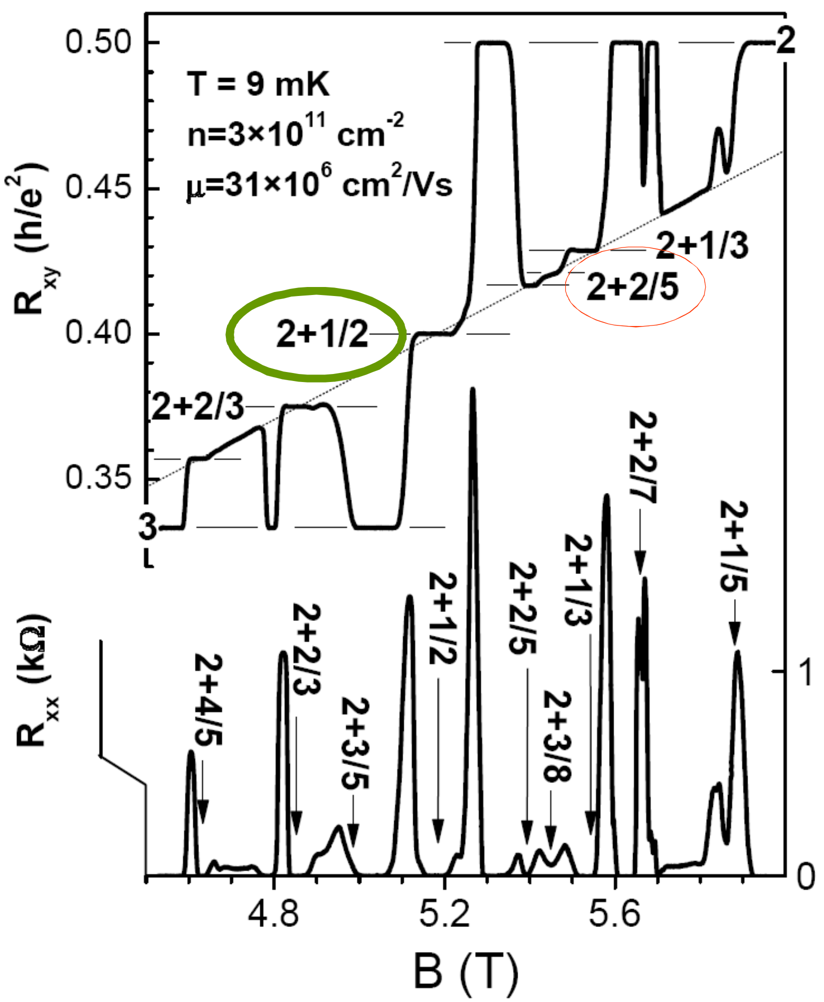

$\nu = 5/2$: Probable Moore-Read Pfaffian state.

Charge *e/4* quasiparticles described by *$SU(2)_2$* Chern-Simons Theory.

Nayak & Wilczek, '96

Possible Non-Abelian FQH States

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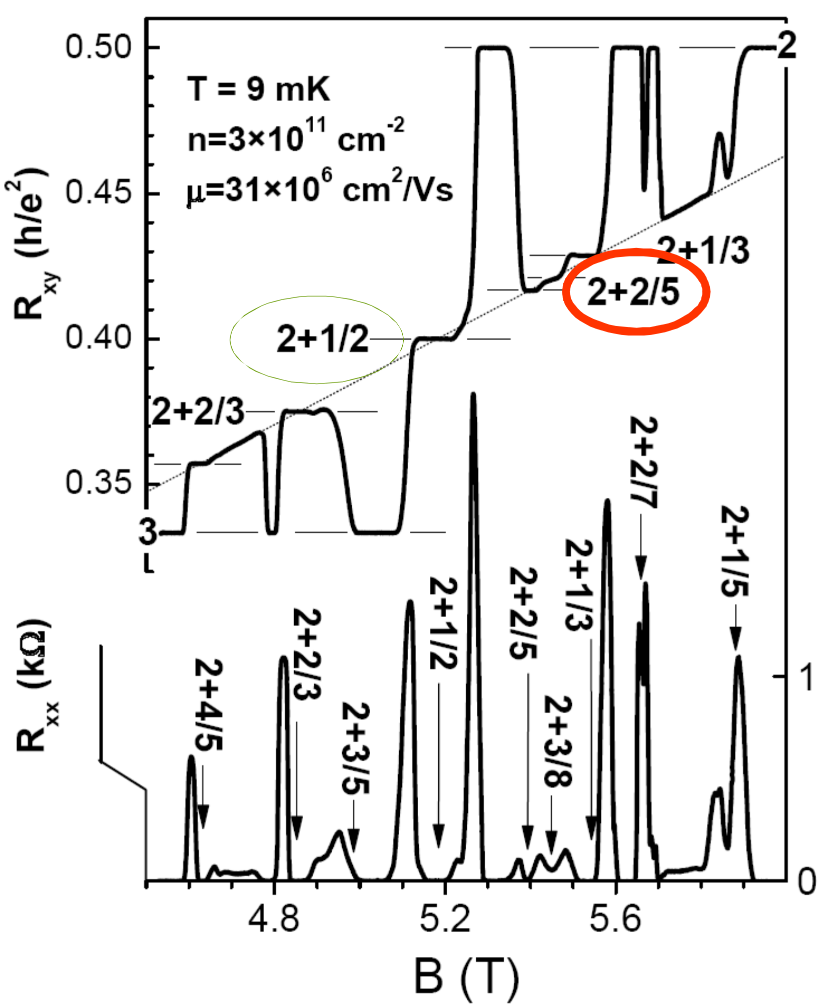

ν= **5/2**: Probable Moore-Read Pfaffian state.

Charge *e/4* quasiparticles described by *SU(2)*$_2$ Chern-Simons Theory.

Nayak & Wilczek, '96

ν = **12/5**: Possible Read-Rezayi "Parafermion" state. Read & Rezayi, '99

Charge *e/5* quasiparticles described by *SU(2)*$_3$ Chern-Simons Theory.
Slingerland & Bais '01

Universal for Quantum Computation!
Freedman, Larsen & Wang '02